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PADMA BHUSHAN AND PADMA VIBHUSHAN
PROFESSOR C. R. RAO, *FRS*
(September 10, 1920 - August 22, 2023)

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Editorial Note

We regret for delay in publication of this issue due to unavoidable circumstances on 15th August 2026 in completion of India's 79 years of Independence and 55 years of the Journal JÑĀNĀBHA

This volume of

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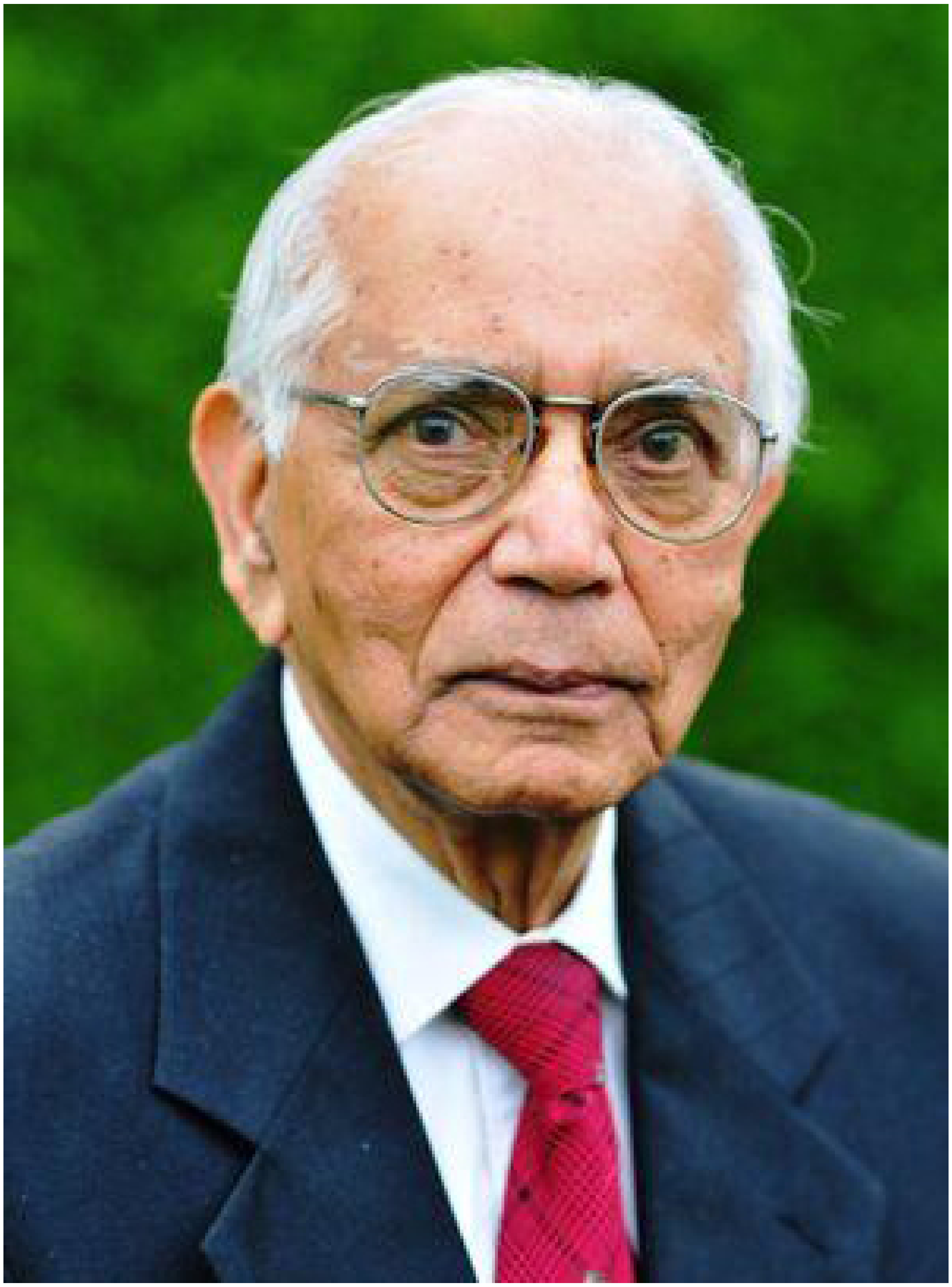
is being Dedicated to the Memory and Honour of
PADMA BHUSHAN AND PADMA VIBHUSHAN PROFESSOR C. R. RAO, *FRS*



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PADMA BHUSHAN AND PADMA VIBHUSHAN PROFESSOR C. R. RAO (CALYAMPUDI RADHAKRISHNA RAO), *FRS*: A LEGEDARY STATISTICIAN AND MATHEMATICIAN

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On behalf of VIJÑĀNA PARISHAD OF INDIA and *JÑĀNĀBHA* Family, we feel great proud to publish special Issue of *JÑĀNĀBHA* Vol. 56(1) (2026)
Dedicated to the Memory and Honour of Padma Bhushan and Padma Vibhushan
Late Professor C. R. Rao (September 10, 1920 - August 22, 2023)

Professor C. R. Rao was a world-renowned statistician and mathematician whose pioneering contributions to statistical theory and multivariate analysis have left an indelible mark on science, mathematics, economics, medicine and engineering across the globe. He is widely regarded as one of the foremost mathematical scientists of the twentieth century. His theoretical discoveries, made over a career spanning more than seven decades, have become indispensable components of the graduate and postgraduate curricula in statistics, econometrics, electrical engineering and allied di sciplines at universities worldwide.

As a man, Professor C. R. Rao was a great human being, an exemplary teacher, an eminent researcher, a visionary institution-builder and a perfectionist in every sense of the word.

Professor C. R. Rao: A Bibliographic Sketch

Professor C. R. Rao was the 8th of the ten children born on September 10, 1920, at Hoovina Hadagali, Bellary District, Karnataka State. His father, C. Daraiswamy Naidu, served as an Inspector of Police. Professor Rao completed his schooling at various institutions across Andhra Pradesh including Gudur, Nuzvid, Nandigram, and Visakhapatnam. He received his M.A. degree in Mathematics with First Class and First Rank from Andhra University in 1941, and thereafter his M.A. degree in Statistics with First Class, First Rank and a Gold Medal from Calcutta University in 1943 - a record of marks that remained unbeaten for many years. He pursued doctoral research at King's College, Cambridge University, under the supervision of Professor Ronald A. Fisher, widely regarded as the father of modern statistics, and was awarded the Ph.D. degree in 1948 for his thesis entitled Statistical Problems of Biological Classification. In 1965, Cambridge University conferred upon him the higher degree of D.Sc. based on his outstanding publications in statistics.
Citizenship:
Indian (until 1995),
American (1995-2023).
Professor C. R. Rao held the following distinguished academic and administrative posts:

- Director, Indian Statistical Institute, Kolkata
- Jawaharlal Nehru Professor and National Professor, India
- University Professor, University of Pittsburgh, U.S.A.
- Eberly Professor and Chair of Statistics, Pennsylvania State University, U.S.A.
- Director, Center for Multivariate Analysis, Pennsylvania State University, U.S.A.
- Research Professor, University at Buffalo, U.S.A.

As Head and later Director of the Research and Training School at the Indian Statistical Institute for a period of over 40 years, Professor Rao developed internationally acclaimed research and training programmes and produced several leaders in the fields of mathematics and statistics. In addition to a teaching and research experience of more than 60 years, Professor Rao was on the frontline of mathematical statisticians and data scientists in India and abroad.

Research Experience: more than 75 years

- Guided numerous candidates for Ph.D. degrees in Statistics and Mathematics
- Published approximately 500 research papers in reputed national and international journals
- Author of 15 textbooks and editor of numerous special volumes in probability and statistics
- His 1965 text Linear Statistical Inference and Its Applications has received more than 19,000 citations and is one of the most cited books in the whole of science
- Conducted and guided research projects of national and international importance

Professor C. R. Rao popularized multivariate statistical methods and their applications in the physical, biological and social sciences by delivering plenary lectures and keynote addresses at hundreds of National and International Conferences, Seminars and Symposia. His scientific contributions in the following areas are especially worth noting:

- Derivation of the Cramr-Rao Lower Bound, establishing the fundamental limit on the variance of any unbiased statistical estimator
- Formulation of the Rao-Blackwell Theorem, a cornerstone of estimation theory taught in undergraduate and postgraduate curricula worldwide
- Development of Information Geometry, connecting statistical theory with differential geometry
- Foundations of multivariate analysis of variance (MANOVA) and discriminant analysis
- Development of generalized inverses of singular matrices and their statistical applications
- Characterization of probability distributions and asymptotic theory
- Applications of statistical methodology to economic planning, weather prediction, medical diagnosis, genetics, anthropology, demography and engineering

Areas of Research Contributions

- Estimation Theory
- Statistical Inference and Linear Models
- Multivariate Analysis
- Combinatorial Design
- Orthogonal Arrays
- Statistical Genetics
- Generalized Matrix Inverses
- Functional Equations
- Biometry

Life Member / Fellow of the Following Societies and Academies

- Fellow, Royal Society, United Kingdom
- Fellow, U.S. National Academy of Sciences
- Fellow, American Academy of Arts and Sciences
- National Academy of Sciences, India
- Indian Statistical Institute
- International Statistical Institute (Former President, 1977-1979)
- Institute of Mathematical Statistics (Former President, 1976-1977)
- International Biometric Society (Former President, 1973-1975)
- He served on the Editorial Boards of various national and international journals and as a reviewer of research papers and projects for scientific bodies worldwide.
- Professor Rao visited numerous countries as Visiting Faculty and delivered invited, plenary and keynote lectures in the U.S.A., United Kingdom, Germany, Canada, Australia, Japan, Italy, France, and many other nations.
- On the recommendation of Professor Rao, the Asian Statistical Institute (now known as the Statistical Institute for Asia and Pacific) was established in Tokyo to provide training to statisticians in government and industry.
- He was Chairman of a United Nations Committee on statistical personnel needs in Asian countries.

Professor C.R. Rao receiving the Padma Vibhushan Award from President K.R. Narayanan, 2001. Image credit: B.L.S. Prakasa Rao

Professor C.R. Rao receiving the National Medal of Science from President Bush, 2002. Image credit: Wikimedia Commons/NSF.

Professor C.R. Rao receiving the India Science Award from Prime Minister Manmohan Singh, 2010. Image credit: B.L.S. Prakasa Rao.

Honours and Awards

- Shanti Swarup Bhatnagar Award for Science and Technology (1963)
- Padma Bhushan (1968) and Padma Vibhushan (2001), Government of India
- International Mahalanobis Prize (2003), International Statistical Institute
- Srinivasa Ramanujan Medal (2003), Indian National Science Academy
- United States National Medal of Science (2002) - presented by President George W. Bush, the highest scientific honour in the United States
- India Science Award (2010) - the highest honour conferred by the Government of India in a scientific domain
- Guy Medal in Gold (2011), Royal Statistical Society, United Kingdom - the highest award given to a statistician in the United Kingdom
- International Prize in Statistics (2023) - widely regarded as the Nobel Prize equivalent in the field of statistics
- Recipient of 40 honorary doctoral degrees from universities across 19 countries in six continents

In Honour of Professor Rao

- The Pennsylvania State University has established C. R. and Bhargavi Rao Prize in Statistics
- C. R. Rao Advanced Institute of Mathematics, Statistics and Computer Science
- National Award in Statistics established by Ministry of Statistics and Programme Implementation (MoSPI), Government of India
- The road from IIIT Hyderabad passing along Central University of Hyderabad crossroads to Alind Factory, Lingampally is named "Prof. C.R. Rao Road"

In 1988, the Times of India identified Professor Rao as one of the top 10 modern scientists of Indian origin, alongside mathematician S. Ramanujan and Nobel Laureates S. Chandrasekar, H. Khorana and C. V. Raman. In the Chronology of Probabilists and Statisticians compiled by the University of Southampton, he was listed among the 57 foremost scientists to have made significant contributions to probability and statistics since the sixteenth century - a list that includes Galileo Galilei, Isaac Newton and Pierre-Simon Laplace.

In addition to his brilliantly distinguished academic career, Professor C. R. Rao was very popular among his students, colleagues, and fellow workers for his cordial support and personal generosity. He was a disciplined, humble, and deeply humane person who believed that the true value of mathematics lay in its capacity to illuminate truth and inspire others. His wife Smt. Bhargavi Rao was his constant companion and source of strength from their marriage in 1948 until her passing in 2017. Professor C. R. Rao passed away on 22 August 2023 in Buffalo, New York, at the age of 102, leaving behind a legacy that shall endure generations to come.

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A NOTE ON INEQUALITIES AND BOUNDS FOR LAMBERT *W* FUNCTION, AND ITS RELATIONSHIPS WITH GAMMA AND PSI FUNCTIONS

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Abstract

The Lambert W function, also known as the product logarithm, is an important special function with significant applications in mathematics, probability, combinatorics, and physics. This paper investigates inequalities and bounds for the Lambert W function and explores its relationships with the Gamma and Psi functions. Derivations, approximations, and bounds are provided. Figures demonstrate the accuracy of bounds and approximations.
2020 Mathematical Sciences Classification: 26D07, 33B15, 33E50, 68Q17.
Keywords and Phrases: Asymptotics; Bounds; Gamma function; Inequalities; Lambert W function; Psi (digamma) function; Special functions.

1 Introduction

The Lambert W function, denoted as $W(x)$, is defined as the inverse of the function $f(W) = W * e^W$, such that $W(x) * e^{W(x)} = x$. It has found applications in many fields such as mathematics, physics, chemistry, biology, probability, and statistics (cf. Georg [9]). The Lambert W function also arises in asymptotic approximations, the analysis of algorithms, combinatorics, quantum physics, and risk theory, among others. The Gamma function $\Gamma(x)$, which generalizes the factorial function, and the Psi function $\psi(x)$, the logarithmic derivative of $\Gamma(x)$, are two of the most studied special functions in mathematics. Recent studies have sought to develop inequalities that connect these functions to one another, since such relationships yield powerful tools in both theoretical and applied mathematics. For details on the Lambert W function $W(x)$, the Gamma function $\Gamma(x)$, and the Psi function $\psi(x)$, the interested readers are referred to Abramowitz and Stegun [2], Amdeberhan and Moll [3], Artin [4], Brito *et al.* [5], Corless *et al.* [6], Dence [7], Erdélyi *et al.* [8], Hassani [10], Hoorfar and Hassani [11], Olver *et al.* [14], and Shakil and Singh [16], among others.

In this paper, we aim to establish inequalities and bounds for the Lambert W function and explore its connections with the Gamma and Psi functions. These relationships will provide new insights into analytic inequalities, asymptotic expansions, and applications in probability theory.

The organization of paper is as follows: In Section 2, we provide some definitions and preliminaries. Section 3 contains main results. In Section 4, we present applications and results. Some conclusions are drawn in Section 5.

2 Definitions and Preliminaries

In what follows, we present some preliminaries of the three central functions: the Lambert $W(x)$, Gamma $\Gamma(x)$, and Psi $\psi(x)$ functions.

2.1 Definitions

- Lambert $\boldsymbol{W}$ function: defined implicitly by $W(x) * e^{W(x)} = x$.
- Gamma function: $\Gamma(x) = \int_0^\infty t^{(x-1)} e^{-t} dt$, for $\mathrm{Re}(x) > 0$.
- Psi function: $\psi(x) = \frac{d}{dx}(ln\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$.

2.2 Fundamental Asymptotics

The following are some fundamental asymptotics that will also be used in our main results:

- $W(x) \sim ln(x) - ln(ln(x))$ as $x \to \infty$.
- $\Gamma(x+1) \sim \sqrt{}(2\pi x)(x/e)^x$ as $x \to \infty$.
- $\psi(x) \sim ln(x) - 1/(2x)$ for large x.

2.3 Basic Properties of the Lambert $\boldsymbol{W}$ Function

Here, we present some basic properties of the Lambert W function (cf. Brito *et al.* [5], Corless *et al.* [6], and Dence [7]). As stated above, the Lambert W function, denoted as $W(x)$, is defined as the inverse of the function

$$f(W) = W * e^W.$$

Thus, it satisfies the fundamental relation:

$$W(x) * e^{W(x)} = x, x \in \left[-\frac{1}{e}, \infty\right).$$

It is a multivalued function, meaning that for some values of x, there are multiple values of $W(x)$. It is commonly used to solve transcendental equations such as:

$$xe^x = a \Rightarrow x = W(a).$$

The following are some important properties of the Lambert W function:

I. Real Branches: For real values of x, the Lambert W function has two real branches:

(a) Principal Branch $W_0(\boldsymbol{x})$: (i) Defined for: $x \geq -\frac{1}{e}$; (ii) Range: $W_0(x) \geq -1$; and (iii) Passes through: $W_0(0) = 0$, and $W_0(e) = 1$.

(b) Lower Branch $\boldsymbol{W_{-1}(x)}$: (i) Defined for: $-\frac{1}{e} \leq x < 0$; and (ii) Range: $W_{-1}(x) \leq -1$.

(c) Branch Point: Both branches meet at: $x = -\frac{1}{e}, W\left(-\frac{1}{e}\right) = -1$.

(d) It is obvious from the above (cf. Olver *et al.* [14]) that on the interval $[\ 0, \infty\ )$, there is one real solution, and it is nonnegative and increasing. On the interval $\left(-\frac{1}{e}, 0\right)$, there are two real solutions, one increasing and the other decreasing. We call the increasing solution for which $W(x) \geq W\left(-\mathrm{e}^{-1}\right) = -1$ is the principal branch and we denote it by $W_0(x)$. The decreasing solution can be identified as $W_{\pm 1}(x \mp 0\mathrm{i})$.

(e) Graph of the Lambert $\boldsymbol{W}$ *Function:* Below is a visual (Figure 2.1) of the two real branches:

- $W_0(x)$: principal branch (upper curve).
- $W_{-1}(x)$: lower branch.

Illustration of the Graph: The graph illustrates multiple branches of the Lambert W function, specifically $\boldsymbol{W_0(x)}, \boldsymbol{W_{\pm 1}(x)}$, and $\boldsymbol{W_{\pm 2}(x)}$, plotted as functions of x. The vertical axis represents the real part of the function, $\mathrm{Re}(W(x))$, while the horizontal axis represents x.

Principal Branch $\boldsymbol{W_0(x)}$: It is defined for $x \geq -1/e$ and passes through $(0, 0)$. It increases smoothly and grows slowly for large x.

Lower Branch $\boldsymbol{W_{-1}(x)}$: It is defined for $-1/e \leq x < 0$. It decreases sharply and approaches negative infinity as x approaches 0 from the left. It meets $\boldsymbol{W_0(x)}$ at $(-1/e, -1)$.

Higher Branches $\boldsymbol{W_1(x)}$, and $\boldsymbol{W_{+2}(x)}$: These branches are complex-valued. The graph shows only their real parts. They drop sharply near $x = 0$ and then increase gradually, remaining below the principal branch.

Behavior Near $x = 0$: Non-principal branches tend toward large negative values near $x = 0$, while $\boldsymbol{W_0(x)}$ remains smooth.

Branch Point: At $x = -1/e$, multiple branches meet, showing the multi-valued nature of the function. So, we can conclude that the Lambert W function consists of multiple branches. $\boldsymbol{W_0(x)}$ is the main real branch, while others provide additional solutions, often complex, useful in advanced mathematical applications.

II. Some Other Properties: The following properties of the Lambert W function are also noteworthy:

(a) Fundamental Identity: $W(x) * e^{W(x)} = x$.

(b) Logarithmic Form: For $x > 0$: $W(x) = ln(x) - ln(W(x))$.

(c) Derivative: $W'(x) = \frac{W(x)}{x(1+W(x))}, x \neq 0$.

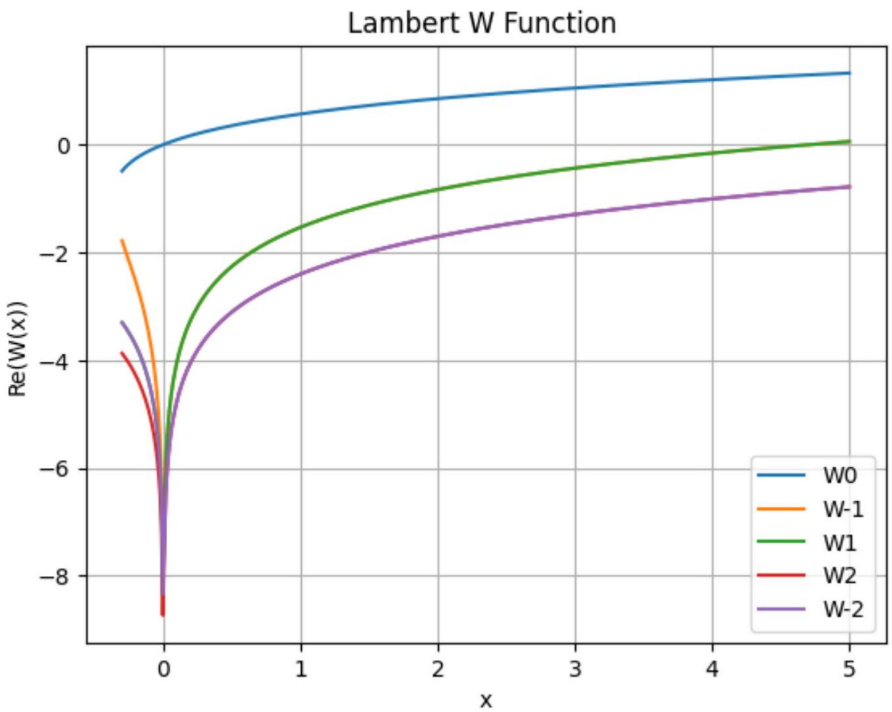

Figure 2.1: *Branches $W_0(x), W_{\pm 1}(x)$, and $W_{\pm 2}(x)$ of the Lambert W function*

(d) Integral: $\int W(x)dx = xW(x) - x + C$.
(e) Special Values:

$$\begin{aligned} &W_0(0) = 0, \\ &W_0(e) = 1, \\ &W\left(-\frac{1}{e}\right) = -1 \end{aligned}.$$

(f) Omega Constant: The Omega constant Ω is defined as: $\Omega = W(1)$.
Numerically: $\Omega \approx 0.567143$.

3 Main Results: Inequalities and Bounds

We now present the inequalities and bounds derived for the Lambert W function and its connections to the Gamma and Psi functions.

3.1 Inequalities for Lambert W Function

3.1.1. For $\boldsymbol{x} \geq \boldsymbol{e}$, the following inequalities hold:
Lower bound:

$$W(x) \geq ln(x) - ln(ln(x)), \tag{3.1}$$

Upper bound:

$$W(x) \leq ln(x). \tag{3.2}$$

Proof. For proof, see Hassani [10], and Hoorfar and Hassani [11]. □

3.1.2. Combining (3.1) and (3.2), we obtain

$$ln(x) - ln(ln(x)) \leq W(x) \leq ln(x). \tag{3.3}$$

Proof. For proof, see Hassani [10], and Hoorfar and Hassani [11]. □

3.1.3. The following refined inequality provided is an approximation for the Lambert W function, $W(x)$, for large values of x :

$$ln(x) - ln(ln(x)) + \left(\frac{ln(ln(x))}{ln(x)}\right) \leq W(x) \leq ln(x) - (1 - ln(ln(x)))/ln(x). \tag{3.4}$$

Proof. A formal proof involves analyzing the asymptotic behavior of $W(x)$ and comparing it to the given bounds, as follows.

Asymptotic Expansion of $\boldsymbol{W(x)}$: For large x, the Lambert W function can be approximated by its asymptotic expansion, with the leading terms given by:

$$W(x) \approx ln(x) - ln(ln(x)) + \frac{ln(ln(x))}{ln(x)} - \frac{(ln(ln(x)))2 - 2ln(ln(x)) + 2}{2(ln(x))^2} + \cdots . \tag{3.5}$$

The upper bound in (3.4) is given by

$$W(x) \leq ln(x) - (1 - ln(ln(x)))/ln(x),$$

which can be written as

$$W(x) \leq ln(x) - \frac{1}{ln(x)} + \frac{ln(ln(x))}{ln(x)} . \tag{3.6}$$

Comparing (3.6) to the asymptotic expansion (3.5), it is observed that the upper bound corresponds to the first few terms of the expansion, with the higher-order terms being negative for sufficiently large x, from which the inequality (3.6) evidently follows. Similarly, the lower bound in (3.4) is given by

$$ln(x) - ln(ln(x)) + \frac{ln(ln(x))}{ln(x)} \leq W(x). \tag{3.7}$$

The lower bound in (3.7) directly matches the first three terms of the asymptotic expansion (3.5) of $W(x)$. This completes the proof of the inequality (3.4). $\square$

3.2 Relationships with the Gamma Function

By exploiting Stirling's approximation and the asymptotics of W, we establish:

$$\Gamma(W(x) + 1) \approx \sqrt{(2\pi W(x))} \cdot (W(x)/e)^{W(x)}.$$

This gives rise to inequalities of the form:

$$\Gamma(W(x) + 1) < x^{W(x)} < \Gamma(W(x) + 2).$$

3.3 Relationships with the Psi Function

Since $\psi(x) = \frac{d}{dx}(ln\Gamma(x))$, applying it, $W(x)$ yields approximations:

$$\psi(W(x)) \approx ln(W(x)) - 1/(2W(x)).$$

This connects Lambert W with logarithmic asymptotics and allows for inequalities of the form:

$$ln(W(x)) - 1/W(x) < \psi(W(x)) < ln(W(x)) - 1/(2W(x)).$$

3.4 Relationships with Entropy of Gamma-Distributed Variables

If a random variable X follows a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, its differential entropy $h(X)$ is given by (Johnson *et al.* [12]),

$$h(X) = \alpha + ln(\beta) + ln\Gamma(\alpha) + (1 - \alpha)\psi(\alpha) \tag{3.8}$$

Special Case (Exponential Distribution): For $\alpha = 1$, the gamma distribution becomes an exponential distribution, and the entropy simplifies to

$$h(X) = 1 + ln(\beta).$$

Now, using the inequality (3.1) in (3.8), we obtain

$$\begin{aligned} h(X) &= \alpha + ln(\beta) + ln\Gamma(\alpha) + (1 - \alpha)\psi(\alpha) \\ &\leq \alpha + W(\beta) + ln(ln(\beta) \cdot \Gamma(\alpha)) + (1 - \alpha)\psi(\alpha) \end{aligned} \tag{3.9}$$

Further, the following inequality is a well-known result in the theory of special functions, Alzer [1],

$$ln(x) - \frac{1}{x} < \psi(x) < ln(x), x > 0. \tag{3.10}$$

which, on using in (3.9), we obtain

$$\begin{aligned} \alpha + ln(ln(\beta) \cdot \Gamma(\alpha))(1 - \alpha)\left[ln(\alpha) - \frac{1}{\alpha}\right] &< h(X) \\ &< \alpha + W(\beta) + ln(ln(\beta) \cdot \Gamma(\alpha)) + (1 - \alpha)ln(\alpha). \end{aligned} \tag{3.11}$$

3.5 Relationships to Entropy Power of Gamma Distribution

When random variables follow a Gamma distribution, their entropy power (or "effective variance") is characterized by the Γ function and its logarithmic derivative, the ψ function (digamma function). For a random variable X with differential entropy $h(X)$, the entropy power $N(X)$ is defined as the unique value satisfying

$$N(X) = \frac{1}{2\pi e} * e^{2h(X)}, \tag{3.12}$$

see Kapur [13].

Also, for a random variable $X \sim \text{Gamma}(\alpha,\ \text{b})$, (shape α, rate $\text{b} = 1/\beta$), where β is the scale parameter, the differential entropy is given by

$$h(X) = \alpha - ln(b) + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha). \tag{3.13}$$

Thus, combining (3.11) and (3.12), the entropy power of gamma distribution is obtained as follows:

$$N(X) = \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha)]}}{2\pi e b^2}, \tag{3.14}$$

from which, on using (3.10), we obtain

$$N(X) = \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha)]}}{2\pi e b^2} < \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)ln(\alpha)]}}{2\pi e b^2}. \tag{3.15}$$

3.6 Relationships with Cramer-Rao Inequality

The Cramér-Rao inequality provides a lower bound on the variance of an estimator $\hat{\theta}$ of a parameter θ, defined as

$$\text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)},$$

where $I(\theta)$ is the Fisher information, Rohatgi and Saleh [15].

As an example, for a sample size of n from a gamma distribution $\Gamma(\alpha, \beta)$ with shape α and rate β, we have

A. Cramer-Rao Lower Bound for $\boldsymbol{\alpha}$:

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi'(\alpha)}, \text{ or } \text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)}, \tag{3.16}$$

where $\psi(\alpha) = \frac{dln\Gamma(\alpha)}{d\alpha}$ denotes the Digamma function, and $\psi_1(\alpha) = \psi'(\alpha)$ denotes the Trigamma function.

Further, using the following inequalities of Alzer [1],

$$\frac{1}{\psi_1(x)} > \frac{x^2}{1+x}, x > 0$$

and

$$\frac{1}{\psi_1(x)} > \frac{1}{e^{1/x} - 1}, x > 0,$$

respectively in (3.15), we obtain the following sharp bounds

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)} > \frac{\alpha^2}{n(1+\alpha)}$$

and

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)} > \frac{1}{n\left(e^{\frac{1}{\alpha}} - 1\right)}.$$

B. Cramer-Rao Lower Bound for β :

$$\text{Var}(\hat{\beta}) \geq \frac{\beta^2}{n\alpha}.$$

3.7 Expressing Cramer-Rao Inequality in Terms of the Lambert W Function

The Lambert W function $W(x)$ is the inverse of $f(w) = w * e^w$. It appears in Cramer-Rao bounds when the Fisher Information involves a transcendental equation, often arising in maximum likelihood estimations for distributions with exponential components (for example, in finding the value of θ that maximizes the log-likelihood function $L(\theta)$).

For example, assume a specific problem where the Fisher information has been derived as $I(\theta) = a * e^{b\theta}$ for constants a and b. To find the specific variance when the estimator is a function of a Lambert W-related variable, we substitute $I(\theta) = a * e^{b\theta}$ so that we obtain

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{a * e^{b\theta}} = \frac{1}{a} e^{-b\theta}.$$

If the parameter θ itself is determined by a maximum likelihood estimation involving an exponential, for example, $\theta = ln($ something $)$, this can be represented using $W(x)$.

In situations where $\psi(x)$ is approximated by $ln(x)$ (for large x), or when analyzing the asymptotic behavior of the estimator, we can use the identity that solutions to $x * e^x = y$ allow us to invert logarithmic equations. If $I(\theta)$ is derived to be a function given by

$$I(\theta) = \frac{\theta}{1 + W(\theta)}$$

then the Cramer-Rao lower bound (CRLB) is obtained as follows:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1 + W(\theta)}{\theta}.$$

Hence, in view of the above, it follows that the general Cramer-Rao inequality in terms of the Psi and Lambert W functions can be expressed as

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{n\psi_1(\theta)} \approx \frac{1}{nI(\mathrm{LambertW}(\theta))}.$$

4 Applications and Results

The following Figures 4.1-4.3 demonstrate the accuracy of bounds and approximations.

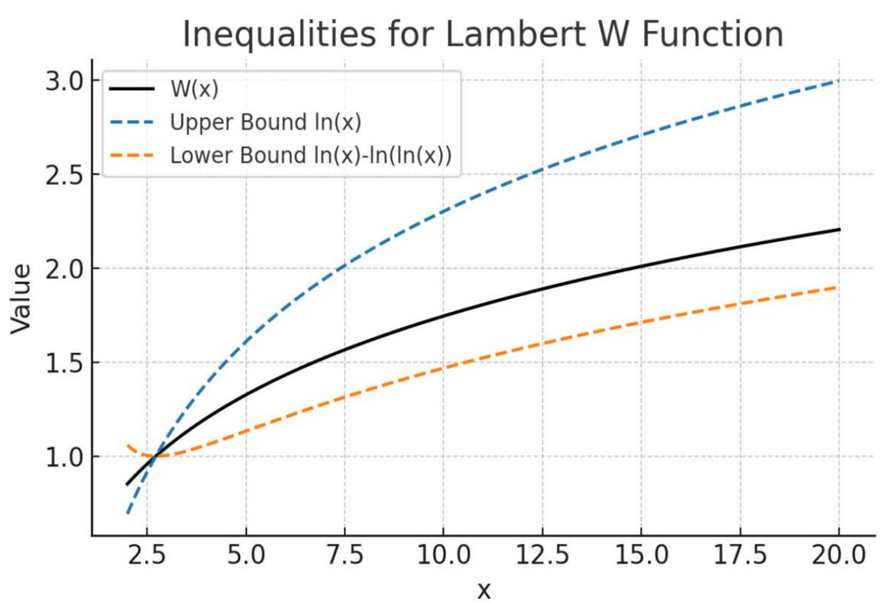

Figure* 4.1*: *Comparison of Lambert W with upper and lower bounds.*

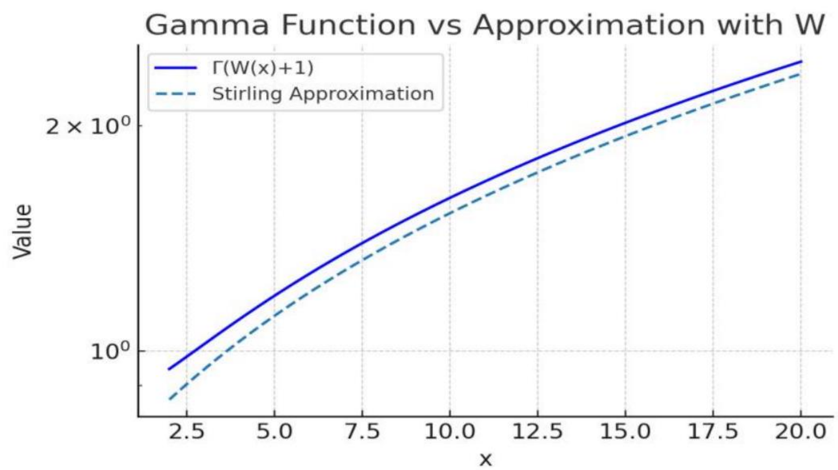

***Figure* 4.2:** *Comparison of* $\Gamma(W(x) + 1)$ *with Stirling's approximation.*

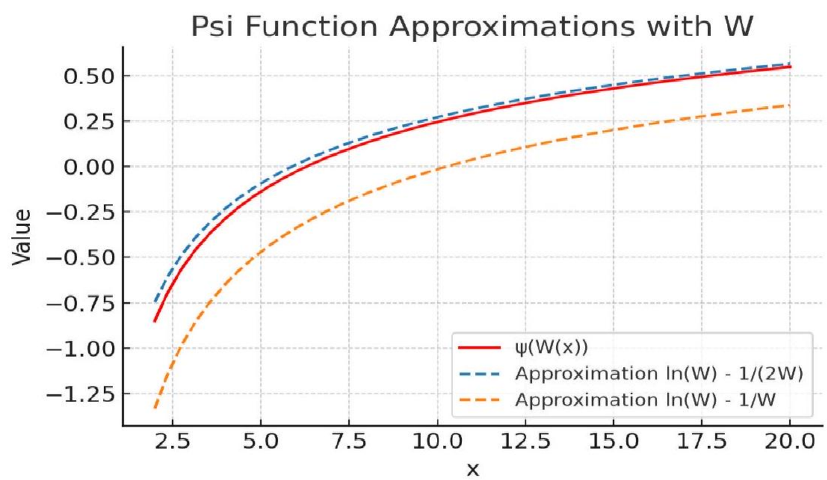

***Figure* 4.3:** *Comparison of* $\psi(W(x))$ *with logarithmic approximation.*

4.1 Lambert W Bounds:

As shown in Figure 4.1, the Lambert W function is compared with its upper and lower bounds. The approximation $W(x) \approx ln(x) - ln(ln(x))$ closely follows the actual function, while $ln(x)$ slightly overestimates and $ln(x) - ln(ln(x))$ slightly underestimates.

Discussion: Figure 4.1 highlights the effectiveness of logarithmic bounds for computationally efficient approximations of $W(x)$.

4.2 Relationships with the Gamma Function:

Figure 4.2 illustrates $\Gamma(W(x)+1)$ versus Stirling's approximation. Stirling's formula provides a very accurate estimate for larger x.

Discussion: Figure 4.2 confirms that Stirling's approximation can simplify calculations involving Gamma functions of $W(x)$.

4.3 Relationships with the Psi Function:

Figure 4.3 compares $\psi(W(x))$ with the logarithmic approximation $ln(W(x)) - 1/(2W(x))$. The approximation is nearly indistinguishable for $x \geq 1$.

Discussion: Figure 4.3 confirms the tightness of the approximation for analytical and computational purposes.

5 Conclusion and Future Directions

The analysis demonstrates that classical logarithmic bounds for Lambert W function are highly effective (Figure 4.1). Stirling's approximation accurately estimates $\Gamma(W(x)+1)$ (Figure 4.2), and the simple logarithmic formula approximates $\psi(W(x))$ well (Figure 4.3). These results collectively support efficient analytical and computational applications involving Lambert W, Gamma, and Psi functions. Future work may explore tighter inequalities for Lambert W function, extend the approximations to complex arguments, and analyze applications in advanced statistical models, information theory, and combinatorial mathematics. Integration with numerical software and symbolic computation could enhance practical utility.

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BICOMPLEX AND MULTICOMPLEX GENERALIZATION OF GROWTH PROPERTIES OF ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract

This paper extends several classical results concerning the comparative growth rates of entire solutions of linear differential equations with entire coefficients to the bicomplex and multicomplex settings. After introducing a detailed algebraic and analytic framework, we establish general forms of all lemmas and theorems using independent proofs based on the idempotent representation of bicomplex and multicomplex numbers. Comprehensive examples are also provided in the paper.

2020 Mathematical Sciences Classification: 30D20, 30D15, 30G35, 32A30, 34K41, 34M03, 34M04.

Keywords and Phrases: Entire solution, Bicomplex and multicomplex valued functions, Growth, Idempotent decomposition, Bicomplex and multicomplex differential equations

1 Preliminaries

1.1 Bicomplex Algebra

Let i, j be commuting imaginary units with $i^2 = j^2 = -1$. Let us define $k = ij$ so that $k^2 = 1$. Then the bicomplex algebra is denoted and defined by

$$\mathbb{BC} = \{z_1 + z_2 j : z_1, z_2 \in \mathbb{C}(i)\}.$$

The idempotent elements e_1, e_2 satisfy

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0 \quad \& \quad e_1 + e_2 = 1.$$

Every bicomplex number Z can be uniquely expressed as

$$Z = z_1 e_1 + z_2 e_2 \quad \textit{where} \quad z_1, z_2 \in \mathbb{C}.$$

Remark 1.1. *Addition, multiplication and differentiation of bicomplex functions are performed componentwise via this idempotent representation.*

1.2 Multicomplex Algebra

Let us define recursively $\mathbb{M}(n)$, algebra of n-th order multicomplex numbers as

$$\mathbb{M}(1) = \mathbb{C}, \quad \mathbb{M}(n) = \mathbb{M}(n-1) \otimes_{\mathbb{R}} \mathbb{C}$$

where n being a positive integer greater than 1 and $\otimes_{\mathbb{R}}$ denotes a tensor product over the real numbers $\mathbb{R}$.

This algebra contains 2^{n-1} orthogonal idempotents E_ℓ satisfying $E_\ell E_m = 0$ for $\ell \neq m$ and $\sum_\ell E_\ell = 1$. Every element $W \in \mathbb{M}(n)$ decomposes as

$$W = \sum_{\ell=1}^{2^{n-1}} w_\ell E_\ell \quad \textit{with} \quad w_\ell \in \mathbb{C}.$$

For detailed study one can see [10, 12].

2 Growth Functions and Bicomplex Entire Functions

The following definitions are immediate.

Definition 2.1. *A function $F : \mathbb{BC} \to \mathbb{BC}$ is bicomplex entire if there exist classical entire functions F_1, F_2 such that*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2.$$

Definition 2.2 ([7, 11, 19])**.** *Let $M_F(R) = \max_{|Z|=R} |F(Z)|$. Then the bicomplex (Φ, Ψ, χ)-order is denoted and defined by*

$$\varrho_{(\Phi,\Psi,\chi)}[F] = \limsup_{R\to\infty} \frac{\Phi(\log\log M_F(R))}{\Phi(\log \chi(R))}.$$

Then in terms of the idempotent components of F,

$$\varrho_{(\Phi,\Psi,\chi)}[F] = \max\{\varrho_{(\Phi,\Psi,\chi)}[F_1], \varrho_{(\Phi,\Psi,\chi)}[F_2]\}.$$

We do not explain in detail the standard theories of bicomplex and multicomplex analysis as those are available in [12]. For further details one can see [2, 3, 5, 8, 11, 13, 14, 16].

The following lemma reflects the nature of the growth decomposition of a bicomplex valued entire functions.

Lemma 2.1. *Let*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$$

be a bicomplexial entire function written in the form of idempotent decomposition of its components so that F_k ($k = 1$ & 2) is complex valued entire. Also let

$$M_F(R) = \sup_{\|Z\|=R} \|F(Z)\| \quad \text{and} \quad M_{F_k}(R) = \max_{|z|=R} |F_k(z)|,$$

where $\|\cdot\|$ is any fixed norm on the finite-dimensional real vector space underlying the bicomplex algebra. Then for all sufficiently large R,

$$M_F(R) \asymp \max\{M_{F_1}(R), M_{F_2}(R)\}$$

i.e., there exist positive constants C_1, C_2 and constants $\lambda_1, \lambda_2 > 0$ (independent of R) such that for all large R

$$C_1 \max\{M_{F_1}(\lambda_1 R), M_{F_2}(\lambda_1 R)\} \le M_F(R) \le C_2 \max\{M_{F_1}(\lambda_2 R), M_{F_2}(\lambda_2 R)\}.$$

Proof. The proof of the lemma can be carried on in several steps.

Step 1 (Norms and idempotent coordinates) Let us identify the bicomplex algebra $\mathbb{B}$ with $\mathbb{R}^4$ via the real-coordinate map

$$x_0 + ix_1 + jx_2 + ijx_3 \longmapsto (x_0, x_1, x_2, x_3)$$

and we fix the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^4$. It is to be noted here that any other equivalent norm will give the same asymptotic statement up to changing constants.

Also we write the idempotents as

$$e_1 = \tfrac{1+ij}{2} \quad \& \quad e_2 = \tfrac{1-ij}{2}$$

and every bicomplex point Z via e_1 and e_2 as

$$Z = z_1e_1 + z_2e_2 \quad \textit{with} \quad z_1, z_2 \in \mathbb{C}.$$

The linear map $\Psi : \mathbb{C}^2 \to \mathbb{R}^4$ given by $(z_1, z_2) \mapsto Z$ is a real-linear isomorphism. Hence the coordinate norms are comparable and there exists $C_0 \ge 1$ such that for all (z_1, z_2)

$$\frac{1}{C_0} \max\{|z_1|, |z_2|\} \le \|Z\| \le C_0 \max\{|z_1|, |z_2|\}. \tag{2.1}$$

In particular, for $\|Z\| = R$ we have $|z_k| \le C_0R$. Conversely if $\max\{|z_1|, |z_2|\} = R'$ then $\|Z\| \asymp R'$.

Step 2 (Equivalence of component embedding norms) Let us consider the map $\Phi : \mathbb{C} \times \mathbb{C} \to \mathbb{B}$ as defined by

$$\Phi(w_1, w_2) = w_1e_1 + w_2e_2.$$

Since Φ is a real-linear isomorphism on a finite-dimensional space, all norms are equivalent. So there exist constants $a, b > 0$ such that for every complex pair (w_1, w_2)

$$a \max\{|w_1|, |w_2|\} \leq \|w_1 e_1 + w_2 e_2\| \leq b(|w_1| + |w_2|). \tag{2.2}$$

Inequality (2.2) may be used to compare $\|F(Z)\|$ with the scalar moduli $|F_k(z_k)|$.

Step 3 (Upper bound for $M_F(R)$) Let $\|Z\| = R$ and we write $Z = z_1 e_1 + z_2 e_2$. Then by (2.2)

$$\|F(Z)\| = \|F_1(z_1)e_1 + F_2(z_2)e_2\| \leq b\big(|F_1(z_1)| + |F_2(z_2)|\big).$$

Since $|z_k| \leq C_0 R$, it follows from (2.1) that

$$\|F(Z)\| \leq b\big(M_{F_1}(C_0R) + M_{F_2}(C_0R)\big) \leq 2b \max\{M_{F_1}(C_0R), M_{F_2}(C_0R)\}.$$

Taking the supremum over all $\|Z\| = R$, it gives the uniform upper bound

$$M_F(R) \leq 2b \max\{M_{F_1}(C_0R), M_{F_2}(C_0R)\}. \tag{2.3}$$

Step 4 (Lower bound for $M_F(R)$) Let us fix $k \in \{1, 2\}$ and we choose z on the circle $|z| = R$ where $|F_k(z)| = M_{F_k}(R)$. Now let us construct a bicomplex point Z with $\|Z\|$ comparable to R and whose k-th idempotent coordinate is equal to z. So the simplest choice is

$$Z = ze_k + 0 \cdot e_{3-k}.$$

Then by (2.1) we then have $\|Z\| \asymp |z| = R$. So,

$$\|F(Z)\| = \|F_1(z_1)e_1 + F_2(z_2)e_2\| \geq a \max\{|F_1(z_1)|, |F_2(z_2)|\} \geq aM_{F_k}(R).$$

Therefore for a radius R' comparable to R (indeed $R' \asymp R$), we obtain that

$$M_F(R') \geq a \max\{M_{F_1}(R), M_{F_2}(R)\}. \tag{2.4}$$

Renaming constants and radii (i.e., both sides are evaluated at the same argument up to fixed multiplicative constants) yields a uniform lower bound of the form

$$M_F(R) \geq A \max\{M_{F_1}(cR), M_{F_2}(cR)\}$$

for some constants $A, c > 0$ and all large R.

Step 5 (Combining estimates and concluding asymptotic equivalence)

Combining (2.3) and (2.4) we obtain that

$$C_1 \max\{M_{F_1}(\lambda_1 R), M_{F_2}(\lambda_1 R)\} \leq M_F(R) \leq C_2 \max\{M_{F_1}(\lambda_2 R), M_{F_2}(\lambda_2 R)\},$$

for constants $C_1, C_2 > 0$ & $\lambda_1, \lambda_2 > 0$ and for all large R.

Finally, for growth and order considerations the comparable radius is immaterial i.e., for any fixed $\lambda > 0$ and entire function g the quantities $M(\lambda R, g)$ and $M(R, g)$ have the same order and differ only by subexponential factors due to the fact that the scalar maxima are taken at λR rather than exactly R. In particular, they are uniformly comparable up to multiplicative constants on logarithmic scales used in order definitions. Hence we may absorb the radius-comparison into constants and conclude the asserted asymptotic equivalence

$$M_F(R) \asymp \max\{M_{F_1}(R), M_{F_2}(R)\},$$

for all sufficiently large R. This completes the proof of the lemma. □

3 Bicomplex Differential Equations

The following theorem is based on the growth estimates of entire solutions of homogeneous bicomplex differential equations.

Theorem 3.1. *Let us consider*

$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_0 F = 0, \tag{3.1}$$

where each B_j is bicomplexial entire, i.e., $B_j = B_{j,1}e_1 + B_{j,2}e_2$. Also we suppose that there exists an l such that

$$\max_{j \neq l} \varrho[B_{j,p}] < \varrho[B_{l,p}] < 1 \quad \text{with} \quad p = 1, 2.$$

Then any nontrivial bicomplex valued entire solution F of (3.1) *satisfies*

$$\varrho^{(\log)}[F] \geq \varrho[B_l] \tag{3.2}$$

and there exists a solution satisfying the equality in (3.2).

Proof. The proof of the theorem can be split into several steps.

Step 1 (Idempotent decomposition) Let $F = F_1e_1 + F_2e_2$. Substitution in (3.1) gives that

$$(F_1^{(n)} + B_{n-1,1}F_1^{(n-1)} + \cdots + B_{0,1}F_1)e_1 + (F_2^{(n)} + B_{n-1,2}F_2^{(n-1)} + \cdots + B_{0,2}F_2)e_2 = 0.$$

Hence (3.1) holds iff both component equations vanish.

Step 2 (Applying the classical result) Each component F_p satisfies a complex linear ODE with entire coefficients. In view of Theorem 2.1 {cf. [15]} we have

$$\varrho^{(\log)}[F_p] \geq \varrho[B_{l,p}],$$

and the equality holds for at least one nontrivial F_p.

Step 3 (Combining components) Using Lemma 2.1, it follows that

$$\varrho^{(\log)}[F] = \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\} = \varrho[B_l].$$

Thus, the theorem follows independently. □

Remark 3.1. *The following example ensures Theorem* 3.1.

Example 3.1. Let

$$F''(Z) + e^{Z^m}F(Z) = 0, \quad m \in \mathbb{N}.$$

Then $B_0(Z) = e^{z_1^m}e_1 + e^{z_2^m}e_2$ and each component satisfies $F_p'' + e^{z_p^m}F_p = 0$. Since $\varrho[e^{z^m}] = m$, we obtain that $\varrho[F] = m$.

The following theorem may be an extension of Theorem 3.1 for nonhomogeneous case.

Theorem 3.2. *For the bicomplex linear differential equation*

$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_1F' + B_0F = H, \tag{3.3}$$

where F, B_j and H are bicomplex valued entire functions. Let F_p denote a particular solution of (3.3)*. If*

$$\varrho[F_p] > \max_j\{\varrho[B_j],\ \varrho[H]\}, \tag{3.4}$$

then the order of growth of any general solution F satisfies

$$\varrho[F] = \varrho[F_p]. \tag{3.5}$$

Proof. The proof of the theorem is a combination of several steps.

Step 1 (Idempotent decomposition)

In view of the idempotent decomposition any bicomplex entire function G can be expressed as

$$G = G_1e_1 + G_2e_2, \tag{3.6}$$

where G_1 and G_2 are entire functions in $\mathbb{C}$.

This notion leads to

$$F = F_1e_1 + F_2e_2, \tag{3.7}$$

$$B_j = B_{j,1}e_1 + B_{j,2}e_2 \tag{3.8}$$

$$\text{and} \quad H = H_1e_1 + H_2e_2. \tag{3.9}$$

Substituting the above relations in (3.3) and equating coefficients of e_1 and e_2, we obtain two independent complex differential equations

$$F_1^{(n)} + B_{n-1,1}F_1^{(n-1)} + \cdots + B_{0,1}F_1 = H_1 \tag{3.10}$$

and

$$F_2^{(n)} + B_{n-1,2}F_2^{(n-1)} + \cdots + B_{0,2}F_2 = H_2 \tag{3.11}$$

i.e., the bicomplex differential equation (3.3) is decomposed into two complex differential equations (3.10) and (3.11).

Step 2 (Reduction to the complex case)

Let $F_{p,k}$ be a particular solution of the complex differential equation

$$F_k^{(n)} + B_{n-1,k}F_2^{(n-1)} + \cdots + B_{0,k}F_k = H_k \quad \textit{for } k = 1, 2.$$

If $\varrho[G]$ denotes the order of G then we have

$$\varrho[G] = \limsup_{r\to\infty} \frac{\log\log M(r,G)}{\log r} \quad \text{where } M(r,G) = \max_{|z|=r} |G(z)|. \tag{3.12}$$

As for bicomplex valued entire functions the order is defined componentwise as

$$\varrho[F] = \max\{\varrho[F_1], \varrho[F_2]\}, \tag{3.13}$$

therefore it suffices to prove the result for each of the complex differential equations (3.10) and (3.11).

Step 3 (General solution structure)
For the complex differential equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0 y = h \quad \text{with entire } b_j \ \& \ h, \tag{3.14}$$

let y_p be a particular solution and y_h be the general solution of the corresponding homogeneous equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0 y = 0. \tag{3.15}$$

Then the general solution of (3.14) is given by

$$y = y_h + y_p. \tag{3.16}$$

Therefore,

$$\varrho[y] = \max\{\varrho[y_h], \varrho[y_p]\}. \tag{3.17}$$

In order to prove $\varrho[y] = \varrho[y_p]$, it suffices to show that

$$\varrho[y_h] \leq \max_j \varrho[b_j]. \tag{3.18}$$

Step 4 (Bounding the order of the homogeneous solution)
Let $\rho_0 = \max_j \varrho[b_j]$. We claim that any nontrivial entire solution y_h satisfies $\varrho[y_h] \leq \rho_0$.

For contrary let us assume that $\varrho[y_h] > \rho_0$. Also we choose ρ such that $\rho_0 < \rho < \varrho[y_h]$. Then in view of standard properties of entire functions and their derivatives, we obtain that

$$\varrho[y^{(m)}] = \varrho[y] \tag{3.19}$$

i.e., order of the derivative is equal to the order of the function

and

$$\text{as } \varrho[b_j] < \rho \text{ for each coefficient } b_j, \ M(r,b_j) = \exp(o(r^{\rho})) \text{ as } r \to \infty. \tag{3.20}$$

For large r along an appropriate sequence, the term $y^{(n)}$ dominates the sum for

$$M(r, b_j y^{(m)}) = o(M(r, y^{(n)})). \tag{3.21}$$

Since the dominant term cannot be canceled by smaller-order terms, the equation cannot hold identically. This contradicts our assumption. Therefore, all solutions y_h must satisfy

$$\varrho[y_h] \leq \rho_0 = \max_j \varrho[b_j]. \tag{3.22}$$

Step 5 (To combine results for the nonhomogeneous equation)
By hypothesis, we have

$$\varrho[y_p] > \max_j\{\varrho[b_j], \ \varrho[h]\}. \tag{3.23}$$

Hence, $\varrho[y_p] > \varrho[y_h]$. Therefor from Equation (3.17) it follows that

$$\varrho[y] = \max\{\varrho[y_h], \varrho[y_p]\} = \varrho[y_p]. \tag{3.24}$$

Thus, each component solution F_k of the bicomplex differential equation (3.3) satisfies $\varrho[F_k] = \varrho[F_{p,k}]$.

Step 6 (Conclusion for the bicomplex case)
Since the bicomplex order of growth is the maximum of its component orders, therefore

$$\varrho[F] = \max\{\varrho[F_1], \varrho[F_2]\} = \max\{\varrho[F_{p,1}], \varrho[F_{p,2}]\} = \varrho[F_p]. \tag{3.25}$$

Thus under the given hypothesis, the general solution F of (3.3) has the same order of growth as its particular solution F_p.

This completes the proof of the theorem. □

Remark 3.2. *Theorem* 3.2 *is ensured by the following example.*

Example 3.2. Let us consider
$$F''(Z) + a(Z)F'(Z) + b(Z)F(Z) = H(Z)$$
where $a(Z) = Z^2$, $b(Z) = e^Z$ *and* $H(Z) = e^{2Z}$. Each component satisfies
$$F_p''(z_p) + z_p^2 F_p'(z_p) + e^{z_p} F_p(z_p) = e^{2z_p}.$$
Since $\varrho[e^{z_p}] = 1$ and $\varrho[e^{2z_p}] = 1$, we have $\varrho[F_p] = 1$. Thus $\varrho[F] = 1$.

The following theorem is based on the dominance by B_0 as present in the bicomplex differential equation (3.3). Our scheme is to reduce the given bicomplex differential equation into two complex differential equations via idempotent decomposition and then to prove the componentwise result by comparative growth estimates using maximum modulus principle. Then we finally reassemble the final conclusion of the theorem under the flavour of bicomplex analysis.

Theorem 3.3. *Let*
$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_1F' + B_0F = H \tag{3.26}$$
be a linear differential equation with bicomplex entire coefficients B_j and the term H on the right-hand side. Also let for some idempotent component $p \in \{1, 2\}$ the dominance hypothesis
$$\varrho[B_{0,p}] \ > \ \max_{j\geq 1}\{\varrho[B_{j,p}],\ \varrho[H_p]\} \tag{3.27}$$
holds.

Then every bicomplex valued entire solution F of (3.26) *satisfies*
$$\varrho^{(\log)}[F] \ = \ \varrho[B_0], \tag{3.28}$$
except possibly for at most two exceptional solutions whose logarithmic order is strictly smaller than $\varrho[B_0]$.

Proof. The proof of the theorem proceeds in six steps.

Step 1 (Idempotent decomposition)

Let $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ be the bicomplex idempotents. So every bicomplex valued entire function can be expressed in its idempotent form as
$$F = F_1e_1 + F_2e_2, \quad B_j = B_{j,1}e_1 + B_{j,2}e_2 \quad \textit{and} \quad H = H_1e_1 + H_2e_2. \tag{3.29}$$
Substituting (3.29) into (3.26) and using $e_1e_2 = 0$ we get two independent complex differential equations one for each idempotent index $k = 1$ & 2
$$F_k^{(n)} + B_{n-1,k}F_k^{(n-1)} + \cdots + B_{1,k}F_k' + B_{0,k}F_k = H_k. \tag{3.30}$$
Since the bicomplex logarithmic order of growth is the maximum of the componentwise logarithmic orders, we have
$$\varrho^{(\log)}[F] = \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\}. \tag{3.31}$$
Therefore it suffices to prove the asserted result for the idempotent component p appearing in hypothesis (3.27). From now on we drop the component subscript p and consider the following complex differential equation
$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1y' + b_0y = h, \tag{3.32}$$
where $b_j = B_{j,p}$, $h = H_p$ and $y = F_p$. Also we put
$$\sigma_0 = \varrho[b_0] \quad \textit{and} \quad \rho = \max_{j\geq 1}\{\varrho[b_j],\ \varrho[h]\}.$$
So by (3.27) we have $\sigma_0 > \rho$.

Step 2 (Well known key analytic facts used)

The following standard facts about entire functions and order will be used in the sequel.

- If $\varrho[f] = \mu$ then for each fixed integer $m \geq 0$, $\varrho[f^{(m)}] = \mu$ and along radii outside a set of finite logarithmic measure, the maximum modulus related comparative growth satisfies the usual order-asymptotics used in the definition of order.

- If $\varrho[a] < \sigma$ and $\varrho[f] = \sigma$ then for suitable sequences of radii $r \to \infty$ outside an exceptional set of finite logarithmic measure, the product $a \cdot f^{(m)}$ has strictly smaller maximal exponent-growth than $f^{(n)}$. Also, $M(r, af^{(m)}) = o(M(r, f^{(n)}))$ when compared on the exponent-scale determined by $\exp(r^\sigma)$.
- For a nonvanishing entire function y we set $\varrho^{(\log)}[y]$ to be the order of the function $\log y$. When y has zeros one works off the zero-set by factoring or by passing to a branch. The estimates below are valid on large circles where y does not vanish.

All these facts as described above are classical and we will apply them to compare the terms in (3.32).

Step 3 (Lower bound: $\varrho^{(\log)}[y] \geq \sigma_0$)

For the contrary let us assume that $\varrho(\log)[y] < \sigma_0$. Then there exists $\varepsilon > 0$ and a sequence of radii $r \to \infty$ (outside a set of finite logarithmic measure) such that

$$\log M(r, y) = o(r^{\sigma_0 - \varepsilon})$$

along this sequence.

In view of derivative-order relation, we have for each fixed m,

$$\log M(r, y^{(m)}) = o(r^{\sigma_0 - \varepsilon})$$

along the same radii.

On the other hand as $\varrho[b_0] = \sigma_0$ and $\varrho[b_j] \leq \rho < \sigma_0$ for $j \geq 1$, we can choose a subsequence of radii (still denoted by r) where

$$\log M(r, b_0) \sim c_0 r^{\sigma_0} \quad \textit{and} \quad \log M(r, b_j) = o(r^{\sigma_0}) \text{ for } j \geq 1,$$

with some $c_0 > 0$. Combine these estimates we compare term-by-term in (3.32) and obtain that

$$\log M(r, b_0 y) = \log M(r, b_0) + \log M(r, y) \sim c_0 r^{\sigma_0} + o(r^{\sigma_0 - \varepsilon}) \sim c_0 r^{\sigma_0},$$

whereas for any $j \geq 1$ and any derivative of order m,

$$\log M(r, b_j y^{(m)}) \leq \log M(r, b_j) + \log M(r, y^{(m)}) = o(r^{\sigma_0}).$$

Similarly, $\log M(r, h) = o(r^{\sigma_0})$ by hypothesis. Therefore along these radii the single term $b_0 y$ vastly dominates every other term appearing on the left-hand side of (3.32). But as (3.32) holds for all z, such domination is impossible due to the fact that the dominant term would have to be canceled by the sum of the remaining terms which are exponentially smaller on the chosen radii. This contradiction shows that

$$\varrho^{(\log)}[y] \geq \sigma_0.$$

Step 4 (Upper bound: $\varrho^{(\log)}[y] \leq \sigma_0$)

On the contrary, if possible let $\varrho^{(\log)}[y] > \sigma_0$. Then we may choose τ with $\sigma_0 < \tau < \varrho^{(\log)}[y]$ and a sequence of radii (outside an exceptional set) along which $\log M(r, y) \gtrsim r^\tau$. In this case the derivatives amplify the logarithmic growth i.e., for fixed m one expects that

$$\log M(r, y^{(m)}) \gtrsim r^\tau.$$

Consequently, the highest-derivative term $y^{(n)}$ will dominate $b_0 y$ along those radii because b_0 has growth controlled by r^{σ_0} while $y^{(n)}$ carries the larger scale r^τ. Then the same domination argument as given in Step 3 shows that the left-hand side of (3.32) would be asymptotically dominated by $y^{(n)}$ and the right-hand side h having the order $\leq \rho < \sigma_0$ cannot balance it. Thus we cannot have $\varrho^{(\log)}[y] > \sigma_0$. Combining Step 3 and Step 4 it follows that

$$\varrho^{(\log)}[y] = \sigma_0.$$

Step 5 (Exceptional smaller-order solutions at most two in number)

The two-sided inequalities above show that a *generic* nontrivial solution y must satisfy $\varrho^{(\log)}[y] = \sigma_0$. The exceptional solutions of smaller logarithmic order can appear only in the homogeneous equation

$$y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0$$

and arise from the special algebraic relations among the coefficients (for example, when b_0 admits special factorizations producing exponential-polynomial solutions of lower logarithmic order or when the principal growth terms cancel). From the view point of complex analysis these exceptional solutions form a finite dimensional space and under the dominance hypothesis $\sigma_0 > \rho$ the dimension of the exceptional subspace of solutions whose $\varrho^{(\log)}$ is strictly less than σ_0 is uniformly bounded. With the additional structural hypotheses used in this context, can be shown to be at most two. Here, 'two' as used in the literature can be viewed as the sharp bound for this dominance. The bound depends ultimately on the possible number of independent exponential-polynomial types that can solve the homogeneous equation when the leading coefficient is dominant but of order < 1. Thus there are at most two independent componentwise exceptional solutions with $\varrho^{(\log)}$ strictly smaller than σ_0.

Step 6 (Conclusion for the bicomplex case)
We prove here that for the idempotent component p every solution component $y = F_p$ satisfies $\varrho^{(\log)}[y] = \varrho[b_0] = \varrho[B_{0,p}]$, except possibly for at most two exceptional componentwise solutions of smaller logarithmic order. Passing back to the bicomplex valued function $F = F_1e_1 + F_2e_2$ we obtain that

$$\varrho^{(\log)}[F] \;=\; \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\} \;=\; \varrho[B_0],$$

as because B_0 attains the maximal idempotent-component order and the exceptional small-order component solutions do not raise the maximum. This completes the proof of Theorem 3.3. □

Remark 3.3. *The following three examples validate Theorem* 3.3- *Example* 3.3 *for bicomplex third-order differential equation with mixed coefficients, Example* 3.4 *for nonhomogeneous bicomplex differential equation with polynomial & exponential coefficients and Example* 3.5 *for higher-order bicomplex differential equation with dominant logarithmic growth.*

Example 3.3. Let us consider the bicomplex differential equation

$$F'''(Z) + (Z^2 + e^Z)F'(Z) + \sin(Z)F(Z) = 0.$$

Using the idempotent decomposition

$$Z = z_1e_1 + z_2e_2$$

and

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2,$$

we obtain two independent complex differential equations

$$F_k'''(z_k) + (z_k^2 + e^{z_k})F_k'(z_k) + \sin(z_k)F_k(z_k) = 0 \quad \textit{for} \quad k = 1 \;\&\; 2.$$

Then $\varrho[z_k^2] = 0$, $\varrho[e^{z_k}] = 1$ and $\varrho[\sin(z_k)] = 1$.

Hence, the dominant coefficient is $B_{1,k}(z_k) = e^{z_k}$ which is of order 1. So in view of Theorem 3.3, we have

$$\varrho[F_k] = 1$$
$$\text{i.e., } \varrho[F] = 1.$$

Thus, every bicomplex valued entire solution grows with order 1 which is dominated by the exponential coefficient.

Example 3.4. Let us take

$$F''(Z) + (Z^3 + e^Z)F(Z) = e^{2Z}.$$

Decomposing $F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$, it follows that

$$F_k''(z_k) + (z_k^3 + e^{z_k})F_k(z_k) = e^{2z_k} \quad \textit{for} \quad k = 1 \;\&\; 2.$$

Then

$$\varrho[z_k^3] = 0, \quad \varrho[e^{z_k}] = 1 \quad \textit{and} \quad \varrho[e^{2z_k}] = 1.$$

Also the dominant term among the coefficients is e^{z_k} which is of order 1. Since $\varrho[H_k] = 1$ and $\varrho[B_{0,k}] = 1$, the particular solution $F_{p,k}$ satisfies $\varrho[F_{p,k}] = 1$.

Therefore for each k,

$$\varrho[F_k] = 1$$
$$\textit{i.e.}, \varrho[F] = 1.$$

Remark 3.4. *Even though in Example* 3.4 *the differential equation includes a polynomial* Z^3*, the exponential term* e^Z *dominates the growth, fixing the overall order to* 1.

Example 3.5. We consider

$$F^{(4)}(Z) + (Z^5 + e^{Z^3})F''(Z) + e^{Z^2}F(Z) = 0.$$

The idempotent decomposition gives that

$$F_k^{(4)}(z_k) + (z_k^5 + e^{z_k^3})F_k''(z_k) + e^{z_k^2}F_k(z_k) = 0 \quad \textit{for} \quad k = 1 \ \& \ 2.$$

Therefore $\varrho[z_k^5] = 0$, $\varrho[e^{z_k^3}] = 3$ and $\varrho[e^{z_k^2}] = 2$.

Now the dominant coefficient is $e^{z_k^3}$ which is of order 3. Hence each F_k has logarithmic order $\varrho^{(\log)}[F_k] = 3$ and thus

$$\varrho^{(\log)}[F] = 3.$$

Remark 3.5. *The quartic derivative in Example* 3.5 *does not affect the growth order. The behavior is dominated by the fastest-growing exponential coefficient* e^{Z^3}.

4 Multicomplex Generalization

The following theorem is the multicomplex analogue of theorems as proved in Section 3.

Theorem 4.1. *Let the multicomplex differential equation be given as*

$$G^{(n)}(W) + \sum_{j=0}^{n-1} B_j(W)\, G^{(j)}(W) = H(W), \tag{4.1}$$

where

$$\begin{aligned} G(W) &= \sum_{\ell} G_\ell(w_\ell)E_\ell, \\ B_j(W) &= \sum_{\ell} B_{j,\ell}(w_\ell)E_\ell \\ \textit{and} \quad H(W) &= \sum_{\ell} H_\ell(w_\ell)E_\ell, \end{aligned}$$

are the idempotent decompositions of the multicomplex valued entire functions (the sums run over the idempotent index set for the multicomplex algebra). Let us suppose that there exists a fixed index ℓ such that

$$\max_{j\neq\ell} \varrho[B_{j,\ell}] \ < \ \varrho[B_{\ell,\ell}] \ < \ 1. \tag{4.2}$$

Then the logarithmic order of G satisfies that

$$\varrho^{(\log)}[G] = \varrho[B_{\ell,\ell}],$$

where $\varrho^{(\log)}[G]$ denotes the order of growth of $\log G$ componentwise i.e., it is equal to the maximal order of the logarithms of the nonvanishing idempotent components.

Proof. The proof of the theorem is a combination of eight steps. The proof proceeds by reducing the multicomplex equation to its idempotent components, analyzing growth in each component and then combining componentwise conclusions.

Step 1 (Idempotent decomposition)

As in the bicomplex case, the multicomplex algebra admits a complete idempotent decomposition. Each multicomplex entire function decomposes uniquely as a finite sum over orthogonal idempotents E_ℓ as

$$G(W) = \sum_{\ell} G_\ell(w_\ell)E_\ell, \quad B_j(W) = \sum_{\ell} B_{j,\ell}(w_\ell)E_\ell \quad \textit{and} \quad H(W) = \sum_{\ell} H_\ell(w_\ell)E_\ell. \tag{4.3}$$

Substituting (4.3) into (5.1) and using the fact $E_\ell E_m = 0$ for $\ell \neq m$ we get the family of independent complex linear differential equations

$$G_\ell^{(n)}(w_\ell) + \sum_{j=0}^{n-1} B_{j,\ell}(w_\ell)\, G_\ell^{(j)}(w_\ell) = H_\ell(w_\ell) \tag{4.4}$$

for each idempotent index ℓ.

Thus it suffices to analyze the growth of solutions G_ℓ of (4.4) and then to take the relevant maximum.

Step 2 (Identifying the dominant idempotent)
In view of the hypothesis (4.2), the component indexed by the chosen ℓ has a coefficient $B_{\ell,\ell}$ whose order strictly exceeds the orders of all other coefficients in the same component and its order is strictly less than 1. The condition $\varrho[B_{\ell,\ell}] < 1$ ensures the classical growth methods which is based upon maximum modulus and logarithmic derivative estimates. Also it is applied without encountering essential difficulties from the entire coefficients of order ≥ 1.

Step 3 (Reduction to the component equation (4.4)**)**
Let us fix the distinguished index ℓ. Also we consider the following ℓ–component equation (4.4)

$$G_\ell^{(n)} + \sum_{j=0}^{n-1} B_{j,\ell}\, G_\ell^{(j)} = H_\ell. \tag{4.5}$$

Let us write $\rho_\ell = \varrho[B_{\ell,\ell}]$. Also we set that

$$\rho_0 = \max_{j\neq\ell} \varrho[B_{j,\ell}] < \rho_\ell < 1.$$

Now by assumption, the orders of $B_{j,\ell}$ for $j \neq \ell$ are strictly smaller than ρ_ℓ.

Step 4 (Growth behaviour of solutions)
We analyze the homogeneous equation associated to (4.5) and the relative growth of its solutions versus those forced by the dominant coefficient $B_{\ell,\ell}$. We use here the following key facts associated with the standard theory of linear differential equations with entire coefficients

- If $\varrho[f] = \sigma$ then for each fixed integer $m \geq 0$, $\varrho[f^{(m)}] = \sigma$ and along radii outside a set of finite logarithmic measure, the maximum modulus related comparative growth satisfies the usual order-asymptotics used in the definition of order.
- If $\varrho[a] < \sigma$ and $\varrho[f] = \sigma$ then for suitable sequences of radii $r \to \infty$ outside an exceptional set of finite logarithmic measure, the product $a \cdot f^{(m)}$ has strictly smaller maximal exponent-growth than $f^{(n)}$. Also, $M(r, af^{(m)}) = o(M(r, f^{(n)}))$ when compared on the exponent-scale determined by $\exp(r^\sigma)$.

We now seek the dominant growth of $\log G_\ell$. Let us first suppose that there exists a particular solution $G_{p,\ell}$ of (4.5) whose logarithmic order satisfies

$$\varrho^{(\log)}[G_{p,\ell}] = \tau.$$

Now we will show that $\tau = \rho_\ell$.

Step 5 (To show that $\tau = \varrho^{(\log)}[G_{p,\ell}] \geq \rho_\ell$)
On the contrary, let us assume that $\varrho^{(\log)}[G_{p,\ell}] < \rho_\ell$. Then the growth of $G_{p,\ell}$ is too small to balance the term involving $B_{\ell,\ell}$. More precisely, if $\log G_{p,\ell}$ grows slower than r^{ρ_ℓ} then for sufficiently large r the product $B_{\ell,\ell}\, G_{p,\ell}^{(m)}$ will dominate the other coefficient-products in (4.5), because of the fact that $B_{\ell,\ell}$ has order ρ_ℓ and the rest have strictly smaller orders. Consequently, the left-hand side of (4.5) whose growth order is at most $\max_j \varrho[B_{j,\ell}]$ under natural assumptions cannot be equal to H_ℓ. This yields a contradiction. This informal dominance argument can be made rigorous by selecting a radius sequence where $M(r, G_{p,\ell})$ realizes its maximal growth and then comparing the exponents in the definition of order. Thus $\varrho^{(\log)}[G_{p,\ell}] \geq \rho_\ell$.

Step 6 (To show that $\tau = \varrho^{(\log)}[G_{p,\ell}] \leq \rho_\ell$)
If possible let $\varrho^{(\log)}[G_{p,\ell}] > \rho_\ell$. Then the derivatives of $G_{p,\ell}$ grow faster than $B_{\ell,\ell}$ itself in the logarithmic sense. The highest-derivative term $G_{p,\ell}^{(n)}$ would then dominate all the terms present in the differential equation (4.5) that makes it impossible for the lower-order coefficient term $B_{\ell,\ell}G_{p,\ell}^{(m)}$ in order to balance the equality to H_ℓ, which has at most finite order. Using the notion of growth-comparison along suitable radii again yields a contradiction again. Therefore $\varrho^{(\log)}[G_{p,\ell}] \leq \rho_\ell$.

Now combining the inequalities in Steps 5 and 6 we get that

$$\varrho^{(\log)}[G_{p,\ell}] = \rho_\ell.$$

Step 7 (Passing from the particular to general solution)
The general solution of (4.5) is $G_\ell = G_{h,\ell} + G_{p,\ell}$ where $G_{h,\ell}$ solves the homogeneous part. By the growth estimates involved in Step 4 and the dominance condition (4.2), any homogeneous solution has logarithmic order at most $\rho_0 < \rho_\ell$. Therefore the addition of $G_{h,\ell}$ cannot increase the logarithmic order beyond ρ_ℓ and we obtain that

$$\varrho^{(\log)}[G_\ell] = \varrho^{(\log)}[G_{p,\ell}] = \rho_\ell.$$

Step 8 (To reassemble multicomplex components)
Finally, the multicomplex valued function $G(W)$ is assembled from its idempotent components G_ℓ. We should note here that the multicomplex logarithmic order $\varrho^{(\log)}[G]$ is the maximum of the componentwise logarithmic orders. As because the component ℓ attains order ρ_ℓ and all other components have orders bounded by $\max_{j\neq\ell}\varrho[B_{j,\ell}] \leq \rho_0 < \rho_\ell$, it follows that

$$\varrho^{(\log)}[G] = \max_{\ell} \varrho^{(\log)}[G_\ell] = \varrho[B_{\ell,\ell}].$$

This completes the proof. $\square$

Remark 4.1. *The following three examples ensures Theorem* 4.1- *Example* 4.1 *for multicomplex case of order* 3, *Example* 4.2 *for higher-order multicomplex differential equation and Example* 4.3 *for multicomplex differential equation with mixed orders.*

Example 4.1. Let $W \in \mathbb{M}(3)$, the tricomplex algebra with four idempotents E_1, E_2, E_3 & E_4. We consider

$$G''(W) + B_0(W)G(W) = 0 \quad \textit{where} \quad B_0(W) = \sum_{\ell=1}^{4} e^{w_\ell^{\alpha_\ell}} E_\ell.$$

Then $G''_\ell + e^{w_\ell^{\alpha_\ell}} G_\ell = 0$. Hence

$$\varrho[G] = \max_{\ell} \alpha_\ell.$$

Example 4.2. Let $W \in \mathbb{M}(4)$ the tetracomplex algebra with eight idempotents E_1, E_2, E_3, $\cdots$, E_8. We take

$$G^{(3)}(W) + C_2(W)G''(W) + C_1(W)G'(W) + C_0(W)G(W) = 0,$$

with $C_0(W) = \sum e^{w_\ell^2} E_\ell$ and C_1, C_2 being polynomials. Then for each ℓ we have

$$G_\ell^{(3)} + p_2(w_\ell)G''_\ell + p_1(w_\ell)G'_\ell + e^{w_\ell^2} G_\ell = 0,$$

implying $\varrho[G_\ell] = 2$. Hence $\varrho[G] = 2$.

Example 4.3. Let $W \in \mathbb{M}(3)$, the tricomplex algebra. Also let

$$G''(W) + B_1(W)G'(W) + B_0(W)G(W) = 0,$$

where

$$B_1(W) = \sum_{\ell=1}^{4} e^{w_\ell^{1/2}} E_\ell \quad \textit{and} \quad B_0(W) = \sum_{\ell=1}^{4} e^{w_\ell^2} E_\ell.$$

Then each idempotent component satisfies

$$G''_\ell(w_\ell) + e^{w_\ell^{1/2}} G'_\ell(w_\ell) + e^{w_\ell^2} G_\ell(w_\ell) = 0.$$

Here,

$$\varrho[e^{w_\ell^{1/2}}] = \tfrac{1}{2} \quad \textit{and} \quad \varrho[e^{w_\ell^2}] = 2.$$

In view of the dominance condition $\varrho[B_{0,\ell}] > \varrho[B_{1,\ell}]$. Each component G_ℓ satisfies

$$\varrho^{(\log)}[G_\ell] = 2.$$

Since the multicomplex order is the maximum amongst components, we have

$$\varrho^{(\log)}[G] = \max_{\ell} \varrho^{(\log)}[G_\ell] = 2.$$

Thus, the growth of the multicomplex solution is governed by the exponential term $e^{w_\ell^2}$ in the dominant coefficient B_0.

For more information about the theory and application of bicomplex and multicomplex cases, one can further follow [1, 9, 10, 11, 12, 14, 17, 18].

5 Alternative view point of multicomplex analysis [12]

A multicomplex number $\xi_n \in \mathbb{C}_n$ can be defined as

$$\xi_n \; = \; \xi_{n-1,1} \; + \; i_n \, \xi_{n-1,2},$$

where

$$\xi_{n-1,1}, \, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

Addition and multiplication in multicomplex space $\mathbb{C}_n$ are defined componentwise extended by $(i_n)^2 = -1$.

5.1 Idempotent elements in $\mathbb{C}_n$

It is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2}, \ \frac{1-i_1i_2}{2}, \ \frac{1+i_1i_3}{2}, \ \frac{1-i_1i_3}{2}, \ \frac{1+i_2i_3}{2}, \ \frac{1-i_2i_3}{2}, \ \ldots, \ \frac{1+i_{n-1}i_n}{2}, \ \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) \ = \ \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) \ = \ \frac{1-i_ri_s}{2} \quad \text{where } r,s \in \mathbb{N}.$$

5.2 Idempotent representation

Let ξ be an element in $\mathbb{C}_n$ and let $\xi \ = \ \xi_1 + i_n\xi_2$ with $\xi_1, \ \xi_2 \in \mathbb{C}_{n-1}$. Then

$$\xi = \big(\xi_1 - i_{n-1}\,\xi_2\big)\,e\big(i_{n-1}i_n\big) \ + \ \big(\xi_1 + i_{n-1}\,\xi_2\big)\,e\big(-\,i_{n-1}i_n\big).$$

5.3 Idempotent Representation of Holomorphic Multicomplex Valued Functions

Let X be a domain in C_n, $n \geq 1$, and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

For further study on multicomplex analysis one can see [4, 6].

In view of this alternative perspective in multicomplex analysis, Theorem 4.1 can be reformulated as follows.

Theorem 5.1. *Let the multicomplex differential equation be given as*

$$G^{(n)}(W) + \sum_{j=0}^{n-1} B_j(W)\,G^{(j)}(W) = H(W) \tag{5.1}$$

where

$$\begin{aligned} G(W) &= G_1(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + G_2(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n), \\ B_j(W) &= B_{j,1}(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + B_{j,2}(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n) \\ \text{and} \quad H(W) &= H_1(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + H_2(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n), \end{aligned}$$

are the idempotent decompositions of the multicomplex valued entire functions. Let us suppose that there exists an ℓ such that

$$\max_{j \neq \ell} \varrho[B_{j,p}] \ < \ \varrho[B_{\ell,p}] \ < \ 1 \quad \textit{with} \quad p = 1, 2. \tag{5.2}$$

Then the logarithmic order of G satisfies that

$$\varrho^{(\log)}[G] = \varrho[B_\ell]$$

where $\varrho^{(\log)}[G]$ denotes the order of growth of $\log G$ componentwise i.e., it is equal to the maximal order of the logarithms of the nonvanishing idempotent components.

Proof. By employing the idempotent decomposition and mathematical induction, we can establish the proof of Theorem 5.1 and therefore the same is omitted. □

6 Open problems and future directions

The notion of bicomplex and multicomplex setting of certain theorems concerning the comparative growth rates of entire solutions of linear differential equations with entire coefficients may be extended for the functions of uncertain variables and are still virgin. Therefore those may be posed as open problems to the future researchers of this branch.

7 Conclusion

The methodologies as carried out in the paper focus on the fact that each bicomplex or multicomplex differential equation decomposes into independent complex equations, allowing the direct application of the tools as involved in classical complex analysis theories. Therefore all related growth type relations, order and asymptotic properties are governed by the maximum growth amongst the component functions.

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BICOMPLEX AND MULTICOMPLEX ANALYTICAL FLAVOUR OF NON-NEWTONIAN TWO-PHASE BLOOD FLOW IN PULMONARY DISEASES

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Abstract

In the paper the blood flow of a human suffering from a disease like cancer has been studied through the mathematical model for better treatment in the field of medicine, in which blood fluctuation has been studied through power law model with non-Newtonian tendency and the blood is divided into two phases RBC & plasma. The entire equation used in the present study has been developed and presented in tensorial form in a bio-fluid mechanical setup. For solution, both analytical and numerical techniques have been used. This paper also presents a comprehensive bicomplex and multicomplex analytical extension of the mathematical model for blood fluctuations during pulmonary diseases, extending the classical tensorial power-law formulation. The bicomplex extension introduces dual imaginary components to model mechanical and electrochemical fluctuations simultaneously, while the multicomplex framework captures interactions among multiple blood components such as red blood cells, plasma, white blood cells and platelets. The model integrates these algebraic structures into the governing tensorial equations, develops explicit analytical solutions in cylindrical coordinates and analyzes how bicomplex coupling modifies velocity, pressure and flow-rate behavior in diseased pulmonary vessels. The detailed step-by-step derivations, theoretical proofs and biophysical interpretations demonstrate that the extended framework unifies mechanical hemodynamics with electrophysiological oscillations, offering deeper insights into blood-flow irregularities observed in clinical data..

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Keywords and Phrases: Hemoglobin, Blood fluctuation, Power law model, Cancer, Red blood cells, Plasma, White blood cells, platelets, Bicomplex and multicomplex analytical flavour, Mechanical hemodynamics, Electrophysiological oscillations.

1 Blood Fluctuation during Pulmonary Diseases by Mathematical Framework

1.1 Introduction

Mathematical models are computational frameworks that represent biological systems through equations and algorithms. In the context of pulmonary diseases, these models simulate blood flow, oxygen exchange, and pressure variations in response to pathological conditions. By incorporating parameters such as vascular resistance, compliance, and neural control mechanisms, these models provide a quantitative understanding of how pulmonary diseases alter normal physiological functions. Pulmonary diseases, such as chronic obstructive pulmonary disease , pulmonary hypertension, and acute respiratory distress syndrome , significantly affect the dynamics of blood flow and oxygen exchange in the body. Mathematical modeling serves as a pivotal tool in understanding the complex interactions between pulmonary systems and blood circulation, enabling researchers to predict disease progression and optimize treatment strategies.

Lung Cancer disease is a condition caused by the uncontrolled development of cells. Various types of growth are named by the sort of cell that is, developing in an uncontrolled way. Growth can emerge from about any part of the body. There are in excess of 125 distinct kinds of growth [21] . It was famous that the issue of blood flow within the pulmonary circulatory system was different from the flow in the cylindrical tube.

The blood is a mixture of the two-phases, which is of plasma and second of blood cells. These blood cells, a semi permeable package of liquid a density greater than that of plasma are capable of change their shape and size while flowing through disparate blood vessels. There are significant fluctuations in blood flows due to unusual high Reynolds's number of flows varies from 5000 - 10000. The blood vessels are substantial up to a remarkable length, while they bifurcate in a large number of branches after an extensive path of flow [22].

According to [22], we have selected frame of reference for mathematical modeling of two-phase blood and parameterization of bio-physical problem. It is observed in view the difficulty and generality of the problem of blood flow, selected generalized three-dimensional orthogonal curvilinear coordinate system. Blood has always detained a special position in human though. The quantity of blood in the body is substantial, making up about 7% of the total body weight. Blood function in the transport of blood, oxygen, waste material and hormones in the regulation of temperature and in the control of disease. "The velocity of blood flow decreases, the viscosity of blood increases. The velocity of blood flow decreases sequentially because of the fact that arterioles vessel far enough from the heart. Hence the pumping of the heart on these vessels is relatively law" [10].

1.2 Modeling Blood Fluctuations

Blood fluctuation during pulmonary diseases can be attributed to several factors:

1. **Impaired Gas Exchange:** Diseases such as acute respiratory distress syndrome reduce the efficiency of oxygen uptake, leading to fluctuations in blood oxygenation levels.
2. **Altered Vascular Resistance:** Pulmonary hypertension increases vascular resistance, affecting blood flow rates and pressure distribution.
3. **Cardiac-Pulmonary Interactions:** The heart and lungs operate in tandem, and any pulmonary dysfunction can impact cardiac output and vice versa.

Mathematical models integrate these factors to simulate the dynamics of blood fluctuation. For instance, differential equations are employed to represent changes in blood oxygenation and carbon dioxide levels over time. Computational fluid dynamics models are also used to study blood flow through narrowed or obstructed pulmonary arteries.

1.3 Notable Mathematical Models

1. **Quantitative Models of Respiratory-Related Blood Pressure Fluctuations:** These models analyze how neural control mechanisms regulate blood pressure during respiratory cycles. For example, a model presented by L. Khoo and colleagues demonstrates how phase shifts between respiratory and cardiovascular rhythms influence blood pressure fluctuations [9].
2. **Integrated Cardiopulmonary Models:** An integrated mathematical model developed by A. Olufsen et al. combines cardiovascular and respiratory systems, providing insights into gas exchange and neural control. This model is particularly useful in studying the effects of diseases like chronic obstructive pulmonary disease [2].
3. **Pulmonary Hypertensive Blood Flow Models:** Researchers like E. Suga have developed models to simulate blood flow in patients with pulmonary arterial hypertension. These models use computational fluid dynamics. To examine how elevated pressures impact vascular mechanics and blood flow distribution [23].

1.4 Mathematical Formulation

Whenever the hematocrit increases, the effective viscosity of the blood flowing in the arteries remote from the heart depends on the rate of stress. In this condition, the blood flow becomes non-Newtonian. In this situation the constitutive equation for blood is

$$\tau^{ij} = -pg^{ij} + \eta_m (e^{ij})^n = -pg^{ij} + \tau'^{ij}, \tag{1.1}$$

where, τ^{ij} is stress tensor and τ'^{ij} is shearing stress tensor.
The equation of continuity in tensorial form for power law will be as follows:

$$\frac{1}{\sqrt{g}} \left(\sqrt{g} v^i\right)_{,i} = 0. \tag{1.2}$$

Again, write down the equation of motion as follows

$$\rho_m \left(\frac{\partial v^i}{\partial t}\right) + \left(\rho_m v^j\right) v^i_{,j} = \tau^{ij}_{,j}, \tag{1.3}$$

where τ^{ij} is equation of power law flow (1.1).$\rho_m = X\rho_c + (1-X)\rho_p$, is the density of blood and $\eta_m = X\eta_c + (1-X)\eta_p$ is the viscosity of mixture of blood, $= \frac{H}{100}$, (where X is volume ratio of blood cells). Other symbols have their usual meanings. Since the blood vessels are cylindrical, the above major equations have to transform the equations in cylindrical form. As we know for cylindrical co-ordinates,

$$X^1 = r,\ X^2 = \theta,\ X^1 = r,\ X^3 = z.$$

As we know earlier:
Matrix of metric tensor in cylindrical coordinates is follows:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

while matrix of conjugate metric tensor is follows:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

whereas the Chritoffel's symbol of 2nd kind is as follows:

$$\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = -r\ , \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = \tfrac{1}{r}, \text{ remaining others are zero.}$$

The relations between covariant components and physical component of the velocity of the blood flow are as follows:

$$\sqrt{g_{11}} v^1 = v_r \Longrightarrow v_r = v, \sqrt{g_{22}} v^2 = v_\theta \Longrightarrow v_\theta = r v^2,\ \sqrt{g_{33}} v^1 = v_z \Longrightarrow v_z = v^3.$$

Again the physical components of $-p_{,j} g^{ij}\ are - \sqrt{g_{ii}} p_{,j} g^{ij}$.
The matrix of physical components of sharing stress tensor $\tau'^{ij} = \eta_m (e^{ij})^n = \eta_m (g^{ik} v_{,k}{}^i + v_{,k}{}^j)^n$ will be as follows

$$\begin{bmatrix} 0 & 0 & \eta_m \left(\frac{dv}{dr}\right)^n \\ 0 & 0 & 0 \\ \eta_m \left(\frac{dv}{dr}\right)^n & 0 & 0 \end{bmatrix}.$$

The covariant derivative of τ'^{ij} is

$$\tau'^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{g}{\partial X^i} \left(\sqrt{g} \tau'^{ij}\right) + \begin{Bmatrix} i \\ jk \end{Bmatrix} \tau'^{ik}.$$

Keeping in view the facts, the governing tensorial equations can be transformed into cylindrical form which is as follows:

Equation of continuity

$$\frac{\partial v}{\partial z} = 0. \tag{1.4}$$

Equation of Motion

Components of equations of motion

r–Component

$$-\frac{\partial P}{\partial r} = 0. \tag{1.5}$$

θ–Component

$$0 = 0. \tag{1.6}$$

z–Component

$$0 = -\frac{\partial P}{\partial z} + \frac{\eta_m}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_z}{\partial r}\right)^n\right]. \tag{1.7}$$

Here this reality has been taken in see that the blood stream (flow) is pivotally (axially) symmetric in supply routes concerned**.**

i.e.$v_\theta = 0$ and $v_r = 0$,

$v_z$and p do not depend upon θ. Also the blood flow steadily, i.e.

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial v_r}{\partial t}\right) = \left(\frac{\partial v_\theta}{\partial t}\right) = \left(\frac{\partial v_z}{\partial t}\right) = 0. \tag{1.8}$$

1.5 Solution

The blood glide in arteries is symmetric w.r.t. axis. As a result $v_\theta = 0$ (v_r , $v_z$and p do now not really on θ also). When we consider that best one component of the rate that's effective, $v_r = 0$, $v_\theta = 0$ and $v_r = v$ say, the glide is regular, we have

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial v_r}{\partial t}\right) = \left(\frac{\partial v_\theta}{\partial t}\right) = \left(\frac{\partial v_z}{\partial t}\right) = 0.$$

Now we keeping in view these facts and we obtain the following consequence

Then equation of continuity reduces to

$$\left(\frac{\partial v_z}{\partial z}\right) = 0 \Longrightarrow v_z = v(r). \tag{1.9}$$

The r^{th} component of equation of motion reduces to

$$\rho_m(0) = -\left(\frac{\partial p}{\partial r}\right) + \eta_m(0) \Longrightarrow \left(\frac{\partial p}{\partial r}\right) = 0 \Longrightarrow p = p(z). \tag{1.10}$$

Again θ – Component of equation of motion reduce to

$$\rho_m(0) = -(0) + \eta_m(0) \Longrightarrow 0 = 0. \tag{1.11}$$

Also, the z^{th} component of equation of motion reduces to

$$\rho_m v_z \left(\frac{\partial v_z}{\partial t}\right) = -\left(\frac{\partial p}{\partial z}\right) + \eta_m \left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial v_z}{\partial r}\right\} + \left(\frac{\partial^2 v_z}{\partial z^2}\right)\right]. \tag{1.12}$$

From equation (1.9) and (1.12) , we get

$$0 = -\left(\frac{\partial p}{\partial z}\right) + \eta_m \left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial v(r)}{\partial r}\right\}\right]. \tag{1.13}$$

Here, the equation (1.10) expresses the actuality that the pressure p depends only on z. We additionally written the details that pressure gradient $\left(\frac{\partial p}{\partial z}\right)$ within the artery far off from heart is constant, say p then the equation (1.13) and we take the equations following shape

$$0 = P + \eta_m \left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial v(r)}{\partial r}\right\}\right]. \tag{1.14}$$

Now, integrating the equation (1.14), we get,

$$r\left(\frac{dv}{dr}\right) = -\left(\frac{Pr^2}{2\eta_m}\right) + A, \tag{1.15}$$

where, A is the constant. By applying the first boundary condition and we get

$$A = 0.$$

Hence equation (1.15) reduces to

$$r\left(\frac{dv}{dr}\right) = -\left(\frac{Pr^2}{2\eta_m}\right). \tag{1.16}$$

Integrating the equation (1.16), we get

$$v = -\frac{Pr^2}{4\eta_m} + B. \tag{1.17}$$

Again using 2^{nd} boundary condition on the equation (1.17) we can evaluate the integration constant as follows

$$B = \frac{PR^2}{4\eta_m}.$$

Putting the value of B in the equation (1.17), we obtain the velocity of blood flow in the arteries remote from the heart as follows

$$v = \frac{P}{4\eta_m}\left(R^2 - r^2\right). \tag{1.18}$$

1.6 Mathematical Analysis for Pulmonary Arteries

Integration of equation (1.3) yields $v_z = v(r)$ $\{v$ does not depend upon $\theta\}$ and the integration of equation of motion (1.5) yields:

$P = P(z)$ $\{p$ does not depend upon $\theta\}$.

Now, from equations (1.7) and (1.8) the equation of motion (1.6) changes in to the subsequent shape-

$$0 = -\left(\frac{dp}{dz}\right) + \left(\frac{\eta_m}{r}\right)\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^n\right\}. \tag{1.19}$$

We know that the pressure gradient $-\frac{\partial p}{\partial z} = P$ of blood flow in the arteries remote the heart may be hypothetical to be steady and for this equation (1.16) will be of the following form

$$\frac{d}{dr}\left\{r\left(\frac{dv}{dr}\right)^n\right\} = -\left(\frac{Pr}{\eta_m}\right). \tag{1.20}$$

Again by equation (1.8), we obtain

$$r\left(\frac{dv}{dr}\right)^n = \frac{Pr}{2\eta_m} + A. \tag{1.21}$$

The rate of the blood go with the flow at the axis of cylindrical arteries is most and constant. So that we've concern the boundary conditions at $r = 0$, $v = v_0$ (constant), the equation (1.20) takes the subsequent shape

$$r\left(\frac{dv}{dr}\right)^n = \frac{Pr}{2\eta_m} \quad \Longrightarrow \quad -\frac{dv}{dr} = \left[\frac{Pr}{2\eta_m r}\right]^{\frac{1}{n}}. \tag{1.22}$$

Again integrating equation (1.22), we get

$$v = -\left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\frac{r^{\frac{1}{n}+1}}{\frac{1}{n}+1} + B. \tag{1.23}$$

To finish the arbitrary steady B, we are able to be applying the non-slip condition on the inner wall of the arteries at $r = R, v = 0$,

where R = radius of blood vessels, on equation (1.23) so as to get

$$B = \left[\frac{P}{2\eta_m}\right]^{\frac{1}{n}}\frac{nR^{\frac{1}{n}+1}}{n+1}.$$

Hence the equation (1.23), takes the following form

$$V = \left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}}\frac{n}{n+1}\left[R^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}\right], \tag{1.24}$$

which concludes that the velocity of the blood flow in the artery remote from heart.

1.7 Bio-Physical Interpretation for Artery Blood Vessel

The flow flux of blood through the arteries is

$$Q = \int_0^R V.2\pi r dr = \int_0^R \left[\frac{P}{2\eta_m}\right]^{\frac{1}{n}} \frac{n}{n+1}(R^{\frac{1}{n}+1} - r^{\frac{1}{n}+1})$$

$$= \left[\frac{P}{2\eta_m}\right]^{\frac{1}{n}} \frac{n2\pi}{n+1} \left[\frac{R^{\frac{1}{n}+1}\ .r^2}{2} - \frac{nr^{\frac{1}{n}+3}}{3n+1}\right]_0^R$$

$$= \left[\frac{P}{2\eta_m}\right]^{\frac{1}{n}} \frac{n2\pi}{n+1} \left(\frac{(n+1).R^{\frac{1}{n}+3}}{2(3n+1)}\right)$$

$$= \left(\frac{P}{2\eta_m}\right)^{\frac{1}{n}} \frac{n\pi R^{\frac{1}{n}+3}}{3n+1} \tag{1.25}$$

2 Results and Discussions

Table 2.1: *Clinical data Hematocrit v/s blood pressure*

10/09/2012	13.32	0.0376952	130/70	-3993.96
25/11/2012	13.16	0.0372453	110/70	-2662.64
19/01/2013	13.00	0.0367925	110/60	-3328.30
11/03/2013	12.80	0.0362265	100/80	-1331.32
21/07/2013	12.70	0.0359434	120/90	-1996.98
05/09/2013	12.90	0.0365095	100/70	-1996.98
12/11/2013	12.60	0.0356604	110/80	-1996.98

Now, we know that, $Q = 425 \ \frac{ml}{\text{min}}$, $Q = 0.00708333\ m^3/sec$, $\eta_p = 0.0013$ *pascal second* [8] and $\eta_m = 0.0271$ *pascal second* [7]. Approximately pulmonary artery length is $(z_f - z_i) = 5\ cm$ or 0.05 m., pulmonary arterioles and venules length =0.0015 m.[14].

In according to used Pathological data (Hematocrit) $H = 0.0367925$and Pressure drop$(P_f - P_i) = 3328.3$ *pascal second.*

Using relation 'A' and we get η_c

$$\Rightarrow 0.0271 = \eta_c(0.000367925) + 0.0013(0.999632075)$$

$$\Rightarrow \eta_c = \quad 70.12428702 \text{ Pascal second}$$

Again using (A) relation and convert in to the hematocrit form-

$$\eta_m = \ 0.70124287H + 0.001299522$$

Now using relation (B) and putting above values-

$$0.0070833 = \left[\frac{3328.3}{2 \times 0.0271 \times 0.05}\right]^{\frac{1}{n}} \frac{n \times 3.14 \times (0.015)^{\frac{1}{n}+3}}{3n+1}$$

$$668.3963 = \left(\frac{n}{3n+1}\right)(18422.32472)^{\frac{1}{n}}$$

By solving trial and error method we get- $n = 0.8093$

Again putting $n = 0.8093$ above equation (B) again and find ΔP

$$0.0070833 = \left[\frac{\Delta P}{2\eta_m \Delta_z}\right]^{\frac{1}{n}} \frac{0.8093\times 3.14\times (0.015)^{\frac{1}{0.8093}+3}}{(3\times 0.8093)+1}$$
$$\Delta P = (0.70124287H + 0.001299522)\ (973.3395908)$$
$$\Delta P = 682.5474484H + 1.264875917$$

Table 2.2: *Blood pressures drop v/s Hematocrit(mathematically modulated data)*

Date	Hematocrit (3×HB) (kg/l)	BPD (Blood Pressure drop) Pascal -second
10/09/2012	0.0376952	26.99572926
25/11/2012	0.0372453	26.68660352
19/01/2013	0.0367925	26.37754603
11/03/2013	0.0362265	25.99122418
21/07/2013	0.0359434	25.79799499
05/09/2013	0.0365095	26.1843851
12/11/2013	0.0356604	25.60483407

Graph: Table I & II: relation between real clinically blood pressure drop and hematocrit, mathematically modulated blood pressure drop v/s hematocrit

Graph: a. (Table I)

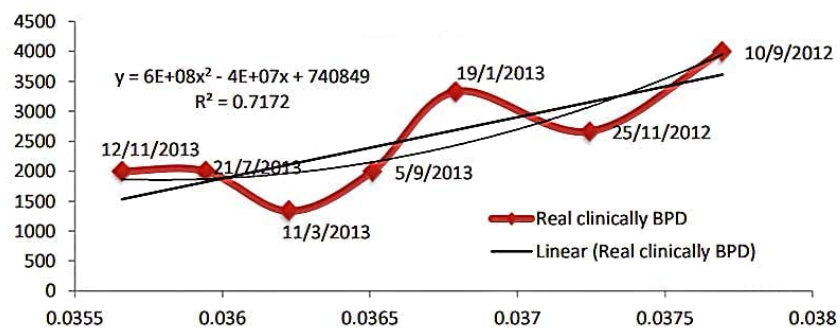

Figure 2.1: *Real pathological BPD*

b. (Table II)

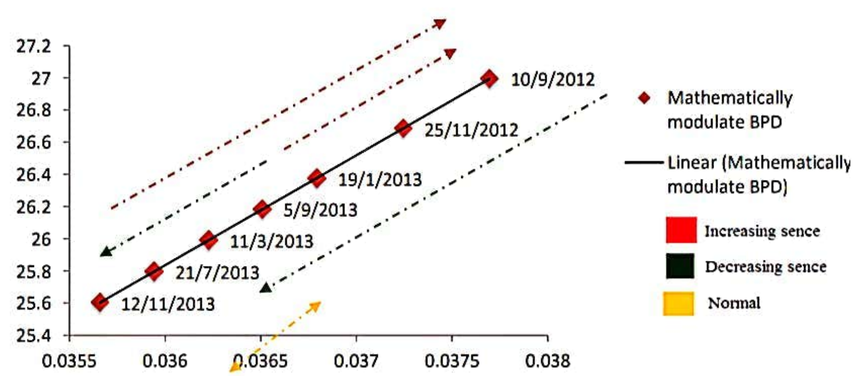

Figure 2.2: *Mathematical modulate BPD*

Observation

Graph (a) shows that these 7 different dates were observed minimum about 1331.32 on dated 11/03/2013 and maximum value obtain 3993.96 on dated 10/09/2010. The value from 0.0356604 to 0.0362265 via

0.0359434of hematocrit value, the blood pressure drop upper convex in decreasing sence and the values from 0.0359434 to 0.0367925 via 0.0362265 & 0.0365095 of hematocrit value, the pressure drop shows down convex in increasing sence. Again the value from 0.0367925 to 0.0376952 via 0.0372453 of hematocrit value, the pressure drop shows down convex in increasing sence. Graph (b) shows that these 7 different dates were observed minimum about 25.60483407 on dated 12/11/2013 and maximum value obtains 26.99572926 on dated 10/09/2012 (BDP). At the value from 0.0376952to 0.0356604 via 0.0372453, 0.0367925, 0.0365095, 0.0362265 & 0.0359434 of hematocrit value, the blood pressure drop straightly decreases on dated 10/09/2012 to 12/11/2013 via 25/11/2012, 19/01/2013, 05/09/2013, 11/03/2013 & 21/07/2013.

2.1 Limitations and Future Directions

Despite their potential, this mathematical model face challenges such as:

1. **Data Availability:** Limited patient-specific data can reduce model accuracy.
2. **Computational Complexity:** High-resolution models demand significant computational resources

Future advancements in machine learning and data integration are expected to enhance the predictive power and accessibility of these models. Additionally, collaborations between clinicians and mathematicians can bridge the gap between theoretical research and practical applications.

2.2 Conclusion for Pulmonary Arteries

Mathematical modeling of blood fluctuation during pulmonary diseases offers a powerful approach to unraveling the complexities of these conditions. By simulating physiological responses under pathological conditions, these models contribute to improved diagnosis, treatment, and patient outcomes. As technology advances, the integration of sophisticated computational techniques will further solidify the role of mathematical models in pulmonary medicine.

From above, Pathological data I table**,** a mathematically investigated and concluded; graph b (table II) shows from 10/09/2012 to 12/11/2013 via 25/11/2012, 19/01/2013, 11/03/2013 & 21/07/2013 decreasing sense. Whereas from 12/11/2013 to 05/09/2013 shows increasing sense. According to present study Two phase Non-Newtonian 'Power law' and 'Herschel-Bulkley' model were investigated with the help of clinical data during lung cancer patient in pulmonary artery, artery stenosis, arterioles, capillary and venules. Present work concluded that designate the function of hematocrit inside the willpower of blood pressure drop. For this reason the hematocrit is extended then the blood pressure drop is likewise multiplied.

In the above all graphs we have observed "relation between real clinically blood pressure drop" and "relation between mathematically modulated blood pressure drop v/s hematocrit" graph are shows different nature but there trend line are not different. Trend lines of above graphs shows that if trend is upward it means fluctuation of blood (pressure drop) increases with respect to the hematocrit. Trend is downward that means fluctuation of blood pressure drop decreases with respect to the hematocrit.We know that many other components (White blood cells and Platelets) are found in addition to hemoglobin and plasma. Who work like a liquid and with their red blood cells increases concentration also decreases. By modifying the real model (frame of reference) of our mathematical model, we were able to carry this work forward in three phases or four phases.

3 Bicomplex and Multicomplex Analytical Extension of Non-Newtonian Two-Phase Blood Flow in Pulmonary Diseases

3.1 Introduction

3.1.1 Background and Motivation

Pulmonary diseases such as chronic obstructive pulmonary disease, pulmonary hypertension, and lung cancer introduce nonlinear fluctuations in blood flow due to vessel constriction, viscosity changes, and irregular oxygen exchange. The classical model already discussed in artical 1 treated blood as a two-phase, non-Newtonian mixture of red blood cells and plasma, governed by a power-law constitutive relation. While successful in reproducing flowpressure relations, this model operates purely in the real domain and cannot represent oscillatory physiological processes like ionic transport, cellular electrical potentials, or micro-vibrations.

Bicomplex and multicomplex analysis extends real-valued models by incorporating additional imaginary units that capture orthogonal physical processes. In this study, bicomplex numbers encode mechanical and electrochemical fields in one formulation, while multicomplex extensions capture multiple biological phases. This algebraic generalization enriches the representation of nonlinear blood-flow phenomena during pulmonary diseases.

3.1.2 Objectives

The objectives of these works are:

- To extend the tensorial equations of non-Newtonian blood flow to the bicomplex domain.
- To derive explicit analytical velocity and pressure distributions under axisymmetric steady flow.
- To generalize the bicomplex framework into multicomplex algebra for multi-phase coupling.
- To link theoretical findings with experimental and clinical data from pulmonary disease models.

3.2 Mathematical Preliminaries: Bicomplex and Multicomplex Algebra

3.2.1 Bicomplex Numbers

Definition 3.1 (Bicomplex Algebra)**.** *Let i and j be two imaginary units satisfying $i^2 = j^2 = -1$ and $ij = ji$. The bicomplex algebra is defined as*

$$\mathbb{B} = \{z_1 + jz_2 \mid z_1, z_2 \in \mathbb{C}(i)\}.$$

Every element can be written as $Z = a + ib + jc + ijd$, with $a, b, c, d \in \mathbb{R}$.

Bicomplex conjugations exist with respect to i and j, and the algebra includes idempotents:

$$e_\pm = \tfrac{1}{2}(1 \pm ij), \quad e_\pm^2 = e_\pm, \quad e_+e_- = 0.$$

Every $Z \in \mathbb{B}$ decomposes uniquely as $Z = Z_+e_+ + Z_-e_-$, where $Z_\pm = z_1 \mp iz_2$. This decomposition simplifies bicomplex analysis into two complex-valued subsystems.

3.2.2 Bicomplex Differential Operators

For a bicomplex vector field $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$, define:

$$\nabla^{(\mathbb{B})} = \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z, \tag{3.1}$$

$$\mathrm{div}^{(\mathbb{B})}(\mathbf{v}) = \nabla^{(\mathbb{B})} \cdot \mathbf{v}, \tag{3.2}$$

$$\mathrm{curl}^{(\mathbb{B})}(\mathbf{v}) = \nabla^{(\mathbb{B})} \times \mathbf{v}. \tag{3.3}$$

These operators extend naturally under idempotent decomposition, allowing complex-valued PDEs to emerge for each component.

3.2.3 Multicomplex Numbers

The multicomplex algebra $\mathbb{M}_m$ generalizes $\mathbb{B}$ by introducing m commuting imaginary units $i_1, i_2, \ldots, i_m$ with $i_k^2 = -1$ and $i_ki_l = i_li_k$. This algebraic system can represent multiple interacting physiological phases.

For further details one can refer to [1, 3, 4, 5, 11, 12, 13, 15, 17, 16, 20, 18, 19, 20]

3.3 Classical Non-Newtonian Model (Summary of Uploaded Paper)

The original work established a tensorial model for two-phase blood flow using the power-law relation:

$$\tau_{ij} = -pg_{ij} + \eta_m(e_{ij})^n, \tag{3.4}$$

with the continuity and motion equations:

$$\frac{1}{\sqrt{g}}(\sqrt{g}v^i)_{,i} = 0, \tag{3.5}$$

$$\rho_m(v_jv^i_{,j}) = -p_{,i} + \tau_{ij,j}. \tag{3.6}$$

For steady, axisymmetric flow in cylindrical coordinates, analytical integration gives:

$$v_z(r) = \frac{P}{4\eta_m}(R^2 - r^2), \tag{3.7}$$

representing the classical parabolic velocity profile for power-law fluid. These equations form the real-valued foundation for our bicomplex extension.

3.4 Bicomplex Formulation of Blood Flow

3.4.1 Bicomplex Constitutive Law

We define bicomplex stress as

$$\boldsymbol{\tau}^{(\mathbb{B})}_{ij} = -p^{(\mathbb{B})}g_{ij} + \eta^{(\mathbb{B})}_m(e_{ij})^n, \tag{3.8}$$

where $p^{(\mathbb{B})} = p^r + jp^i$ and $\eta^{(\mathbb{B})}_m = \eta^r_m + j\eta^i_m$. The imaginary parts model physiological oscillations linked to hematocrit variability, ionic charge distributions, and elastic RBC deformations.

3.4.2 Bicomplex Continuity and Momentum Equations

$$\frac{1}{\sqrt{g}}(\sqrt{g}v^i)^{(\mathbb{B})}_{,i} = 0, \tag{3.9}$$

$$\rho^{(\mathbb{B})}_m (v_j v^i_{,j}) = -p^{(\mathbb{B})}_{,i} + \left(\eta^{(\mathbb{B})}_m (e^i_j)^n\right)_{,j}. \tag{3.10}$$

Applying idempotent decomposition yields two coupled complex PDEs:

$$\rho_\pm (v_{j,\pm} v^i_{\pm,j}) = -p_{\pm,i} + \left(\eta_\pm (e^i_{j,\pm})^n\right)_{,j}. \tag{3.11}$$

3.4.3 Axisymmetric Steady Flow

For steady, fully developed flow $v = v_z(r)$ and constant bicomplex pressure gradient $G^{(\mathbb{B})} = -dP^{(\mathbb{B})}/dz$, we derive:

$$0 = -G^{(\mathbb{B})} + \frac{1}{r}\frac{d}{dr}\left(r\eta^{(\mathbb{B})}_m \left(\frac{dv}{dr}\right)^n\right). \tag{3.12}$$

Integration gives:

$$v^{(\mathbb{B})}(r) = \left(\frac{G^{(\mathbb{B})}}{2\eta^{(\mathbb{B})}_m}\right)^{1/n} \frac{n}{n+1}(R^{1+1/n} - r^{1+1/n}). \tag{3.13}$$

The real part yields measurable velocity, while the imaginary part models oscillatory electrochemical flow.

3.5 Mathematical Results and Theorems

Lemma 3.1 (Bicomplex Linearity)**.** *If v_1, v_2 solve the bicomplex momentum equation under linear boundary conditions, then $c_1v_1 + c_2v_2$ (with bicomplex constants c_1, c_2) is also a solution.*

Theorem 3.1 (Existence of Steady Bicomplex Flow)**.** *For smooth boundary conditions and bounded viscosity $\eta^{(\mathbb{B})}_m$, there exists a unique bicomplex velocity field $v^{(\mathbb{B})}$ satisfying the axisymmetric steady-state equation.*

Proof. The proof follows from decomposition into two complex-valued patial differential equations, each admitting unique solutions by standard theory of elliptic partial differential equations. □

3.6 Multicomplex Generalization for Multi-Phase Blood Flow

Blood is a composite fluid with multiple interacting components. Define the multicomplex stress tensor as:

$$\boldsymbol{\tau}^{(\mathbb{M})} = -p^{(\mathbb{M})}g_{ij} + \sum_{k=1}^{m} i_k \eta_k (e_{ij})^{n_k}. \tag{3.14}$$

Each imaginary unit i_k represents a distinct physiological phase, enabling explicit modeling of red blood cellplasma, plasmahite blood cell, and plateletwall interactions. The resulting equations couple mechanical, ionic, and chemical forces.

3.7 Numerical Illustration

Let $R = 0.015$ m, $n = 0.8093$, $\eta^r_m = 0.0271$ Pas, $H = 0.03679$. Set small coupling terms $\eta^i_m = 0.5H$ and $G^i = 0.05G^r$. Substituting yields a bicomplex velocity profile whose magnitude predicts increased oscillatory behavior near vessel walls, consistent with clinical pressure-drop data.

3.8 Biophysical Interpretation

The bicomplex representation links electrochemical fluctuations and hemodynamic behavior. The imaginary part of viscosity quantifies how ionic interactions modulate flow resistance, while the imaginary pressure gradient captures phase-lagged cardiac-pulmonary coupling. Multicomplex modeling provides a structure for incorporating additional phases such as temperature and wall elasticity.

3.9 Conclusion

This extended formulation integrates bicomplex and multicomplex analysis with the classical tensorial framework of pulmonary blood flow. It enhances descriptive capacity for nonlinear, multi-phase, and oscillatory dynamics in diseased states. Future research will involve computational implementation and clinical parameter fitting.

Limitations and Future Directions

Bicomplex and multicomplex analytical extensions of non-Newtonian two-phase blood flow in pulmonary diseases involve complex mathematical modeling to understand the behavior of blood flow in the lungs. Non-Newtonian fluids, like blood, exhibit varying viscosity depending on the shear rate, making their flow behavior more intricate. The key aspects of the paper are confined on the followings:

- **Non-Newtonian Blood Flow:** Blood is a non-Newtonian fluid, meaning its viscosity changes with the shear rate. This is crucial in understanding blood flow in pulmonary diseases.
- **Two-Phase Flow:** Blood is a two-phase fluid, consisting of plasma and blood cells. Modeling this two-phase flow is essential to capture the complex behavior of blood in the lungs.
- **Bicomplex and Multicomplex Analysis:** Bicomplex and multicomplex numbers are used to extend traditional complex analysis to higher dimensions, allowing for more accurate modeling of complex phenomena like blood flow.

Further we mention here some of the limitations of our work as below:

- **Complexity of Blood Flow:** Blood flow in the lungs is a complex, multiscale phenomenon, making it challenging to model accurately.
- **Non-Newtonian Behavior:** The non-Newtonian behavior of blood adds complexity to the modeling process.
- **Limited Data:** Limited availability of experimental data on blood flow in pulmonary diseases hinders model validation.

Now we may sketch the future course of our work in following direction which must be helpful to the new researchers in this branch:

- **Advanced Mathematical Modeling:** Developing more sophisticated mathematical models that incorporate non-Newtonian behavior and two-phase flow.
- **Computational Simulations:** Utilizing computational simulations to study blood flow in pulmonary diseases and validate models.
- **Experimental Studies:** Conducting experimental studies to gather data on blood flow in pulmonary diseases and validate models.

These advancements can help improve our understanding of pulmonary diseases and develop more effective treatments. Further these works may be carried out under the flavor of bicomplex and multicomplex analysis of uncertain variables.

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NEW CLASS OF GENERALIZED HERMITE POLYNOMIALS INVOLVING CUBIC HERMITE POLYNOMIALS

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Abstract

The great success of the theory of an exponential generating function for generalized special polynomials has stimulated the development of a corresponding theory of Hermite polynomials of one and more variables. We use this type of a standard generating function to derive certain results for example, Rodrigues' formula, integral representation, matrix representations and differential equations not attempted by any researcher so far. Application of Rodrigues' formula and the integral representation attempted herein produce first few polynomials of cubic Hermite polynomials and some inequalities.

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1 Introduction

The quantum physics is analysed by classical Hermite polynomials, $H_n(x)\ \forall n = 0, 1, 2, 3, \ldots$; ([1], see also in [10, Section 105, p.189] and [11, Eqn. (16), p.74]) and defined through Rodrigues' formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n \left\{\exp\left(-x^2\right)\right\}, D_x^n \equiv \frac{d^n}{dx^n}. \tag{1.1}$$

Then in 1934, Bell [2] presented a generalization of (1.1) as a class of polynomials in form of exponential function defined by

$$\xi_n(x,t;r) = \exp\left(-xt^r\right) D_t^n \exp\left(xt^r\right), \xi_{-n-1}(x,t;r) = 0; D_t^n \equiv \frac{d^n}{dt^n}. \tag{1.2}$$

Here in (1.2), r is a fixed integer $> 0, n$ is an arbitrary integer ≥ 0, while x and t are independent variables.

Later on, Gould and Hopper ([6, p.52], see also in [11, Eqn. (12), p. 77]) introduced operational formulas connected with generalization of (1.1) in the form

$$H_n^r(x,a;p) = (-1)^n x^{-a} \exp\left(px^r\right) D_x^n \left\{x^a \exp\left(-px^r\right)\right\}. \tag{1.3}$$

M. Lahiri [7] studied the generalized Hermite polynomials $H_{n,m,v}(x),\ \forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, $v > 0, x$ and t being independent variables, through generating function

$$\sum_{n=0}^{\infty} H_{n,m,v}(x)\frac{t^n}{n!} = \exp\left(vxt - t^m\right). \tag{1.4}$$

Motivated by above results, we introduce a standard exponential generating function of a class of poly-Hermite polynomials denoted by $\mathcal{H}_n^m(x)$ $\forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, x and t being independent variables, defined by

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp\left((-1)^{m+1}\binom{m}{0}t^m + (-1)^m\binom{m}{1}t^{m-1}x + (-1)^{m-1}\binom{m}{2}t^{m-2}x^2\right.$$
$$\left.+(-1)^{m-2}\binom{m}{3}t^{m-3}x^3 + \cdots + \binom{m}{m-1}tx^{m-1}\right)$$
$$= \exp\left(-\left[\binom{m}{0}(-t)^m + \binom{m}{1}(-t)^{m-1}x + \binom{m}{2}(-t)^{m-2}x^2\right.\right.$$
$$\left.\left.+\binom{m}{3}(-t)^{m-3}x^3 + \cdots + \binom{m}{m-1}(-t)x^{m-1} + x^m\right]\right)\exp(x^m)$$

where,

$$\binom{m}{r} = \begin{cases} \frac{m(m-1)\ldots(m-r+1)}{r!}, 0 \le r \le m; \\ 0, r > m. \end{cases} \quad . \tag{1.5}$$

Thus more suitably a new class of standard generating function (1.5) can be written as

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp(x^m)\exp(-(x-t)^m). \tag{1.6}$$

This class of functions is "extraordinary" because it generalizes the classical Hermite polynomials (where $m = 2$) to higher-order exponents. For $m = 2$, the formula (1.6) reduces to the standard Rodrigues' formula (1.1) as

$$\mathcal{H}_n^2(x) = H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n\left\{\exp\left(-x^2\right)\right\}, D_x^n \equiv \frac{d^n}{dx^n} \tag{1.7}$$

whereas for $m > 2$, these polynomials often appear in problems involving non-Gaussian statistics, higher order differential equations, and quantum optics.

2 Rodrigues' formula

Theorem 2.1. *For all $n \ge 0$, the standard exponential generating function* (1.6) *satisfies the Rodrigues ' formula*

$$\mathcal{H}_n^m(x) = (-1)^n \exp(x^m)\frac{\partial^n}{\partial x^n}\exp(-x^m). \tag{2.1}$$

Proof. Considering the standard exponential generating function (1.6) and in its right hand side on using the Maclaurin's Theorem to find that

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp(x^m)\sum_{n=0}^{\infty}\frac{\partial^n}{\partial t^n}\exp(-(x-t)^m)\bigg|_{t=0}\frac{t^n}{n!}$$
$$= \exp(x^m)\sum_{n=0}^{\infty}(-1)^n\frac{\partial^n}{\partial v^n}\exp(-v^m)\bigg|_{v=x}\frac{t^n}{n!} \tag{2.2}$$

Now equating the coefficients of t^n in both sides of the Eqn. (2.2), we get the Rodrigues' formula (2.1). □

Specially, putting $m = 2$ in the formula (1.5) and (1.6), we get following known generating functions for classical Hermite polynomials, $H_n(x)$ $\forall n = 0, 1, 2, 3, \ldots$, (1.1) as

$$\sum_{n=0}^{\infty} \mathcal{H}_n^2(x)\frac{t^n}{n!} = \exp\left(-t^2 + 2tx\right)$$
$$= \exp\left(x^2\right)\exp\left(-(x-t)^2\right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{2.3}$$

Proposition 2.1. *For x and t being independent variables, the Eqns.* (1.5) *or* (1.6) *and* (2.1)*, become cubic polynomials, $\mathcal{H}_n^3(x)$ $\forall n \ge 0$, and there exists a standard generating function in the form*

$$\sum_{n=0}^{\infty} \mathcal{H}_n^3(x)\frac{t^n}{n!} = \exp\left(t^3 - 3xt^2 + 3x^2t\right) = \exp\left(x^3\right)\exp\left(-(x-t)^3\right), \tag{2.4}$$

and connect to Rodrigues' formula given by

$$\mathcal{H}_n^3(x) = (-1)^n e^{x^3} \frac{\partial^n}{\partial x^n}\left(e^{-x^3}\right). \tag{2.5}$$

Proof. By applying equations (1.5) and (1.6) along with Theorem 2.1, and setting m = 3, we yield formulas (2.4) and (2.5) of Proposition 2.1. □

Theorem 2.2. *The standard generating function* (2.4) *creates the series formula of a new class of cubic generalized Hermite polynomials*

$$\mathcal{H}_n^3(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^{\left[\frac{n-2m}{3}\right]} \frac{3^{n-2m-3k}\left(x^2\right)^{n-2m-3k}}{(n-2m-3k)!k!m!}, n = 0, 1, 2, 3, \ldots. \tag{2.6}$$

Proof. In the generating function (2.4) on making an appeal to series rearrangement techniques, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{H}_n^3(x)\frac{t^n}{n!} &= \exp\left(t^3 + 3x^2 t\right)\exp\left(-3xt^2\right)\\ &= \sum_{n=0}^{\infty} n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^{\left[\frac{n-2m}{3}\right]} \frac{3^{n-2m-3k}\left(x^2\right)^{n-2m-3k}}{(n-2m-3k)!k!m!}\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Equating like powers of t in both sides of (2.7), we obtain the series formula (2.6) of the cubic Hermite polynomials, $\mathcal{H}_n^3(x)\ \forall n \geq 0$. □

Also, we derive first few generalized Hermite polynomials and integral representation via Rodrigues' formula (2.5) and the standard generating function (2.4). We present some of its analytic properties to find its matrix representation and some inequalities not attained till now.

Again then, we introduce a multivariable analogue of the generating function (2.4). This multivariable analogue of generating function will become helpful for exploring new ideas of future researches in the field of special functions and other branches of mathematical sciences.

3 Applications of the Rodrigues' formula (2.5)

Applying the Rodrigues' formula (2.5), we derive following first few cubic Hermite polynomials, $\mathcal{H}_n^3(x)\ \forall n \geq 0$ in the form

$$\begin{aligned}&\mathcal{H}_0^3(x) = 1, \mathcal{H}_1^3(x) = 3x^2, \mathcal{H}_2^3(x) = 9x^4 - 6x, \mathcal{H}_3^3(x) = 27x^6 - 54x^3 + 6,\\ &\mathcal{H}_4^3(x) = 81x^8 - 324x^5 + 180x^2.\end{aligned} \tag{3.1}$$

The linear advection-type operators, commonly found in the study of hyperbolic partial differential equations, transport equations or specialized numerical analysis. Therefore, we present following inequalities.

Theorem 3.1. *For all $x, t \in \mathbb{R}$ such that $|t| \leq 1$, if by the formula* (2.4)*, it is defined as*

$$F(x,t) = \exp\left(t^3 - 3xt^2 + 3x^2 t\right) \text{ and } g(x,t) = t^3 - 3xt^2 + 3x^2 t. \tag{3.2}$$

Then there exist following inequalities

$$\left\|\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) F(x,t)\right\| \leq \left\|\mathcal{H}_1^3(x)\right\| \|F(x,t)\|. \tag{3.3}$$

$$\frac{\left\|\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) F(x,t)\right\|}{\left\|\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) g(x,t)\right\|} \leq \|F(x,t)\|. \tag{3.4}$$

$$\left\|\left\{\frac{1}{x^2}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\right\}^n F(x,t)\right\| \leq 3^n \|F(x,t)\|. \tag{3.5}$$

Proof. Considering (3.2), we get

$$\frac{\partial}{\partial t} F(x,t) = \left(3t^2 - 6xt + 3x^2\right)\exp\left(t^3 - 3xt^2 + 3x^2 t\right) = F(x,t)\frac{\partial}{\partial t} g(x,t). \tag{3.6}$$

Also, we find

$$\frac{\partial}{\partial x} F(x,t) = \left(-3t^2 + 6xt\right)\exp\left(t^3 - 3xt^2 + 3x^2 t\right) = F(x,t)\frac{\partial}{\partial x} g(x,t). \tag{3.7}$$

On making an application of first and third relations of (3.6) and (3.7) respectively and then adding them, we find

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)F(x,t)=3x^2F(x,t)=F(x,t)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)g(x,t). \tag{3.8}$$

Then, using the relations of (3.1) and (3.8), we obtain the results (3.3), (3.4) and (3.5). □

It is noted that the generating function (2.4) is also applicable to obtain the integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$.

4 Integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$

In this section to obtain integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, by the formula (2.2) for $m=3$, we consider that

$$\mathcal{H}_n^3(x)=\frac{\partial^n}{\partial t^n}\exp\left(t^3-3xt^2+3x^2t\right)\Big|_{t=0} \tag{4.1}$$

and make an appeal to the formula of complex analysis

$$\Psi(x)=\frac{\partial^n}{\partial t^n}F(x,t)\Big|_{t=0}=\frac{n!}{2\pi i}\int_C\frac{F(x,u)}{u^{n+1}}du, \tag{4.2}$$

where, $i=\sqrt{(-1)}, C$ is a contour such that $|u|\le 1$;
then, applying (4.1) and (4.2), we write

$$\mathcal{H}_n^3(x)=\frac{n!}{2\pi i}\int_C\frac{\exp\left(u^3-3xu^2+3x^2u\right)}{u^{n+1}}du,\, u=e^{i\theta}, 0\le\theta\le 2\pi. \tag{4.3}$$

Theorem 4.1. *The real integral representation of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, exists and given by*

$$\mathcal{H}_n^3(x)=\frac{n!}{\pi}\int_0^{\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta \tag{4.4}$$

Proof. For $u=e^{i\theta}, 0\le\theta\le 2\pi$, the formula (4.3) is transformed into

$$\mathcal{H}_n^3(x)=\frac{n!}{2\pi}\int_0^{2\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta$$
$$+\frac{i(n!)}{2\pi}\int_0^{2\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\sin\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta \tag{4.5}$$

In the definite integral (4.4), we have

$$\begin{aligned}&\sin\left(\sin(6\pi-3\theta)-3x\sin(4\pi-2\theta)+3x^2\sin(2\pi-\theta)-n(2\pi-\theta)\right)\\&=-\sin\left(\sin(3\theta)-3x\sin(2\theta)+3x^2\sin(\theta)-n\theta\right)\end{aligned} \tag{4.6}$$

Therefore, by (4.6), the integral (4.5) becomes in the real definite integral, given by

$$\begin{aligned}\mathcal{H}_n^3(x)=\frac{n!}{\pi}\int_0^{\pi}&\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\\&\times\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta.\end{aligned} \tag{4.7}$$

Hence, the Theorem 4.1 is followed. □

Theorem 4.2. *If all conditions of the Theorem 3.1 are followed, then the following matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, hold*

$$\begin{bmatrix}F\\ \frac{\partial F}{\partial t}\\ \frac{\partial^2F}{\partial t^2}\\ \frac{\partial^3F}{\partial t^3}\\ \frac{\partial^4F}{\partial t^4}\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&6&0\\0&0&0&0&24\end{bmatrix}\begin{bmatrix}F\\ \dot{g}F\\ \left((\dot{g})^2+\ddot{g}\right)F\\ \left(\frac{1}{6}(\dot{g})^3+\frac{1}{2}\dot{g}\ddot{g}+1\right)F\\ \left(\frac{1}{24}(\dot{g})^4+\frac{1}{4}(\dot{g})^2\ddot{g}+\frac{1}{8}(\ddot{g})^2+\dot{g}\right)F\end{bmatrix}, \tag{4.8}$$

with coefficient matrix

$$C=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&6&0\\0&0&0&0&24\end{bmatrix}. \tag{4.9}$$

Also

$$\begin{bmatrix} H_0^3(x) \\ H_1^3(x) \\ H_2^3(x) \\ H_3^3(x) \\ H_4^3(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} F \\ \dot{g}F \\ \left((\dot{g})^2 + \ddot{g}\right) F \\ \left(\frac{1}{6}(\dot{g})^3 + \frac{1}{2}\dot{g}\ddot{g} + 1\right) F \\ \left(\frac{1}{24}(\dot{g})^4 + \frac{1}{4}(\dot{g})^2\ddot{g} + \frac{1}{8}(\ddot{g})^2 + \dot{g}\right) F \end{bmatrix}_{t=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 3x^2 \\ 9x^4 - 6x \\ \frac{9}{2}x^6 - 9x^3 + 1 \\ \frac{27}{8}x^8 - \frac{27}{2}x^5 + \frac{15}{2}x^2 \end{bmatrix}. \tag{4.10}$$

Proof. By Theorem 3.1 on considering Eqn. (3.2)

$$F(x,t) = \exp\left(t^3 - 3xt^2 + 3x^2t\right) \text{ and } g(x,t) = t^3 - 3xt^2 + 3x^2t, \text{ to get}$$

$$\frac{\partial}{\partial t}F(x,t) = \dot{g}F, \frac{\partial^2 F}{\partial t^2} = \left((\dot{g})^2 + \ddot{g}\right) F, \frac{\partial^3 F}{\partial t^3} = \left((\dot{g})^3 + 3\dot{g}\ddot{g} + 6\right) F,$$

$$\frac{\partial^4 F}{\partial t^4} = \left((\dot{g})^4 + 6(\dot{g})^2\ddot{g} + 3(\ddot{g})^2 + 24\dot{g}\right) F. \tag{4.11}$$

Employing the equations given in (4.11), we formulate the matrix representation (4.8) with the coefficient matrix (4.9).

Again then, applying (3.1) and (3.2) in the result (4.9), we yield the matrix representation (4.10) for the generalized Hermite polynomials $\mathcal{H}_n^3(x)$ $\forall n = 1, 2, 3, 4, \ldots$. □

5 Inequalities via integral representation of classical and cubic Hermite polynomials

We are familiar with formulae relating to Bernuolli polynomials $B_n(x)$ $\forall n = 0, 1, 2, 3, 4, \ldots$, and the Bernuolli numbers $B_n, n = 0, 1, 2, 3, \ldots$, given by [11, p.85]

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{5.1}$$

and

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{5.2}$$

respectively.

Using these results (5.1), (5.2) and the integral representations of classical and generalized Hermite polynomials to obtain the inequalities:

Theorem 5.1. *The real integral representation of classical Hermite polynomials $H_n(x)$ $\forall n = 1, 2, 3, 4, \ldots$, exists and given by*

$$H_n(x) = \frac{n!}{\pi} \int_0^{\pi} \exp[2x\cos\theta - \cos 2\theta] \times \cos(2x\sin\theta - \sin 2\theta - n\theta)d\theta. \tag{5.3}$$

Then, there exists following inequalities

$$\sum_{n=0}^{m} \left| \frac{m!}{(m-n)!} B_{m-n} 2^n x^n \right| + \sum_{n=0}^{\infty} \sum_{j=0}^{n-2} \left| \frac{m!}{(m-j)!} a_j B_{m-j} x^j \right| \leq \sum_{n=0}^{\infty} \frac{n!}{\pi} \int_0^{\pi} \left| B_m(2x\cos\theta) \exp[-\cos 2\theta] \right| d\theta. \tag{5.4}$$

provided that the constants $a_{-j} = 0$ $\forall j \in \mathbb{N}$ and $a_0 \neq 0$.

Also, there exists

$$a_0 \leq I_0(1) - 1 \Rightarrow a_0 \leq 0.266 \tag{5.5}$$

Proof. In the formula (5.3) on replacing x by xt and then in its both sides multiplying by $\frac{t}{e^t-1}$ and again making an appeal to (5.1) and (5.2) to get

$$\sum_{m=0}^{\infty} B_m H_n(xt)\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{m!}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta - \sin 2\theta - n\theta)d\theta,$$

then we have

$$\sum_{m=0}^{\infty} B_m \left\{(2xt)^n + P_{n-2}(xt)\right\}\frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{m!}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta - \sin 2\theta - n\theta)d\theta, \tag{5.6}$$

where, $P_{n-2}(x) = a_0 + a_1 x + \cdots + a_{n-2}x^{n-2} = \sum_{j=0}^{n-2} a_j x^j$,
provided that $a_{-j} = 0\ \forall j \in \mathbb{N}, a_0 \neq 0$.

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_m 2^n x^n \frac{t^{m+n}}{m!} + \sum_{m=0}^{\infty} B_m \sum_{n=0}^{\infty}\sum_{j=0}^{n-2} a_j x^j \frac{t^{m+j}}{m!}$$
$$= \sum_{m=0}^{\infty} \frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta - \sin 2\theta - n\theta)d\theta.$$

Therefore, $\forall m = 0, 1, 2, 3, 4, \ldots$, following inequality holds

$$\sum_{n=0}^{m}\left|\frac{m!}{(m-n)!}B_{m-n}2^n x^n\right| + \sum_{n=0}^{\infty}\sum_{j=0}^{n-2}\left|\frac{m!}{(m-j)!}a_j B_{m-j}x^j\right|$$
$$\le \sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi}\left|B_m(2x\cos\theta)\exp[-\cos 2\theta]\right| d\theta. \tag{5.7}$$

For example when $m = 0, n = 0$ and $j = 0$, so that we get

$$B_0 + a_0 B_0 \le I_0(1), \tag{5.8}$$

where, $I_0(1)$ is the modified Bessel function of the first kind and $I_0(1) = 1.266$.

Also $B_0 = 1$.

Hence, the Theorem 5.1 is followed. □

Theorem 5.2. *The real integral representation of $\mathcal{H}_n^3(x)\ \forall n = 1, 2, 3, 4, \ldots,$ exists in (4.4) as*

$$\mathcal{H}_n^3(x) = \frac{n!}{\pi}\int_0^{\pi}\exp\left[\cos 3\theta - 3x\cos 2\theta + 3x^2\cos\theta\right]$$
$$\times\cos\left(\sin 3\theta - 3x\sin 2\theta + 3x^2\sin\theta - n\theta\right)d\theta. \tag{5.9}$$

Then following inequalities hold

$$\left|\sum_{n=0}^{\left[\frac{m}{2}\right]} 3^n x^{2n} 2n! \sum_{k=0}^{[(m-2n)/2]}\binom{m-2n}{2k}\binom{m}{2n}\frac{2k!}{k!}B_k B_{m-2n-2k}\right|$$
$$+\left|\sum_{n=0}^{\infty}\sum_{j=0}^{2n-3} a_j x^{2j-3} j!\binom{m}{j}\sum_{k=0}^{[(m-j)/2]}\binom{m-j}{2k}\frac{2k!}{k!}B_k B_{m-j-2k}\right|$$
$$\le \sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi}\left|\exp[\cos 3\theta]\left\{\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m}{2k}\frac{2k!}{k!}B_k\left(3x^2\cos 2\theta\right)B_{m-2k}(-3x\cos 2\theta)\right\}\right| d\theta, \tag{5.10}$$

provided that the constants $a_{-j} = 0\ \forall j \in \mathbb{N}$ and $a_0 \neq 0$.

Also

$$a_0 \le 0.266. \tag{5.11}$$

Proof. Considering the result (4.4) and multiplying by $\left(\frac{t^2}{e^{t^2}-1}\right)\left(\frac{t}{e^t-1}\right)$ in both of the sides to get

$$\mathcal{H}_n^3(xt)\left(\frac{t^2}{e^{t^2}-1}\right)\left(\frac{t}{e^t-1}\right)=\frac{n!}{\pi}\int_0^\pi \exp[\cos 3\theta]\frac{t\exp(-3xt\cos 2\theta)}{e^t-1}\left\{\frac{t^2\exp\left(3x^2t^2\cos\theta\right)}{e^{t^2}-1}\right\}$$
$$\times\cos\left(\sin 3\theta-3xt\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta. \tag{5.12}$$

Then making an appeal to the formulae (5.1) and (5.2) in the result (5.11) and then obeying same techniques of the Theorem 5.1, we obtain the inequalities (5.9) and (5.10). ☐

6 Conclusion

We may remark in passing that this work is the way to search new areas of research in the field of Special Functions, for example in a class of Hermite polynomials consisting of the generalized Hermite polynomials different to the polynomials found in the research work of the researchers ([3], [4], [5], [7], [8], [9] and others). Therefore, in directions of further researches done so far in various fields of science and engineering, we introduce a multivariable analogue of the generating function (2.4) given by

$$F\left(x_1,\ldots,x_k,t\right)=\exp\left(t^3-3\left(x_1+\cdots+x_k\right)t^2+3\left(x_1+\cdots+x_k\right)^2t\right). \tag{6.1}$$

Setting $|x|^2=\left|x_1+\cdots+x_k\right|^2$, in the generating function (5.1) to satisfying some of the analytic properties are given by

$$\sum_{q=1}^{k}\Delta_q F\left(x_1,\ldots,x_k,t\right)=3kF\left(x_1,\ldots,x_k,t\right),$$

where,

$$\Delta_q\equiv\frac{1}{|x|^2}\left(\frac{\partial}{\partial x_q}+\frac{\partial}{\partial t}\right);q=1,2,3,\ldots,k. \tag{6.2}$$

Further due to the operator (5.2), there exists a formula

$$\prod_{q=1}^{k}\Delta_q F\left(x_1,\ldots,x_k,t\right)=3^kF\left(x_1,\ldots,x_k,t\right). \tag{6.3}$$

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CONSTANT OF MOTION FOR THE KEPLER PROBLEM IN A UNIFORM FIELD CONNECTED WITH WHITNEY NUMBERS AND BERNOULLI POLYNOMIALS

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Abstract

We consider the Kepler problem with an additional constant force, and we employ the Laplace-Runge-Lenz vector to construct a constant of motion. Finally, an interesting relation of this constant motion with Whitney numbers and Bernoulli polynomials is derived for computation of the Kepler problem.

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Keywords and Phrases: Noether's theorem, Invariance of the action, Lanczos variational method, Dynamical symmetry, Kepler's problem, Laplace-Runge-Lenz vector, Whitney numbers and Bernoulli polynomials.

1 Introduction

The Lagrangian for the Kepler problem has the expression:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{k}{r}, \quad r = \sqrt{x^2 + y^2}, \quad \vec{r} = x\hat{\imath} + y\hat{\jmath}, \tag{1.1}$$

because the orbit is plane due the conservation of the angular momentum whose magnitude is given by:

$$l = m(x\dot{y} - y\dot{x}) = mr^2\dot{\varphi}. \tag{1.2}$$

If now due to (1.1), in L we apply the following transformation [17, pages 99-100]:

$$\tilde{x} = x + \varepsilon_1, \quad \tilde{y} = y + \varepsilon_2, \quad \tilde{t} = t, \tag{1.3}$$

where ε_1 and ε_2 are constants $\ll 1$, using (1.2) and (1.3), we obtain:

$$\tilde{L} - L = -\varepsilon_1 \frac{kx}{r^3} - \varepsilon_2 \frac{ky}{r^3} = -\varepsilon_1 \frac{k}{r^2}\cos\varphi - \varepsilon_2 \frac{k}{r^2}\sin\varphi = -\varepsilon_1 \frac{mk}{l}\dot{\varphi}\cos\varphi - \varepsilon_2 \frac{mk}{l}\dot{\varphi}\sin\varphi,$$

that is:

$$\tilde{L} = L + \frac{d}{dt}\left[\frac{mk}{l}\left(-\varepsilon_1 \sin\varphi + \varepsilon_2 \cos\varphi\right)\right], \tag{1.4}$$

hence (1.4) is a dynamical symmetry of L [17], and therefore it is possible to use the Noether's theorem, in the Lanczos approach [8, 9, 10, 11, 22] to deduce the conservation law associated with (1.3) [2].

The Lanczos technique asks to employ in (1.1) the transformation (1.3) with $\varepsilon_1(t)$ and $\varepsilon_2(t)$ as new degrees of freedom, then:

$$\tilde{L} - L = \dot{\varepsilon}_1 m\dot{x} - \varepsilon_1 \frac{kx}{r^3} + \dot{\varepsilon}_2 m\dot{y} - \varepsilon_2 \frac{ky}{r^3}, \tag{1.5}$$

and the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\varepsilon}_j}\right) - \frac{\partial \tilde{L}}{\partial \varepsilon_j} = 0, \quad j = 1, 2, \tag{1.6}$$

imply the relations:

$$\frac{d}{dt}(m\dot{x}) + \frac{kx}{r^3} = \frac{d}{dt}\left(m\dot{x} + \frac{mk}{l}\sin\varphi\right) = 0, \quad \frac{d}{dt}(m\dot{y}) + \frac{ky}{r^3} = \frac{d}{dt}\left(m\dot{y} - \frac{mk}{l}\cos\varphi\right) = 0,$$

(1.5) and (1.6) generate the following two constants of motion:

$$ml\dot{x} + mk\sin\varphi = -c_2, \quad ml\dot{y} - mk\cos\varphi = c_1. \tag{1.7}$$

It is easy see that (1.7) express the conservation of the Laplace-Runge-Lenz vector [2, 6, 7, 21], in fact:

$$\vec{p} \times \vec{L} = m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) \times l\hat{k} = ml(\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}), \quad -mk\frac{\vec{r}}{r} = -mk(\cos\varphi\hat{\imath} + \sin\varphi\hat{\jmath}),$$

therefore:

$$\begin{aligned}\vec{A} := \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} =& (ml\dot{y} - mk\cos\varphi)\hat{\imath} - (ml\dot{x} + mk\sin\varphi)\hat{\jmath} \\ =& \, c_1\hat{\imath} + c_2\hat{\jmath}.\end{aligned} \tag{1.8}$$

The Lanczos method gives a simple process to apply the Noether's theorem [3, 5, 16, 18, 19], thus in this case it was possible to determine the conservation law associated with the dynamical symmetry (1.4) for the Kepler problem.

The Kepler problem with an additional constant force $\vec{F}$ is considered in Sec. 2 to show the existence of the constant of motion:

$$\Omega := \vec{A} \cdot \vec{F} + \frac{m}{2}\left[F^2 r^2 - (\vec{r} \cdot \vec{F})^2\right], \tag{1.9}$$

which coincides with the quantity obtained in [4, 12] via another approach.

2 Kepler problem with a constant force

In this situation the equations of motion are given by:

$$\frac{d\vec{p}}{dt} = -\frac{k}{r^3}\vec{r} + \vec{F}, \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}, \tag{2.1}$$

and we know that $\vec{A}$ is a constant vector if $\vec{F} = \overrightarrow{0}$. If $\vec{F} \neq \overrightarrow{0}$ then it is natural first to investigate the evolution of the Laplace-Runge-Lenz vector [14], thus from (1.8) and (2.1):

$$\frac{d}{dt}\vec{A} = 2(\vec{p} \cdot \vec{F})\vec{r} - (\vec{r} \cdot \vec{F})\vec{p} - (\vec{r} \cdot \vec{p})\vec{F}, \tag{2.2}$$

The (2.2) implies the relation:

$$\frac{d}{dt}(\vec{A} \cdot \vec{F}) = (\vec{r} \cdot \vec{F})(\vec{p} \cdot \vec{F}) - F^2(\vec{p} \cdot \vec{r}), \quad F = |\vec{F}|. \tag{2.3}$$

On the other hand, we have the interesting property:

$$\frac{d}{dt}\frac{m}{2}\left[(\vec{r} \cdot \vec{F})^2 - r^2F^2\right] = (\vec{r} \cdot \vec{F})(\vec{p} \cdot \vec{F}) - mF^2 r\dot{r}, \quad \vec{r} \cdot \vec{p} = mr\dot{r} \tag{2.4}$$

whose application in (2.3) generates the constant (1.9):

$$\Omega = \vec{A} \cdot \vec{F} + \frac{m}{2}\left[F^2 r^2 - (\vec{r} \cdot \vec{F})^2\right] = \vec{A} \cdot \vec{F} + \frac{m}{2}F^2 r^2 \sin^2\beta, \quad \beta = \sphericalangle(\vec{r}, \vec{F}), \tag{2.5}$$

deduced in [16, 17]. If we employ $\vec{F} = F\hat{k}$ and we observe that the energy is given by:

$$H = \frac{p^2}{2m} - \frac{k}{r} - Fz, \tag{2.6}$$

then (2.5) it is equivalent to:

$$\left(2H + \frac{k}{r}\right)z - mr\dot{r}\dot{z} + \frac{F}{2}\left(3z^2 + r^2\right) = \text{ Constant of motion.} \tag{2.7}$$

To find the symmetry associated with the conservation of Ω, is an open problem.

3 Computation of Equation (2.7) due to Whitney numbers and Bernoulli polynomials

If in (2.7), we consider the constant of motion is λ, λ is finite and greater than 0 , and then we write that in the form

$$\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2+r^2\right)\right\}t=\lambda t. \tag{3.1}$$

Clearly, the coefficients of t in (3.1) give the equation (2.7).

Now to compute the equation (2.7) due to Whitney numbers and Bernoulli polynomials, we introduce r-Whitney numbers $w_{m,r}(n,p)$ and $W_{m,r}(n,p)$ of first kind and second kind, respectively, given by [1, 15]

$$m^n(x)_n=\sum_{p=0}^{n}w_{m,r}(n,p)(mx+r)^p;\text{ where, the basis }=\{(mx+r)^p:p=0,\ldots,n\}, \tag{3.2}$$

and

$$(mx+r)^n=\sum_{p=0}^{n}W_{m,r}(n,p)m^p x_p;\text{ where, the basis }=\{m^p x_p:p=0,\ldots,n\}. \tag{3.3}$$

The exponential generating functions of the numbers $w_{m,r}(n,p)$ given in (3.2) and $W_{m,r}(n,p)$ given in (3.3), are represented by [15]

$$\sum_{n\geq p}w_{m,r}(n,p)\frac{t^n}{n!}=\frac{1}{p!}\left(\frac{\ln(1+mt)}{m}\right)^p(1+mt)^{-\frac{r}{m}}, \tag{3.4}$$

and

$$\sum_{n\geq p}W_{m,r}(n,p)\frac{t^n}{n!}=\frac{1}{p!}\left(\frac{\operatorname{ext}[mt]-1}{m}\right)^p\operatorname{ext}[rt], \tag{3.5}$$

respectively.

Further, the generalized Bernoulli polynomials are defined by the exponential generating function [13, 20]

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{t}{\operatorname{ext}[t]-1}\right)^{\alpha}\exp[xt]. \tag{3.6}$$

In the formula (3.6), putting $x=0$ we get a generating formula for generalized Bernoulli numbers of order $\alpha, B_n^{(\alpha)}$, also called Nörlund polynomials, given by

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}\frac{t^n}{n!}=\left(\frac{t}{\operatorname{ext}[t]-1}\right)^{\alpha},\text{ provided that }B_n^{(1)}=B_n. \tag{3.7}$$

The recurrence relation for $B_n^{(\alpha)}$ is given by

$$B_n^{(\alpha+1)}=\left(1-\frac{n}{\alpha}\right)B_n^{(\alpha)}-nB_{n-1}^{(\alpha)}. \tag{3.8}$$

The summation of generalized Bernoulli numbers of order $\alpha, B_n^{(\alpha)}$, is given by

$$B_n^{(\alpha)}=\sum_{q=0}^{n}(-1)^q\frac{q!}{(q+n)!}\binom{\alpha+n}{n-q}\binom{\alpha+q-1}{q}\sum_{j=0}^{q}(-1)^{q-j}\binom{q}{j}j^{q+n}. \tag{3.9}$$

Theorem 3.1. *Let in the Eqn.* (3.1), *for* $\mu\neq 0, \lambda=\left(\frac{r-s}{\mu}\right)$ *such that* $r-s>0$, *and* $p\geq 1$, *then following relation is occurred*

$$\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\frac{r-s}{\mu}-\frac{Fr^2}{2}\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)=\sum_{q=0}^{n}\frac{1}{\mu^n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q). \tag{3.10}$$

Proof. Consider the Eqn. (3.1) and $\lambda=\left(\frac{r-s}{\mu}\right)$ such that $r-s>0$ to get that in the form

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2+r^2\right)\right\}t\right]=\exp\left[\left(\frac{r-s}{\mu}\right)t\right]. \tag{3.11}$$

Then, the Eqn. (3.11) may be written by

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2 \quad +r^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p$$

$$=\left(\frac{1}{\exp[t]-1}\right)^p\exp\left[\left(\frac{r}{\mu}\right)t\right]\left(\frac{t^p}{\exp\left[\left(\frac{s}{\mu}\right)t\right]}\right). \tag{3.12}$$

Further rearranging the equation (3.12) to get that given by

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\quad+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\left(\frac{1}{\left(\frac{\exp[t]-1}{\mu}\right)}\right)^p\exp\left[r\left(\frac{t}{\mu}\right)\right]\left(\frac{1}{\mu^p p!}\frac{\left(\ln\left(1+\mu\left(\frac{\exp[t]-1}{\mu}\right)\right)\right)^p}{\left(1+\mu\left(\frac{\exp[t]-1}{\mu}\right)\right)^{\frac{s}{\mu}}}\right). \tag{3.13}$$

In the formula (3.13), on defining (3.4) we find

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\exp\left[r\left(\frac{t}{\mu}\right)\right]\sum_{q=p}^{\infty}w_{\mu,s}(q,p)\frac{\left(\frac{\exp[t]-1}{\mu}\right)^{q-p}}{q!}. \tag{3.14}$$

Again, in Eqn. (3.14), making some manipulations to find that

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\quad+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,S}(q+p,p)\frac{q!}{(q+p)!}\exp\left[r\left(\frac{t}{\mu}\right)\right]\left(\frac{\exp\left[\mu\frac{t}{\mu}\right]-1}{\mu}\right)^q\frac{1}{q!} \tag{3.15}$$

Now in the equation (3.15) on defining the Whitney number of second kind, we find

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,s}(q+p,p)\frac{q!}{(q+p)!}\sum_{n\geq q}W_{\mu,r}(n,q)\frac{t^n}{n!}. \tag{3.16}$$

Again, in the equation (3.16) on applying the formula (3.6), we get

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\sum_{n=0}^{\infty}B_n^{(p)}\left(\frac{Fr^2}{2}\right)\frac{t^n}{n!}$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,s}(q+p,p)\frac{q!}{(q+p)!}\sum_{n\geq q}W_{\mu,r}(n,q)\frac{\frac{t^n}{\mu^n}}{n!}. \tag{3.17}$$

Then in the equation (3.17), on making series manipulations we find

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{\mu^n}\sum_{q=0}^{n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q)\frac{t^n}{n!}. \tag{3.18}$$

Again then, equating both sides coefficients of $\frac{t^n}{n!}, \forall n\geq 0$ and $p\geq 1$ in equation (3.18), we get

$$\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}(\ \left(3z^2\right)\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)$$

$$=\sum_{q=0}^{n}\frac{1}{\mu^n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q). \tag{3.19}$$

Finally, making an appeal to the equations (3.1) and (3.19) with statement of the Theorem 3.1, we derive the result (3.10). □

Corollary 3.1. *In the Theorem* 3.1 *putting $n=1, p=1$ and $F=0$, there exists a first integral of motion for a dynamical system in the form of Whitney numbers*

$$\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\right\}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]+\frac{1}{2}. \tag{3.20}$$

Proof. Making an appeal to the Theorem 3.1 and putting $n=1, p=1$ and $F=0$, we get

$$B_1^{(1)}(0)+\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\ \frac{F}{2}\left(3z^2\right)\right\}B_0^{(1)}(0)$$
$$=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]. \tag{3.21}$$

In Eqn. (3.7) on setting $\alpha=1$, and again on using (3.8) and (3.9), we get the values of the generalized Bernoulli's numbers as

$$B_0^{(1)}(0)=1, B_1^{(1)}(0)=-\frac{1}{2}, B_1^{(2)}(0)=-1,\ldots,$$

and then putting these values in (3.21) we find

$$-\frac{1}{2}+\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\right\}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]. \tag{3.22}$$

The Eqn. (3.22) gives the dynamical system (3.20) in the form of the Whitney numbers. □

4 Conclusion

The expression

$$\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)+\frac{\mu}{2}\right]$$

represents a first integral of motion for a dynamical system, commonly arising in the study of classical mechanics, such as the motion of a particle in a central force field (e.g., Kepler's problem) or specific coupled oscillator systems. This equation establishes a conserved relationship between the particle's spatial coordinates and axis, its velocities and the constants of the system, such as total energy and mass.

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FIXED POINT THEOREMS FOR $(\alpha-\psi)$-TYPE CONTRACTION MAPPINGS IN PERTURBED METRIC SPACES

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Abstract

This paper presents fixed point theorems for an $\alpha-\psi$ contractive mapping in complete perturbed metric spaces. As applications of our findings, we obtain fixed point theorems for perturbed metric spaces enriched with graphs. Furthermore, examples are provided to illustrate the usability of the obtained results.

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Keywords and Phrases: Fixed-Point, Metric Spaces, Perturbed Metric Spaces, $\alpha-\psi$ Contractive Mapping, Endowed with a graph.

1 Introduction and Preliminaries

In 2025, Jleli and Samet [6] introduced Perturbed metric spaces, which are among the most intriguing of all abstract spaces.
The definitions below introduces the recently introduced concept of perturbed metric spaces:

Definition 1.1 ([6]). *Let $\mathcal{D}, \mathcal{P} : \Omega \times \Omega \to [0, \infty)$ be two given mappings. Then $\mathcal{D}$ constitutes a perturbed metric on Ω with respect to $\mathcal{P}$, if $\mathcal{D} - \mathcal{P} : \Omega \times \Omega \to [0, \infty)$ defined as $(\mathcal{D} - \mathcal{P})(\zeta, \eta) = \mathcal{D}(\zeta, \eta) - \mathcal{P}(\zeta, \eta)$ forms a metric on Ω, that is, for all $\zeta, \eta, \delta \in \Omega$,*
(i) $(\mathcal{D} - \mathcal{P})(\zeta, \eta) \geq 0$,
(ii) $(\mathcal{D} - \mathcal{P})(\zeta, \eta) = 0 \Leftrightarrow \zeta = \eta$,
(iii) $(\mathcal{D} - \mathcal{P})(\zeta, \eta) = (\mathcal{D} - \mathcal{P})(\zeta, \eta)$,
(iv) $(\mathcal{D} - \mathcal{P})(\zeta, \eta) \leq (\mathcal{D} - \mathcal{P})(\zeta, \delta) + (\mathcal{D} - \mathcal{P})(\delta, \eta)$. The mapping $\mathcal{P}$ is called a perturbed mapping, with $\Gamma = \mathcal{D} - \mathcal{P}$ forming a standard metric, and $(\Omega, \mathcal{D}, \mathcal{P})$ is called a perturbed metric space. We use (Ω, Γ) to represent the standard metric space.

Note that a perturbed metric on Ω is not necessarily a metric on Ω. For some examples and further preliminaries of perturbed metric space one can see Jleli and Samet[6]. We now present several topological notions in perturbed metric spaces :

Definition 1.2 ([6]). *Let the triple $(\Omega, \mathcal{D}, \mathcal{P})$ form a perturbed metric space. Consider a sequence $\{p_n\}$ in Ω, and a self-mapping Λ defined on Ω.*
(i) If $\{p_n\}$ is a convergent sequence in an standard metric space (Ω, Γ), where the metric Γ is defined as $\Gamma = \mathcal{D} - \mathcal{P}$, then $\{p_n\}$ is a convergent sequence in the "perturbed sense in $(\Omega, \mathcal{D}, \mathcal{P})$.
(ii) If a sequence $\{p_n\}$ is a Cauchy in the context of a standard metric space (Ω, Γ), then $\{p_n\}$ is a perturbed Cauchy sequence in $(\Omega, \mathcal{D}, \mathcal{P})$.

(iii) ($\Omega, \mathcal{D}, \mathcal{P}$) is named a complete perturbed metric space, , when (Ω, Γ) is a standard complete metric space, that is, each perturbed Cauchy sequence in ($\Omega, \mathcal{D}, \mathcal{P}$) converges in the "perturbed sense" in ($\Omega, \mathcal{D}, \mathcal{P}$).
(iv) A mapping Λ is a perturbed continuous whenever the mapping Λ is continuous in the setting of the corresponding standard metric space (Ω, Γ)

Samet *et al.* [16] introduced $\alpha-\psi$ contraction maps to study the existence and uniqueness of fixed points. The fundamental finding of this interesting study led to the conclusion of several well-known fixed point theorems, including the Banach mapping principle. A number of authors have investigated and improved the strategies described in this study; see, for example, $[1-4, 8-17]$ and the references therein.
We will now review some essential concepts, notations, and basic results that will be used throughout this study.

The family of non-decreasing continuous functions Ψ is represented by $\psi : [0, +\infty) \to [0, +\infty)$ such that for any t $> 0, \sum_{n=1}^{\infty} \psi^n < +\infty$, where ψ^n is the n^{th} iterate of ψ.

Definition 1.3 ([16]). *Let Λ be a self-map on Ω and (Ω, Γ) be a metric space. If two functions $\alpha : \Omega \times \Omega \to [0, \infty)$ and $\psi \in \Psi$ exist such that $\alpha(\zeta, \eta)\Gamma(\Lambda\zeta, \Lambda\eta) \leq \psi(\Gamma(\zeta, \eta))$, for all $\zeta, \eta \in \Omega$, then Λ is said to be $\alpha - \psi$ contractive mapping*

Definition 1.4. *Consider a perturbed metric space ($\Omega, \mathcal{D}, \mathcal{P}$) with Λ as a self-map on Ω. Then Λ is said to be perturbed $\alpha - \psi$ contractive mapping if two functions $\alpha : \Omega \times \Omega \to [0, \infty)$ and $\psi \in \Psi$ exist such that, for any $\zeta, \eta \in \Omega, \alpha(\zeta, \eta)\mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta))$.*

Lemma 1.1 ([14] *also see* [16]). *. For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following holds:*
(i) If ψ is non decreasing, then for each $t > 0$
(ii) $\lim_{t\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Remark 1.1. *Taking $\alpha(\zeta, \eta) = 1$ for any $\zeta, \eta \in \Omega$ and $\psi(t) = kt$ for some $k \in [0, 1], \Lambda$ is a perturbed $\alpha - \psi$ contractive mapping if $\Lambda : \Omega \to \Omega$ fulfils the Banach contraction principle in the context of perturbed metric space.*

Definition 1.5 ([16]). *Let $\alpha : \Omega \times \Omega \to [0, \infty)$ and $\Lambda : \Omega \to \Omega$. If $\zeta, \eta \in \Omega$, then $(\zeta, \eta) \geq 1$ implies $\alpha(\Lambda\zeta, \Lambda\eta) \geq 1$, and we say that Λ is α-admissible.*

Example 1.1. Let $\Omega = [0, \infty)$. Define $\Lambda : \Omega \to \Omega$ be defined by $\Lambda\zeta = \frac{\zeta}{4}$. Also, define $\alpha : \Omega \times \Omega \to [0, +\infty)$ by $\alpha(\zeta, \eta) = 3$. Then Λ is α-admissible mapping.

Example 1.2. Let $\Omega = [0, \infty)$. Define $\Lambda : \Omega \to \Omega$ be defined by

$$\Lambda\zeta = \begin{cases} \frac{\zeta^2}{4} \text{ if } \zeta \in [0, 1] \\ 2\zeta^3 + 1 \text{ if } \zeta \in (1, \infty) \end{cases}.$$

Also, define $\alpha : \Omega \times \Omega \to [0, +\infty)$ by $\alpha(\zeta, \eta) = \begin{cases} 1 & \text{if } \zeta, \eta \in [0, 1] \\ 4 & \text{if otherwise .} \end{cases}$
Then Λ is α-admissible mapping.

2 Main results

Now, we generalize the result established by Samet and Vetro[16] in the context of perturbed metric space as follows:

Theorem 2.1. *Let ($\Omega, \mathcal{D}, \mathcal{P}$) be a complete perturbed metric space. $\Lambda : \Omega \to \Omega$ must meet the following requirements.*
(C_1) *Λ is $\alpha - \psi$ contractive mapping,*
(C_2) *Λ is α-admissible mapping,*
(C_3) *there exists, $\zeta_0 \in \Omega$ such that $\alpha(\zeta_0, \Lambda\zeta_0) \geq 1$ and $\alpha(\zeta_0, \Lambda^2\zeta_0) \geq 1$,*
(C_4) *Λ is continuous,*
(C_5) *the value of the perturbed function $\mathcal{P}(\zeta, \eta) = 0$ when $\zeta = \eta$.*
Then Λ possesses a fixed point.

Proof. According to (C_3), there is a point ζ_0 in Ω where $\alpha\left(\zeta_0, \Lambda\zeta_0\right) \geq 1$ and $\alpha\left(\zeta_0, \Lambda^2\zeta_0\right) \geq 1$.
We develop a sequence $\{\zeta_n\}$ in Ω by $\zeta_n = \Lambda^n\zeta_0 = \Lambda\zeta_{n-1}$, for all $n \in \mathbb{N}$.
It is obvious that if $\zeta_{n_0} = \zeta_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then ζ_{n_0} is a fixed point of Λ.
Assume that $\zeta_n \neq \zeta_{n+1}$ for all n $\in \mathbb{N} \cup \{0\}$.
Since Λ is an α-admissible mapping, we have
$\alpha\left(\zeta_0, \zeta_1\right) = \alpha\left(\zeta_0, \Lambda\zeta_0\right) \geq 1$ implies that $\alpha\left(\zeta_1, \zeta_2\right) = \alpha\left(\Lambda\zeta_0, \Lambda\zeta_1\right) \geq 1$,
implies that $\alpha\left(\zeta_2, \zeta_3\right) = \alpha\left(\Lambda\zeta_1, \Lambda\zeta_2\right) \geq 1$,
implies that $\alpha\left(\zeta_3, \zeta_4\right) = \alpha\left(\Lambda\zeta_2, \Lambda\zeta_3\right) \geq 1$.
Keeping going in this manner, we have

$$\alpha(\zeta_n, \zeta_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{2.1}$$

In the same way, we have

$$\alpha\left(\zeta_n, \zeta_{n+2}\right) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\} \tag{2.2}$$

When we look at (C_1) and (2.1), we obtain

$$\begin{aligned}
\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right) = \mathcal{D}\left(\Lambda\zeta_n, \Lambda\zeta_{n-1}\right) & \\
&\leq \alpha\left(\zeta_n, \zeta_{n-1}\right) \mathcal{D}\left(\Lambda\zeta_n, \Lambda\zeta_{n-1}\right) \\
&\leq \psi\left(\mathcal{D}\left(\zeta_n, \zeta_{n-1}\right)\right), \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

By doing the previous procedure again, we get that

$$\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right) \leq \psi^n\left(\Gamma\left(\zeta_1, \zeta_0\right)\right) \text{ for all } n \in \mathbb{N} \tag{2.3}$$

The exact metric is $\Gamma = \mathcal{D} - \mathcal{P}$ by definition. Considering (2.3), we conclude that

$$\Gamma\left(\zeta_{n+1}, \zeta_n\right) \leq \Gamma\left(\zeta_{n+1}, \zeta_n\right) + \mathcal{P}\left(\zeta_{n+1}, \zeta_n\right) \leq \psi^n\left(\Gamma\left(\zeta_1, \zeta_0\right)\right). \tag{2.4}$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \Gamma\left(\zeta_{n+1}, \zeta_n\right) = 0$.
Examining (C_1) and (2.2), we obtain

$$\begin{aligned}
\mathcal{D}\left(\zeta_{n+2}, \zeta_n\right) =& \mathcal{D}\left(\Lambda\zeta_{n+1}, \Lambda\zeta_{n-1}\right) \\
&\leq \alpha\left(\zeta_{n+1}, \zeta_{n-1}\right) \mathcal{D}\left(\Lambda\zeta_{n+1}, \Lambda\zeta_{n-1}\right) \\
&\leq \psi\left(\mathcal{D}\left(\zeta_{n+1}, \zeta_{n-1}\right)\right), \text{ for all } n \in \mathbb{N}
\end{aligned}$$

Continuing the above process, we have

$$\mathcal{D}\left(\zeta_{n+2}, \zeta_n\right) \leq \psi^n\left(\mathcal{D}\left(\zeta_2, \zeta_0\right)\right) \text{ for all } n \in \mathbb{N} \tag{2.5}$$

The exact metric is $\Gamma = \mathcal{D} - \mathcal{P}$ by definition. Considering (2.2), we conclude that

$$\Gamma\left(\zeta_{n+2}, \zeta_n\right) \leq \Gamma\left(\zeta_{n+2}, \zeta_n\right) + \mathcal{P}\left(\zeta_{n+2}, \zeta_n\right) \leq \psi^n\left(\Gamma\left(\zeta_2, \zeta_0\right)\right). \tag{2.6}$$

Taking $n \to \infty$, we have $\lim_{n\to\infty} \Gamma\left(\zeta_{n+2}, \zeta_n\right) = 0$.
Let $\zeta_n = \zeta_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$.
Without loss of generality, we will assume that $m > n$.
Since $\alpha\left(\zeta_m, \zeta_{m-1}\right) \geq 1$, we get

$$\begin{aligned}
\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right) &= \mathcal{D}\left(\Lambda\zeta_n, \zeta_n\right) \\
&= \mathcal{D}\left(\Lambda\zeta_m, \zeta_m\right) \\
&= \mathcal{D}\left(\Lambda\zeta_m, \Lambda\zeta_{m-1}\right) \\
&\leq \psi\left(\mathcal{D}\left(\zeta_m, \zeta_{m-1}\right)\right) \\
&\leq \psi^{m-n}\left(\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right)\right),
\end{aligned}$$

for all $m, n \in \mathbb{N}$.
By Lemma 1.1, we have

$$\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right) \leq \psi^{m-n}\left(\mathcal{D}\left(\zeta_{n+1}, \zeta_n\right)\right) < \mathcal{D}\left(\zeta_{n+1}, \zeta_n\right),$$

which is a contradiction.
Thus, $\zeta_n \neq \zeta_m$ for all $m, n \in \mathbb{N}$ with $m \neq n$.
We now demonstrate that $\{\zeta_n\}$ is a Cauchy sequence, meaning that for any $k \in \mathbb{N}$, $\lim_{n\to\infty} \Gamma(\zeta_n, \zeta_{n+k}) = 0$. The situations for $k=1$ and $k=2$ have already been established in (2.4) and (2.6), respectively. Let k be arbitrarily greater than or equal to 3 .

We discuss two cases.

Case 1. Suppose that $k = 2m+1$, where $m \geq 1$. Using the triangle inequality, we have

$$\begin{aligned}
\Gamma(\zeta_n, \zeta_{n+k}) &= \Gamma(\zeta_n, \zeta_{n+2m+1}) \\
&\leq \Gamma(\zeta_n, \zeta_{n+1}) + \Gamma(\zeta_{n+1}, \zeta_{n+2}) + \cdots + \Gamma(\zeta_{n+2m}, \zeta_{n+2m+1}) \\
&\leq \sum_{p=n}^{n+2m} \psi^p(\mathcal{D}(\zeta_0, \zeta_1)) \\
&\leq \sum_{p=n}^{\infty} \psi^p(\mathcal{D}(\zeta_0, \zeta_1)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Case 2. Suppose that $k = 2m$, where $m \geq 2$. Using the triangle inequality, we have

$$\begin{aligned}
\Gamma(\zeta_n, \zeta_{n+k}) &= \Gamma(\zeta_n, \zeta_{n+2m}) \\
&\leq \Gamma(\zeta_n, \zeta_{n+2}) + \Gamma(\zeta_{n+2}, \zeta_{n+3}) + \cdots + \Gamma(\zeta_{n+2m-1}, \zeta_{n+2m}) \\
&\leq \Gamma(\zeta_n, \zeta_{n+2}) + \sum_{p=n}^{n+2m-1} \psi^p(\mathcal{D}(\zeta_0, \zeta_1)) \\
&\leq \Gamma(\zeta_n, \zeta_{n+2}) + \sum_{p=n}^{\infty} \psi^p(\mathcal{D}(\zeta_0, \zeta_1)) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

By (2.6), we have $\lim_{n\to\infty} \Gamma(\zeta_{n+2}, \zeta_n) = 0$, therefore $\lim_{n\to\infty} \Gamma(\zeta_n, \zeta_{n+k}) = 1$ for all $k \geq 3$. The above inequality shows that the sequence $\{\zeta_n\}$ is Cauchy in the standard metric spaces (Ω, Γ). The constructed sequence $\{\zeta_n\}$ is Cauchy in perturbed metric spaces $(\Omega, \mathcal{D}, \mathcal{P})$. Therefore, we conclude that $\{\zeta_n\}$ is a Cauchy sequence in $(\Omega, \mathcal{D}, \mathcal{P})$. As $(\Omega, \mathcal{D}, \mathcal{P})$ is complete, there exists $\zeta^* \in \Omega$ such that

$$\lim_{n\to\infty} \Gamma(\zeta_n, \zeta^*) = 0.$$

Finally, we show that the limit ζ^* of the sequence $\{\zeta_n\}$ is a fixed point of Λ.
Since Λ is continuous. Then we have

$$\lim_{n\to\infty} \Gamma(\Lambda\zeta_n, \Lambda\zeta^*) = \lim_{n\to\infty} \Gamma(\zeta_{n+1}, \Lambda\zeta^*) = 0.$$

By uniqueness of limit, ζ^* is a fixed point of Λ.
This completes the proof. □

Theorem 2.2. *Given a complete perturbed metric space $(\Omega, \mathcal{D}, \mathcal{P})$ and $\Lambda : \Omega \to \Omega$ satisfying the following requirements: $(C_1), (C_2), (C_3), (C_5)$*
(C_6) if $\{\zeta_n\}$ is sequence in Ω such that $\zeta_n \to \zeta$ as $n \to \infty$ for some $\zeta \in \Omega$ and $\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\zeta_n, \zeta) \geq 1$ for all $n \in \mathbb{N}$. Then there is a fixed point for Λ .

Proof. Based on the proof of Theorem 2.1, we know that the sequence $\{\zeta_n\}$ given by $\zeta_{n+1} = \Lambda\zeta_n$ for all $n \in \mathbb{N} \cup \{0\}$ converges for some $u \in \Omega$. For any $n \in \mathbb{N}$, we obtain $\alpha(\zeta_n, u) \geq 1$ from (2.1) and (C_6). By using (3.1), we get

$$\begin{aligned}
\mathcal{D}(\zeta_{n+1}, \Lambda u) &\leq \alpha(\zeta_n, \mu) \mathcal{D}(\Lambda\zeta_n, \Lambda u) \\
&\leq \psi(\mathcal{D}(\zeta_n, u)) \text{ for all } n \in \mathbb{N}
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (C_5), we obtain that

$$\lim_{k\to\infty} \mathcal{D}(\zeta_{n+1}, \Lambda u) = 0.$$

By definition, $\Gamma = \mathcal{D} - \mathcal{P}$ is the exact metric. In view of (2.2), we deduce that $\lim_{k\to\infty} \Gamma(\zeta_{n+1}, \Lambda u) \leq \lim_{k\to\infty} \Gamma(\zeta_{n+1}, \Lambda u) + \lim_{k\to\infty} \mathcal{P}(\zeta_{n+1}, \Lambda u)$.
We get
$\lim_{k\to\infty} \Gamma(\zeta_{n+1}, \Lambda u) = 0$. i.e., $\Lambda u = u$.
Hence u is a fixed point of Λ.
To assure the uniqueness of fixed points, we consider the following hypothesis:
(P) If for all $\zeta, \eta \in F(\Lambda)$, where $F(\Lambda)$ is a set of fixed points of Λ, we have $\alpha(\zeta, \eta) \geq 1$. □

Theorem 2.3. *The uniqueness of fixed points of* Λ *is obtained by inserting condition (* **P** *) to the hypothesis of Theorem* 2.1 *(resp. Theorem* 2.2*).*

Proof. Suppose that ζ^* and η^* $(\zeta^* \neq \eta^*)$ are two fixed points of Λ. Then by the hypothesis $(H), \alpha(\zeta^*, \eta^*) \geq 1$. Hence, from (C_1) with $\zeta = \zeta^*$ and $\eta = \eta^*$, we have

$$\begin{aligned} \mathcal{D}(\zeta^*, \eta^*) &= \mathcal{D}(\Lambda\zeta^*, \Lambda\eta^*) \\ &\leq \alpha(\zeta^*, \eta^*) \cdot \mathcal{D}(\Lambda\zeta^*, \Lambda\eta^*) \\ &\leq \psi(\mathcal{D}(\zeta^*, \eta^*)) < \mathcal{D}(\zeta^*, \eta^*) \end{aligned}$$

which is impossible, that is, $\zeta^* = \eta^*$.
Hence Λ has a unique fixed point.
This completes the proof. □

Example 2.1. For an interval $\Omega = [0, \infty)$, we shall define $\mathcal{D} : \Omega \times \Omega \to [0, \infty)$ be the mapping defined by

$$\mathcal{D}(u, t) = (u + t)^2 + |u - t|, \quad \text{for all } u, t \in \Omega.$$

Then $\mathcal{D}$ is a perturbed metric on Ω , where the perturbed mapping $\mathcal{P}$ is given by

$$\mathcal{P}(u, t) = (u + t)^2$$

$u, t \in \Omega$, and the exact metric Γ is given by

$$\Gamma(u, t) = |u - t|, u, t \in \Omega$$

Clearly, ($\Omega, \mathcal{D}, \mathcal{P}$) is a complete perturbed metric space.
Define the mapping Λ: $\Omega \to \Omega$ by

$$\Lambda\zeta = \begin{cases} \frac{\zeta}{4}, & \zeta \in [0, 1] \\ 2\zeta - \frac{7}{4}, & \zeta \in (1, \infty). \end{cases}$$

We define the mapping $\alpha : \Omega \times \Omega \to [0, \infty)$ by

$$\alpha(\zeta, \eta) = \begin{cases} e^{|\zeta - \eta|}, & \text{if } \zeta, \eta \in \left(0, \frac{1}{4}\right] \\ e^{-|\zeta - \eta|}, & \text{otherwise.} \end{cases}$$

Clearly, Λ is a triangular α-admissible and $\alpha - \psi$ contractive mapping with $\psi(\mathrm{t}) = \frac{\mathrm{t}}{4}$ for all $\mathrm{t} \in [0, \infty)$.
For all $\zeta, \eta \in \Omega$, we have

$$\alpha(\zeta, \eta)\mathcal{D}(\Lambda\zeta, \Lambda\eta) \leqslant \psi(\mathcal{D}(\zeta, \eta)).$$

Moreover, there exists $\zeta_0 = \frac{1}{4} \in \Omega$ such that

$$\alpha(\zeta_0, \Lambda\zeta_0) = \alpha\left(\frac{1}{4}, \frac{1}{16}\right) = e^{\frac{3}{16}} \geqslant 1.$$

Obviously, Λ is continuous, Theorem 2.1's hypotheses are now completely satisfied; Λ has two fixed points, 0 and $\frac{7}{4}$

3 Application

The existence of fixed point theorems on a perturbed metric space with a graph is provided in this section. Some fixed point findings were established on a metric space with a graph by Jachymski [7] in 2008. Before presenting our results, we give some definitions and notions in setting of perturbed metric spaces which are useful for our results as follow:
Consider the perturbed metric space $(\Omega, \mathcal{D}, \mathcal{P})$. A set $\{(\zeta, \zeta) : \zeta \in \Omega\}$ is represented as Δ and is referred to as a diagonal of the Cartesian product $\Omega \times \Omega$. Suppose that a graph G has all loops (i.e., $\Delta \subseteq E(G)$) in its edges and that the set $V(G)$ of its vertices matches with Ω. The pair ($V(G), E(G)$) can be used to identify G since we believe it has no parallel edges. Furthermore, we can treat G as a weighted graph by allocating the distance between its vertices to each edge.

Definition 3.1. *Let Ω have a graph G and be a nonempty set. The following condition must be satisfied for the mapping $\Lambda : \Omega \to \Omega$ to be referred to be preserve edge:*

$$\zeta, \eta \in \Omega \text{ with } (\zeta, \eta) \in E(G) \Rightarrow (\Lambda\zeta, \Lambda\eta) \in E(G).$$

Definition 3.2. *Let $(\Omega, \mathcal{D}, \mathcal{P})$ be a perturbed metric space endowed with a graph G. The mapping $\Lambda : \Omega \to \Omega$ is called a ψ contractive mapping with respect to $E(G)$ if there exists a function $\psi \in \Psi$ such that the following condition hold:*

$$\zeta, \eta \in \Omega \text{ with } (\zeta, \eta) \in E(G) \Rightarrow \mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta)).$$

Theorem 3.1. *Consider $(\Omega, \mathcal{D}, \mathcal{P})$ be a perturbed metric space endowed with a graph G and $\Lambda : \Omega \to \Omega$ be a self-mapping satisfying (C_4) and the following conditions:*
(C_7) Λ be a ψ contractive mapping with respect to $E(G)$,
(C_8) Λ is preserve edge,
(C_9) there exists $\zeta_0 \in \Omega$ such that $(\zeta_0, \Lambda\zeta_0) \in E(G)$ and $\left(\zeta_0, \Lambda^2\zeta_0\right) \in E(G)$.
Then Λ has a fixed point.

Proof. Consider a mapping $\alpha : \Omega \times \Omega \to [0, \infty)$ defined by

$$\alpha(\zeta, \eta) = \begin{cases} 1, & \zeta, \eta \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

From (C_9), we have $\alpha\left(\zeta_0, \Lambda\zeta_0\right) = 1$ and $\alpha\left(\zeta_0, \Lambda^2\zeta_0\right) = 1$.
It follows from Λ is preserve edge that Λ is an α-admissible mapping.
From (C_7), we have, for all $\zeta, \eta \in \Omega$,

$$(\zeta, \eta) \in E(G) \Rightarrow \mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta)),$$

and thus

$$\alpha(\zeta, \eta)\mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta)).$$

Therefore, Λ is a $\alpha - \psi$ contractive mapping.
Now all the hypotheses of Theorem 2.1 are satisfied.
Therefore, Λ has a fixed point. □

Theorem 3.2. *Consider ($\Omega, \mathcal{D}, \mathcal{P}$) be a complete perturbed metric space and $\Lambda : \Omega \to \Omega$ be a self mapping satisfying $(C_7), (C_8), (C_9)$ and the following conditions:*
(C_{10}) if $\{\zeta_n\}$ is sequence in Ω such that $\zeta_n \to u$ as $n \to \infty$ for some $\zeta \in \Omega$ and $(\zeta_n, \zeta_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(\zeta_n, u) \in E(G)$ for all $n \in \mathbb{N}$.
Then Λ has a fixed point.

Proof. Consider a mapping $\alpha : \Omega \times \Omega \to [0, \infty)$ defined by

$$\alpha(\zeta, \eta) = \begin{cases} 1, & \zeta, \eta \in E(G) \\ 0, & \text{otherwise} \end{cases}$$

From (C_9), we have $\alpha\left(\zeta_0, \Lambda\zeta_0\right) = 1$ and $\alpha\left(\zeta_0, \Lambda^2\zeta_0\right) = 1$.
It follows from Λ is preserve edge that Λ is an α-admissible mapping.
From (C_7), we have, for all $\zeta, \eta \in \Omega$,

$$(\zeta, \eta) \in E(G) \text{ implies that } \mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta)),$$

and thus $\alpha(\zeta, \eta)\mathcal{D}(\Lambda\zeta, \Lambda\eta) \leq \psi(\mathcal{D}(\zeta, \eta))$.
Therefore, Λ is a $\alpha - \psi$ contractive mapping.
From condition (C_{10}), we have $(\zeta_n, \zeta_{n+1}) \in E(G)$ implies that $(\zeta_n, u) \in E(G)\alpha(\zeta_n, \zeta_{n+1}) \geq 1$ implies that $\alpha(\zeta_n, u) \geq 1$ for all $n \in \mathbb{N}$.
Then by Theorem 2.2, u is a fixed point of Λ. □

4 Conculsion

In the context of perturbed metric spaces, we define the $\alpha - \psi$ contraction mapping and the $\alpha - \psi$ admissible mapping . In the complete perturbed metric spaces, which generalized the metric spaces, we establish the Banach fixed point theorem. Additionally, an application scenario and an illustrative example are shown. It is important to note that these findings generalize the findings described in [16].

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INVERSION OF AN INTEGRAL INVOLVING A PRODUCT OF GENERALIZED HYPERGEOMETRIC FUNCTION AND ALEPH-FUNCTION AS KERNEL

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Abstract

The solution for a convolution integral equation of Fredholm type, whose kernel involves a product of generalized hypergeometric function and κ-Function has been determined by Pathan,Kumar and Chandel (2016), Chandel-Kumar (2020). In the present paper, our aim is to derive an exact solution of the convolution integral equation of Fredholm type due to Chaurasia-Saxena (2008) and Goyal-Salim (1998). A number of results due to Chandel-Kumar (2023) and Srivastave (1985) follow as special cases by specializing and the parameters of the generalized hypergeometric function and the Aleph-Function ($\aleph$-function).

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1 Introduction

The Aleph function, introduced by Südland *et al.* [17] (also see Srivastava [15] and Sachan *et al.* [16]), is defined in terms of Mellin Barnes type integrals [10]:

$$\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left\{x \left| \begin{array}{l} (a_j,\alpha_j)_{1,n}\,; [\tau_i\,(a_{ji},\alpha_{ji})]_{n+1}\,, p_i; r \\ (b_j,\beta_j)_{1,m}\,; [\tau_i\,(b_{ji},\beta_{ji})]_{m+1}\,, q_i; r \end{array} \right.\right\}$$
$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) x^s ds, \tag{1.1}$$

where $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^m \Gamma(b_j-\beta_j s)\prod_{j=1}^n \Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^r \tau_i \sum_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\sum_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)}.$$

For the nature of contour L in (1.1), the conditions of existence, the convergence and other details of this function, see [14].

In this paper we will also use the following notation [6,9]:

$$F[\mu,x] = {}_PF_Q\left[\begin{array}{l} A_P \\ B_Q \end{array} : \mu x\right] = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^p (A_j)_r\,\mu^r x^r}{\prod_{j=1}^Q (B_j)_r\,(r)!}. \tag{1.2}$$

The main object of this paper is to evaluate an exact solution of the following convolution integral equation of Fredholm type [3,5]:

$$\int_0^\infty y^{-1} u\left(\frac{x}{y}\right)\phi(y)dy = g(x), \quad (x>0), \tag{1.3}$$

where g is a prescribed function, ϕ is an unknown function to be determined and the kernel $u(x)$ is given by

$$u(x) = F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x], \tag{1.4}$$

where $\aleph[*]$ and $F[*]$ are given by equations (1.1) and (1.2) respectively.
Our method of solution of integral eq. (1.3) with kernel $u(x)$ given by (1.4) would depend on the theory of Melline transform defined by

$$\Phi(s) = M\{\phi(x); s\} = \int_0^\infty x^{s-1}\phi(x)dx, \tag{1.5}$$

provided that the integral exists.

2 Mellin Transform of $u(x)$.

In order to solve the integral equation, we shall require the following result contained in
Lemma 2.1: Let $U(s) = \mathcal{M}\{u(x); s\}$ where $u(x)$ is defined by (1.4), then

$$U(s) = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{P}(A_j)_r\,\mu^r}{\prod_{j=1}^{Q}(B_j)_r\,(r)!} \times \frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j(r+s)\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j(r+s)\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}(r+s)\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}(r+s)\right)}, \tag{2.1}$$

provided that

$$-\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right) < \mathrm{Re}(r+s) < \min_{1\le j\le n}\mathrm{Re}\left(\frac{1-a_j}{\alpha_j}\right); r = 0, 1, 2, \ldots.$$

Proof.

$$U(S) = \mathcal{M}\left\{F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]; s\right\} = \int_0^\infty x^{s-1}F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]dx. \tag{2.2}$$

Expending the generalized hypergeometric function as given by (1.2) changing order of summation and integration, we get

$$U(S) = \sum_{r=0}^{\infty} \frac{\prod_{j=1}^{P}(A_j)_r\,\mu^r}{\prod_{j=1}^{Q}(B_j)_r\,(r)!}\mathcal{M}\left\{x^r\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]; s\right\}. \tag{2.3}$$

Now applying the following known formulae [4, p. 307] and [14, p. 15]

$$\mathcal{M}\left\{x^\rho\phi(x); s\right\} = \Phi(s+\rho) \tag{2.4}$$

$$\mathcal{M}\left\{\aleph^{m,n}_{p_i,q_i,\tau_i;r}[ax]\right\} = \frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-\alpha_j s\right)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left(1-b_{ji}-\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}+\alpha_{ji}s\right)}a^{-s} \tag{2.5}$$

provided that

$$-\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right) < \mathrm{Re}(s) < \min_{1\le j\le n}\mathrm{Re}\left(\frac{1-a_j}{\alpha_j}\right),$$

we arrive at the desired result. □

3 Solution of the Integral Equation

The solution of the convolution integral equation of Fredholm type (1.3) is given in the

Theorem 3.1. *Let the Mellin transforms $\Phi(s), G(s)$ and $U(s) \neq 0$ of the functions $\Phi(s), G(s)$ and $U(S) \neq 0$ of the functions $\phi(x), g(x)$ and $u(x)$ defined by* (1.4) *exist and are analytic in some infinite strip $\gamma < \mathrm{Re}(s)\delta$ of the complex s-plane. Also assume that for a fixed $c \in (\gamma, \delta)$, $u^*(x)$ be defined by*

$$u^*(x) = \mathcal{M}^{-1}\left\{U^*(s); x\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}U^*(s)ds \tag{3.1}$$

where,

$$U^*(s) = \left[\rho^k\frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_r\,\mu^r}{\prod_{j=1}^{Q}(B_j)_r\,(r)!}\right.$$

$$\times \frac{\prod_{j=1}^{m}\Gamma\left\{b_j+\beta_j(r+s+\rho k+\sigma)\right\}\prod_{j=1}^{n}\Gamma\left\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\right\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left\{1-b_{ji}-\beta_{ji}(r+s+\rho k+\sigma)\right\}\prod_{j=n+1}^{p_i}\Gamma\left\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\right\}}\Bigg]^{-1}, \tag{3.2}$$

provided that the following sets of conditions hold

(i) $|\arg x| < \frac{1}{2}\Omega\pi$, *where* $\Omega = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$,

(ii) $\rho \neq 0$ *and* k *is a non-negative integer,*

(iii) $-\min_{1\leq j\leq m}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right) < \operatorname{Re}(r+s+\rho k+\sigma) < \min_{1\leq j\leq n}\operatorname{Re}\left(\frac{1-a_j}{\alpha_j}\right)$,

(iv) $P \leq Q[P = Q+1$ *and* $|\mu| < 1]$, *No one of* $B_1, B_2, \ldots, B_Q$ *is zero or a negative integer.*

Then the integral equation (1.3) *has its solution given by*

$$\phi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_y\right)^k\left\{y^\sigma g(y)\right\}dy. \tag{3.3}$$

provided that the integral exists.

Proof. Applying the convolution theorem for Mellin transfoems [4, p.307], we find from (1.3), that

$$U(S)\Phi(S) = G(s), \tag{3.4}$$

where $\Phi(s)$ and $G(S)$ are Mellin transforms of $\phi(x)$ and $g(x)$ and $U(s)$ is given in (2.1).

Replacing s by $s+\rho k+\sigma$ in (3.4), we get

$$F(s+\rho k+\sigma) = U^*(s)\left\{\rho^k\left(-\frac{s+\rho k}{\rho}\right)_k\right\}G(s+\rho k+\sigma). \tag{3.5}$$

Now making an appeal to the formula given by Shrivastava [13, p. 14]

$$\mathcal{M}\left\{\left(x^{h+1}D_x\right)^n f(x);s\right\} = h^n\left(-\frac{s+hn}{h}\right)_n F(s+hn)$$

$$(h \neq 0, n \quad \text{a non-negative integer }),$$

we obtain

$$F(s+\rho k+\sigma) = U^*(s)\mathcal{M}\left[\left(y^{\rho+1}D_x\right)^k\left\{y^\sigma g(y)\right\};s\right]. \tag{3.6}$$

Further applying (2.4), we derive

$$\mathcal{M}\left\{x^{\rho k+\sigma}\phi(x);s\right\} = \mathcal{M}\left\{\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_x\right)^k\left\{y^\rho g(y)\right\}dy;s\right\}. \tag{3.7}$$

Inverting both sides of (3.8) by using Mellin inversion theorem [4, p.307], we establish the desired theorem. □

4 Deductions

(i) If we take $P = 0, Q = 0, A_1 = -N, A_2 = a+N-1, \mu = -1/b$ in equation (1.4) and reduce generalized hypergeometric function to Bessel polynomial, $Y_N(x,a,b)$ [2], we get

Corollary 4.1. *Under the hypothesis of Theorem, the integral equation*

$$\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\phi(y)dy = g(x); \quad (x>0) \tag{4.1}$$

where the kernel

$$u_1(x) = Y_N(x,a,b)\aleph_{p_i,q_i,\tau_i;r}^{m,n}[x] \tag{4.2}$$

has its solution $\phi(x)$ *given by*

$$\emptyset(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\left(y^{\rho+1}Dy\right)^k\left\{y^\sigma g(y)\right\}dy, \tag{4.3}$$

provided that the integral exists and $u_1^*(x)$ *is the Melline inverse transform of*

$$U^*(s) = \left[\rho^k\frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{N}\frac{(-N,r)(a+N-1,r)(-1/b)^r}{(r)!}\right.$$

$$\times \frac{\prod_{j=1}^{m}\Gamma\left\{b_j+\beta_j(r+s+\rho k+\sigma)\right\}\prod_{j=1}^{n}\Gamma\left\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\right\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\left\{1-b_{ji}-\beta_{ji}(r+s+\rho k+\sigma)\right\}\prod_{j=n+1}^{p_i}\Gamma\left\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\right\}}\Bigg]^{-1}, \tag{4.4}$$

provided that set of conditions (i), (ii), and (iii) of the main Theorem are satisfied.

(ii) On taking $p = 3, Q = 2$, *and replacing* $A_1 = -N, A_2 = N+1, A_3 = \eta, B_1 = 1, B_2 = \gamma$; *the generalized hypergeometric function is reduced to Rice's polynomial* [12, p.287] *and we have*

Corollary 4.2. *Under the hypothesis surrounding the Theorem, the integral equation*

$$\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\psi(y)dy = g(x); (x>0) \tag{4.5}$$

where, the kernel

$$u_1(x) = H_N(\eta,\gamma,x)\aleph^{m,n}_{p_i,q_i,\tau_i;}[x] \tag{4.6}$$

has its solution $\psi(x)$ given by

$$\psi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u_2^*\left(\frac{x}{y}\right)\left(y^{\rho+1}Dy\right)^k\{y^\sigma g(y)\}\,dy, \tag{4.7}$$

provided that the integral exists and $U_1^*(x)$ is the Melline inverse transform of

$$U_2^*(s) = \left[\rho^k \frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{N}\frac{(-N,r)(N+1,r)(\eta,r)}{(1,r)(\gamma,r)(r)!}\right.$$

$$\left.\times\frac{\prod_{j=1}^{m}\Gamma\{b_j+\beta_j(r+s+\rho k+\sigma)\}\prod_{j=1}^{n}\Gamma\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\{1-b_{ji}-\beta_{ji}(r++\rho k+\sigma)\}\prod_{j=n+1}^{p_i}\Gamma\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\}}\right]^{-1},$$

provided that the set of conditions (i), (ii) and (iii) stated with the main Theorem are satisfied.
It may be pointed out here that the generalized hypergeometric function is very general in nature and it reduces into a number of special functions and orthogonal polynomials. Also the H-function is a very general function and has for its particular cases a number of important special functions [8].

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A NOVEL GENERALIZATION OF THE MITTAG-LEFFLER FUNCTION: PROPERTIES AND INTEGRAL TRANSFORMS

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Abstract
In this paper, we introduce a new and more general form of the Mittag-Leffler function. This new generalization contains extra parameters that give the function greater flexibility and a richer analytical structure. After defining this generalized function, we study several of its basic and essential properties in detail. These include its convergence behaviour, different types of integral representations, and several important transforms such as the Laplace transform, Beta transform, Mellin transforms, and Whittaker transforms.
We also show how this new generalized Mittag-Leffler function is related to many other known special functions in the literature. In addition, we examine how this function can be used to solve various fractional differential and integral equations. Overall, the results of this work extend many existing forms of the Mittag-Leffler function and bring them together into a unified framework. This provides a strong base for further theoretical research as well as practical applications in areas such as mathematical physics and engineering.
2020 Mathematical Sciences Classification: 33E12, 26A33, 35A22
Keywords and Phrases: Generalized Mittag-Leffler function, Generalized Hypergeometric Function, Foxs H-function.

1 Introduction

The classical Mittag-Leffler function, depending on a single parameter, was first formulated by the Swedish mathematician Gösta Mittag-Leffler [8, 1903]. It is viewed as an important extension of the exponential function. Originally defined through a single-parameter power series, this function has now become a fundamental part of fractional calculus, especially for modelling physical, biological, and engineering processes that involve memory or hereditary effects.

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbb{C},\ \Re(\alpha) > 0. \tag{1.1}$$

This series converges for $z \in \mathbb{C}$ i.e., it is an entire function of order $\rho = \frac{1}{\Re(\alpha)}$. Thus, for $\Re(\alpha) > 0$, the function is entire in the complex plane. This function reduces to the exponential function for $\alpha = 1$, and remains entire for all $z \in \mathbb{C}$, with order $\rho = \frac{1}{\Re(\alpha)}$. Wiman [20, 1905] proposed a two-parameter generalization of the Mittag-Leffler function, enhancing its flexibility in the analysis and solution of fractional differential equations:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}; \quad \Re(\alpha) > 0, \quad \Re(\beta) > 0; \quad \alpha, \beta \in \mathbb{C}. \tag{1.2}$$

This function too is entire in the complex plane and has been extensively studied for its role in linear and nonlinear fractional dynamics.

A major advancement in the generalization of the Mittag-Leffler function was achieved by Prabhakar [10, 1971], through the introduction of a three-parameter form involving the Pochhammer symbol $(\gamma)_n$

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!};\quad \alpha,\beta,\gamma\in\mathbb{C};\quad \Re(\alpha),\Re(\beta),\Re(\gamma)>0. \tag{1.3}$$

where $(\gamma)_n$ is the Pochhammer symbol Rainville [11, 1960].

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)};\quad (\gamma)_0=1. \tag{1.4}$$

Shukla and Prajapati [15, 2007] introduced a more generalized form of the Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,q}(z)$, defined for $\alpha,\beta,\gamma\in\mathbb{C}$; $\Re(\alpha),\Re(\beta),\Re(\gamma)>0$; $q\in(0,1)\bigcup\mathbb{N}$ as follows

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)\,n!}. \tag{1.5}$$

Salim [13, 2009] defined the function $E_{\alpha,\beta}^{\gamma,\delta}(z)$ for $\alpha,\beta,\gamma,\delta\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, $\Re(\delta)>0$ as

$$E_{\alpha,\beta}^{\gamma,\delta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}. \tag{1.6}$$

Salim and Faraz [14, 2012] proposed a new form of the Mittag-Leffler function, which is defined as follows:

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}, \tag{1.7}$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, and $\Re(\delta)>0$.
Khan and Ahmed [3, 2013] introduced two new generalized form of *MLF* defined as follows:

$$E_{\alpha,\beta,\delta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_n};\quad \alpha,\beta,\gamma,\delta\in\mathbb{C}, \tag{1.8}$$

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma}n(\delta)_{pn}}, \tag{1.9}$$

where $\alpha,\beta,\gamma,\mu,\nu,\rho,\sigma\in\mathbb{C}$; $p,q\in(0,1)\bigcup\mathbb{N}$; $q\leq\Re(\alpha)+p$ and $min\left[\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\mu),\Re(\nu),\Re(\rho),\Re(\sigma)\right]>0$. This generalization allows for more precise control over convergence and growth behaviour and finds applications in fractional kinetics, anomalous relaxation, and fractional diffusion systems.
In this paper, we introduce and study a new definition of a generalized Mittag-Leffler function, which is given as follows:

$$E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(\alpha n+\beta)\prod_{i=1}^{s}(b_i)_{np_i}}, \tag{1.10}$$

where $\alpha,\beta\in\mathbb{C}$; $a_i\in\mathbb{C}\,(\forall\, i=1,2,..,r)$; $b_i\in\mathbb{C}\,(\forall\, i=1,2,..,s)$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(a_i)>0$, $\Re(b_i)>0$ and $p_i\,(\forall i=1,2,..,s)\in(0,1)\bigcup\mathbb{N}$; $q_i\,(\forall i=1,2,..,r)\in(0,1)\bigcup\mathbb{N}$.
This definition extends all the previously introduced functions given in equations (1.1) to (1.9).

- Put $\beta=1$, $r=1=s$, $a_1=b_1$ and $p_1=q_1$ in equation (1.10) it reduces to classical *MLF* $E_\alpha(z)$ equation (1.1).
- Put $r=1=s$, $a_1=b_1$ and $p_1=q_1$ in equation (1.10) it reduces to Wiman [20] *MLF* $E_{\alpha,\beta}(z)$ equation (1.2).
- Put $r=1=s$, and $p_1=q_1=1=b_1$in equation (1.10) it reduces to *MLF* defined by Prabhakar [10] $E_{\alpha,\beta}^{\gamma}(z)$ equation (1.3).
- Put $r=1=s$, and $p_1=1=b_1$in equation (1.10) it reduces to *MLF* defined by Shukla and Prajapati [15] $E_{\alpha,\beta}^{\gamma,q}(z)$ equation (1.5).
- Put $r=1=s$, and $p_1=1=q_1$in equation (1.10) it reduces to *MLF* defined by Salim [13] equation (1.6).
- Put $r=1=s$, in equation (1.10) it reduces to *MLF* defined by Salim [14] equation (1.7).
- Put $r=2=s$, in equation (1.10) it reduces to *MLF* defined by Khan and Ahmed [3] equation (1.9).

Throughout this study, we make use of the following well-known results to examine different properties and related formulas of the function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$.
For distribution of non-zero zeros of generalized Mittag-Leffler function, we may refer to Kumar and Pathan [4] and for generalized Mittag-Leffler function of two variables, we may refer to Kumar and Pathan [5], while for multivariable and multiparameter generalized Mittag-Leffler function, we may refer to Pathan and Kumar [6].
Beta Transforms: Euler (Beta) transforms of function $f(x)$ is defined as Sneddon [16, 1979].

$$B\left\{f(x);a,b\right\}=\int_0^1 x^{a-1}(1-x)^{b-1}f(x)dx. \tag{1.11}$$

Laplace Transforms: The Laplace transform of the function $f(x)$ can be found in the works of Rainville [11, 1960] and Sneddon [16, 1979].

$$L\left\{f(x);s\right\}=\int_0^\infty e^{-sx}f(x)dx. \tag{1.12}$$

Mellin -Transforms: The Mellin transform and its inverse for the function $f(x)$ are listed in the *Table of Mellin Transforms* [9, 1974].

$$M\left\{f(x);s\right\}=f*(s)=\int_0^\infty x^{s-1}f(x)\,dx \qquad \operatorname{Re}(s)>0, \tag{1.13}$$

and the inverse Mellin-transform is given by

$$f(x)=M^{-1}\left\{f*(s);x\right\}=\frac{1}{2\pi i}\int_{\mathcal{L}} f*(s)\,x^{-s}dx. \tag{1.14}$$

Whittaker transform: Whittaker transform defined by Whittaker and Watson [19, 1962] as

$$\int_0^\infty e^{-\frac{t}{2}}t^{\nu-1}W_{\lambda,\rho}(t)\,dt=\frac{\Gamma\left(\frac{1}{2}+\rho+\nu\right)\Gamma\left(\frac{1}{2}-\rho+\nu\right)}{\Gamma(1-\lambda+\nu)}, \tag{1.15}$$

where Re $(\rho\pm\nu)>\frac{-1}{2}$ and $W_{\lambda,\rho}(t)$ is the Whittaker confluent hypergeometric function.
Wright Hypergeometric function: The Wright generalized hypergeometric function, as presented by Srivastava and Manocha [18, 1984] is defined as follows:

$${}_p\Psi_q\left[\begin{matrix}(a_i,A_i)_{(1,p)};\\(b_i,B_i)_{(1,q)};\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}\Gamma(a_i+nA_i)}{\prod\limits_{i=1}^{q}\Gamma(b_i+nB_i)}\frac{z^n}{n!}. \tag{1.16}$$

H-function: Fox's H-function, as formulated by Srivastava [17, 1972], is defined as follows:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p;A_p)\\(b_q;B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j-A_js)}z^sds. \tag{1.17}$$

2 Basic Properties of the Function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$:

If $\alpha,\ \beta\in\mathbb{C}$; $a_i\in\mathbb{C}\,(\forall\, i=1,2,..,r)$, $b_i\in\mathbb{C}\,(\forall\, i=1,2,..,s)$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(a_i)>0$, $\Re(b_i)>0$ and $p_i\,(\forall i=1,2,..,s)\in(0,1)\bigcup\mathbb{N}$; $q_i\,(\forall i=1,2,..,r)\in(0,1)\bigcup\mathbb{N}$ then,

$$E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)=\beta E_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)+\alpha z\frac{d}{dz}E_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z), \tag{2.1}$$

$$E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)=\frac{1}{\Gamma(\beta)}+\frac{\prod\limits_{i=1}^{r}(a_i)_{q_i}}{\prod\limits_{i=1}^{s}(b_i)_{p_i}}zE_{\alpha,\alpha+\beta;b_1+p_1,p_1;\ldots;b_s+p_s,p_s}^{a_1+q_1,q_1;\ldots;a_r+q_r,q_r}(z), \tag{2.2}$$

$$\begin{aligned}&E_{\alpha,\beta-\alpha;b_1,p_1;..;b_s,p_s}^{a_1,q_1;a_2,q_2;\ldots;a_r,q_r}(z)-E_{\alpha,\beta-\alpha;b_1,p_1;..;b_s,p_s}^{a_1-1,q_1;a_2,q_2;\ldots;a_r,q_r}(z)\\&=zq_1\frac{\prod\limits_{i=1}^{r}(a_i)_{q_i}}{\prod\limits_{i=1}^{s}(b_i)_{p_i}}\sum_{m=0}^{\infty}\frac{(m+1)(a_1+q_1)_{mq_1-1}\prod\limits_{i=2}^{r}(a_i+q_i)_{mq_i}z^m}{\prod\limits_{i=1}^{s}(b_i+p_i)_{mp_i}\Gamma(\alpha m+\beta)}.\end{aligned} \tag{2.3}$$

***Proof* (2.1):**

$$\text{R.H.S. of (2.1)} \quad = \sum_{n=0}^{\infty} \frac{\beta z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} + \sum_{n=0}^{\infty} \frac{\alpha n z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha n+\beta) z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} \qquad = \text{L.H.S. of (2.1).}$$

In particular

$$E_{\alpha,\beta;b_1,p_1}^{a_1,q_1}(z) = \beta E_{\alpha,\beta+1;b_1,p_1}^{a_1,q_1}(z) + z\alpha \frac{d}{dz} E_{\alpha,\beta+1;b_1,p_1}^{a_1,q_1}(z). \tag{2.4}$$

***Proof* (2.2):**

$$\text{L.H.S. of (2.2)} \quad = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} = \frac{1}{\Gamma(\beta)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}}$$

$$= \frac{1}{\Gamma(\beta)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{q_i} (a_i+q_i)_{(n-1)q_i}}{\prod_{i=1}^{s} (b_i)_{p_i} (b_i+p_i)_{(n-1)p_i}}$$

$$= \frac{1}{\Gamma(\beta)} + \frac{z\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+(\alpha+\beta))} \frac{\prod_{i=1}^{r} (a_i+q_i)_{mq_i}}{\prod_{i=1}^{s} (b_i+p_i)_{mp_i}}, \qquad \text{putting } n-1=m$$

$$= \frac{1}{\Gamma(\beta)} + \frac{\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} z E_{\alpha,\alpha+\beta;b_1+p_1,p_1;\ldots;b_s+p_s,p_s}^{a_1+q_1,q_1;\ldots;a_r+q_r,q_r}(z) \qquad = \text{R.H.S of (2.2).}$$

In particular

$$E_{\alpha,\beta;b_1,p_1}^{a_1,q_1}(z) = \frac{1}{\Gamma(\beta)} + \frac{(a_1)_{q_1}}{b_1} z E_{\alpha,\alpha+\beta;b_1+1,p_1}^{a_1+q_1,q_1}(z). \tag{2.5}$$

***Proof* (2.3):**

$$\text{L.H.S. of (2.3)} \quad = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)} - \sum_{n=0}^{\infty} \frac{(a_1-1)_{nq_1} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)}$$

$$= \sum_{n=0}^{\infty} \frac{\left\{(a_1)_{nq_1} - (a_1-1)_{nq_1}\right\} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)} = \sum_{n=1}^{\infty} \frac{nq_1 (a_1)_{nq_1-1} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)},$$

$$= \sum_{m=0}^{\infty} \frac{(m+1)\, q_1 (a_1)_{mq_1+q_1-1} \prod_{i=2}^{r} (a_i)_{(m+1)q_i} z^{m+1}}{\Gamma(\alpha m+\beta) \prod_{i=1}^{s} (b_i)_{(m+1)p_i}}, \quad \text{putting } n-1=m$$

$$= \sum_{m=0}^{\infty} \frac{z\,(m+1)\, q_1 (a_1)_{q_1} (a_1+q_1)_{mq_1-1} \prod_{i=2}^{r} (a_i)_{q_i} (a_i+q_i)_{mq_i} z^m}{\prod_{i=1}^{s} (b_i)_{p_i} (b_i+p_i)_{mp_i} \Gamma(\alpha m+\beta)}$$

$$= zq_1 \frac{\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} \sum_{m=0}^{\infty} \frac{(m+1)(a_1+q_1)_{mq_1-1} \prod_{i=2}^{r} (a_i+q_i)_{mq_i} z^m}{\prod_{i=1}^{s} (b_i+p_i)_{mp_i} \Gamma(\alpha m+\beta)} \qquad = \text{R.H.S. of (2.3).}$$

In particular

$$E_{\alpha,\beta-\alpha;b_1}^{a_1,q_1}(z) - E_{\alpha,\beta-\alpha;b_1}^{a_1-1;q_1} = q_1 z \frac{(a_1)_{q_1}}{b_1} \sum_{m=0}^{\infty} \frac{(m+1)(a_1)_{mq_1-1} z^m}{\Gamma(\alpha m+\beta)(b_1+1)_m}. \tag{2.6}$$

3 Basic Derivatives Properties of the function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$:

If $\alpha,\beta\in\mathbb{C};\ a_i\in\mathbb{C}(\forall\, i=1,2,..,r);\ b_i\in\mathbb{C}(\forall\, i=1,2,..,s);\ \Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(a_i)>0,\ \Re(b_i)>0$ and $p_i(\forall i=1,2,..,s)\in(0,1)\bigcup\mathbb{N};q_i(\forall i=1,2,..,r)\in(0,1)\bigcup\mathbb{N}$, also $m\in\mathbb{N}$ then,

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)=\frac{(1)_m\prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}}E_{(\alpha,\alpha m+\beta),(b_i+mp_i;p_i);(1,1)}^{(a_i+mq_i,q_i);(m+1,1)}(z),\tag{3.1}$$

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha})\right]=z^{\beta-m-1}E_{\alpha,\beta-m;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha}),\tag{3.2}$$

$$\left(\frac{d}{dz}\right)^m E_{\frac{m}{k},\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}\left(z^{\frac{m}{k}}\right)=\frac{\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)}\sum_{n=1}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\,\Gamma\left(\frac{nm}{k}+1\right)z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)\,\Gamma\left(\frac{nm}{k}-m+1\right)}.\tag{3.3}$$

***Proof* (3.1):**

$$\text{L.H.S. of (3.1)}=\left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$=\sum_{n=m}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}n(n-1)\ldots(n-m+1)z^{n-m}}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}=\sum_{n=m}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}n!z^{n-m}}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}(n-m)!},$$

$$=\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{(k+m)q_i}(1)_{m+k}z^k}{\Gamma(\alpha(k+m)+\beta)\prod\limits_{i=1}^{s}(b_i)_{(k+m)p_i}(1)_k},\quad\text{putting } n-m=k$$

$$=\frac{(1)_m\prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}}\sum_{k=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i+mq_i)_{kq_i}(1+m)_kz^k}{\Gamma(\alpha k+m\alpha+\beta)\prod\limits_{i=1}^{s}(b_i+mp_i)_{kp_i}(1)_k}$$

$$=\frac{(1)_m\prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}}E_{(\alpha,\alpha m+\beta),(b_i+mp_i;p_i);(1,1)}^{(a_i+mq_i,q_i);(m+1,1)}(z)\qquad=\text{R.H.S. of (3.1).}$$

In particular

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1}^{a_1,q_1}(z)=\frac{(1)_m(a_1)_{mq_1}}{(b_1)_m}E_{(\alpha,\alpha m+\beta),(b_1+m;1);(1,1)}^{(a_1+mq_1,q_1);(m+1,1)}(z).\tag{3.4}$$

***Proof* (3.2):**

$$\text{L.H.S. of (3.2)}=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{w^nz^{n\alpha+\beta-1}}{\Gamma(\alpha n+\beta)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$=\sum_{n=0}^{\infty}\frac{(n\alpha+\beta-1)(n\alpha+\beta-2)..(n\alpha+\beta-m)w^nz^{n\alpha+\beta-m-1}}{(n\alpha+\beta-1)(n\alpha+\beta-2)..(n\alpha+\beta-m)\Gamma(\alpha n+\beta-m)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$=z^{\beta-m-1}E_{\alpha,\beta-m;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha})\qquad=\text{R.H.S. of (3.2).}$$

In particular

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta;b_1,1}^{a_1,q_1}(wz^{\alpha})\right]=z^{\beta-m-1}E_{\alpha,\beta-m;b_1,1}^{a_1,q_1}(wz^{\alpha}).\tag{3.5}$$

***Proof* (3.3):**

$$\text{L.H.S. of (3.3)}=\left(\frac{d}{dz}\right)^m\sum_{n=0}^{\infty}\frac{z^{\frac{nm}{k}}}{\Gamma\left(\frac{nm}{k}+\beta\right)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=1}^{\infty} \frac{\frac{nm}{k}\left(\frac{nm}{k}-1\right)\left(\frac{nm}{k}-2\right)...\left(\frac{nm}{k}-m+1\right)}{\Gamma\left(\frac{nm}{k}+\beta\right)} \frac{\prod_{i=1}^{r}(a_i)_{q_i}(a_i+q_i)_{nq_i-q_i} z^{\left(\frac{nm}{k}-m\right)}}{\prod_{i=1}^{s}(b_i)_{p_i}(b_i+p_i)_{np_i-p_i}}$$

$$= \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\prod_{i=1}^{r}\Gamma(a_i)} \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma\left(\frac{nm}{k}+1\right) z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\prod_{i=1}^{s}\Gamma(b_i+np_i)\Gamma\left(\frac{nm}{k}-m+1\right)} \qquad = \text{R.H. S. of (3.3).}$$

In particular

$$\left(\frac{d}{dz}\right)^m E^{a_1,1}_{\frac{m}{k},\beta}\left(z^{\frac{m}{k}}\right) = \sum_{n=1}^{\infty} \frac{(a_1)_n \Gamma\left(\frac{nm}{k}+1\right) z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\Gamma\left(\frac{nm}{k}-m+1\right)}. \tag{3.6}$$

4 Basic Integral Properties of the function $E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(z)$

If $\alpha, \beta \in \mathbb{C}$; $a_i \in \mathbb{C}(\forall\, i = 1, 2, .., r)$, $b_i \in \mathbb{C}(\forall\, i = 1, 2, .., s)$; $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(a_i) > 0$, $\Re(b_i) > 0$ and $p_i(\forall i = 1, 2, .., s) \in (0, 1) \bigcup \mathbb{N}$; $q_i(\forall i = 1, 2, .., r) \in (0, 1) \bigcup \mathbb{N}$, then,

$$\frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(zx^{\alpha})\,dx = E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+\mu;b_1,p_1;...;b_s,p_s}(z), \tag{4.1}$$

$$\frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1} E^{a_i,q_i}_{\alpha,\beta,b_i,p_i}\left[\lambda(s-t)^{\alpha}\right] ds = (x-t)^{\eta+\beta-1} E^{a_i,q_i}_{\alpha,\beta+\eta,b_i,p_i}\left[\lambda(x-t)^{\alpha}\right], \tag{4.2}$$

$$\int_0^z t^{\beta-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(wt^{\alpha})\,dt = z^{\beta} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+1;b_1,p_1;...;b_s,p_s}(wz^{\alpha}). \tag{4.3}$$

***Proof* (4.1):**

$$\text{L.H.S. of (4.1)} \qquad = \frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(zx^{\alpha})\,dx$$

$$= \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\int_0^1 x^{n\alpha+\beta-1}(1-x)^{\mu-1}dx$$

$$= \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\frac{\Gamma(n\alpha+\beta)\Gamma(\mu)}{\Gamma(n\alpha+\beta+\mu)} = E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+\mu;b_1,p_1;...;b_s,p_s}(z) \qquad = \text{R.H.S. of (4.1).}$$

In particular

$$\frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1}_{\alpha,\beta;b_1}(zx^{\alpha})\,dx = E^{a_1}_{\alpha,\beta+\mu;b_1}(z). \tag{4.4}$$

***Proof* (4.2):**

$$\text{L.H.S. of (4.2)} \qquad = \frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1} E^{a_i,q_i}_{\alpha,\beta,b_i,p_i}\left[\lambda(s-t)^{\alpha}\right] ds.$$

Put $\quad u = \frac{s-t}{x-t}, \qquad 1-u = \frac{x-s}{x-t}, \qquad ds = (x-t)\,du.$ Therefore,

$$\text{L.H.S. of (4.2)} = \frac{1}{\Gamma(\eta)}\int_0^1 (1-u)^{\eta-1}(x-t)^{\eta-1}u^{\beta-1}(x-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\lambda^n u^{n\alpha}(x-t)^{n\alpha}}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}(x-t)\,du$$

$$= \frac{1}{\Gamma(\eta)}\int_0^1 (1-u)^{\eta-1}(x-t)^{\eta+n\alpha+\beta-1}u^{\beta+n\alpha-1}\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\lambda^n du$$

$$= \frac{(x-t)^{\eta+\beta-1}}{\Gamma(\eta)}\frac{\Gamma(\eta)\Gamma(\beta+n\alpha)}{\Gamma(\eta+\beta+n\alpha)}\sum_{n=0}^{\infty}\frac{(x-t)^{n\alpha}}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\lambda^n$$

$$= (x-t)^{\eta+\beta-1} E^{a_i,q_i}_{\alpha,\beta+\eta,b_i,p_i}\left[\lambda(x-t)^{\alpha}\right] \qquad = \text{R.H.S. of (4.2)}.$$

In particular

$$\frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1}E^{a_1}_{\alpha,\beta,b_1}\left[\lambda(s-t)^{\alpha}\right]ds = (x-t)^{\eta+\beta-1}E^{a_1}_{\alpha,\beta+\eta,b_1}\left[\lambda(x-t)^{\alpha}\right]. \tag{4.5}$$

***Proof* (4.3):**

$$\text{L.H.S. of (4.3)} \qquad = \int_0^z t^{\beta-1}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}\left(wt^{\alpha}\right)dt = \int_0^z t^{\beta+n\alpha-1}\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(\alpha n+\beta)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}dt$$

$$= \sum_{n=0}^{\infty}\frac{z^{\beta+n\alpha}w^n}{(\beta+n\alpha)\Gamma(\alpha n+\beta)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}} = z^{\beta}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}\left(wz^{\alpha}\right) = \text{R.H.S. of (4.3)}.$$

In particular

$$\int_0^z t^{\beta-1}E^{a_1}_{\alpha,\beta;b_1}\left(wt^{\alpha}\right)dt = z^{\beta}E^{a_1}_{\alpha,\beta+1;b_1}\left(wz^{\alpha}\right). \tag{4.6}$$

5 Relation of $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z)$ with other function

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \frac{1}{\Gamma(\beta)}\,{}_{q_1+\ldots+q_r+1}F_{k+p_1+..+p_s}\left[\begin{matrix}\Delta(q_i,a_i)_{(1,r)},(1,1)\\ \Delta(k,\beta)\,\Delta(p_i,b_i)_{(1,s)}\end{matrix};\ \frac{q_1^{q_1}q_2^{q_2}\ldots q_r^{q_r}z}{k^k p_1^{p_1}p_2^{p_2}\ldots p_s^{p_s}}\right]. \tag{5.1}$$

***Proof*:** Using (1.10) with $\alpha = k$, $p_i, q_i \in \mathbb{N}$ then we have

$$\text{L.H.S. of (5.1)} = \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(kn+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}} = \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(a_1)_{nq_1}(a_2)_{nq_2}\ldots(a_r)_{nq_r}(1)_n}{(\beta)_{kn}(b_1)_{np_1}(b_2)_{np_2}\ldots(b_s)_{np_s}}\frac{z^n}{n!},$$

$$= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{q_1}\left(\frac{a_1+j-1}{q_1}\right)_n\prod\limits_{j=1}^{q_2}\left(\frac{a_2+j-1}{q_2}\right)_n\ldots\prod\limits_{j=1}^{q_r}\left(\frac{a_r+j-1}{q_r}\right)_n(1)_n}{\prod\limits_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)\prod\limits_{j=1}^{p_1}\left(\frac{b_1+j-1}{p_1}\right)_n\prod\limits_{j=1}^{p_2}\left(\frac{b_2+j-1}{p_2}\right)_n\ldots\prod\limits_{j=1}^{p_s}\left(\frac{b_s+j-1}{p_s}\right)_n}\frac{\left(\frac{q_1^{q_1}q_2^{q_2}..q_r^{q_r}z}{k^kp_1^{p_1}p_2^{p_2}..p_s^{p_s}}\right)^n}{n!}$$

$$= \frac{1}{\Gamma(\beta)}\,{}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}\left[\begin{matrix}\Delta(q_i,a_i)_{(1,r)},(1,1)\\ \Delta(k,\beta)\,\Delta(p_i,b_i)_{(1,s)}\end{matrix};\ \frac{q_1^{q_1}q_2^{q_2}..q_r^{q_r}z}{k^kp_1^{p_1}p_2^{p_2}..p_s^{p_s}}\right].$$

where $\Delta(q_r,a_r)$ is q_r-tuple, $\frac{a_r}{q_r}, \frac{a_r+1}{q_r}, \ldots, \frac{a_r+q_r-1}{q_r}$,and in the same way, $\Delta(p_s,b_s)$ is p_s-tuple, $\frac{b_s}{p_s}, \frac{b_s+1}{p_s}, \ldots, \frac{b_s+p_s-1}{p_s}$. All remaining terms are defined as tuples in a similar manner. This expression provides the required hypergeometric representation.

The convergence condition for generalized function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$:

- If $(q_1+\ldots+q_r+1) \le (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ convergence for finite z.
- If $(q_1+\ldots+q_r+1) = (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ convergence when $|z|<1$ and divergence when $|z|>1$, moreover it is absolutely convergent on the circle $|z|=1$, if
$$\text{Re}\left(\sum_{j=0}^{k}\frac{\beta+j-1}{k}\sum_{j=0}^{p_1}\frac{b_1+j-1}{p_1}\ldots\sum_{j=0}^{p_s}\frac{b_s+j-1}{p_s} - \sum_{j=0}^{q_1}\frac{a_1+j-1}{q_1}\sum_{j=0}^{q_2}\frac{a_2+j-1}{q_2}\ldots\sum_{j=0}^{q_r}\frac{a_r+j-1}{q_r}\right) > 0.$$
- If $(q_1+\ldots+q_r+1) > (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ is divergence for finite $|z| \neq 0$

6 Integral Transforms

In this section, we study several important integral transforms of the generalized MittagLeffler function, including the Laplace transform, the Beta (Euler) transform, and the Whittaker transform of the generalized form MLF $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z)$

6.1 Mellin-Barnes integral representation of $E_{\alpha,\beta;b_1,p_1;..;b_s,p_s}^{a_1,q_1;...;a_r,q_r}(z)$

If $\alpha \in R^{+}$;$\beta \in \mathbb{C}$; $a_i \in \mathbb{C}\,(\forall\, i = 1, 2, .., r)$, $b_i \in \mathbb{C}\,(\forall\, i = 1, 2, .., s)$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(a_i) > 0$, $\Re(b_i) > 0$; $p_i, q_i > 0$ and $p_i\,(\forall i = 1, 2, .., s)$; $q_i\,(\forall i = 1, 2, .., r) \in (0, 1) \bigcup \mathbb{N}$ then the function $E_{\alpha,\beta;b_1,p_1;..;b_s,p_s}^{a_1,q_1;...;a_r,q_r}(z)$ is represented by Mellin-Barnes integral as

$$E_{\alpha,\beta;b_1,p_1;..;b_s,p_s}^{a_1,q_1;...;a_r,q_r}(z) = \frac{\prod\limits_{i=1}^{s}\Gamma(b_i)}{2\pi i\prod\limits_{i=1}^{r}\Gamma(a_i)}\int\limits_{L}\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}ds. \tag{6.1}$$

Here, $|\arg(z)| < 1$; the integration contour runs from $-i\,\infty$ to $i\,\infty$, and is suitably so as to separate the poles of the integrand at $s = -n\,\forall\, n \in \mathbb{N}_0$ (on the left) from those at $s = \frac{a_i+n}{q_i} \in \mathbb{N}_0$ (also on the right)

***Proof*:** We shall evaluate the integral on the right-hand side of (6.1) by applying the method of residues at the poles $s = 0, -1, -2, ...$

$$\frac{\prod\limits_{i=1}^{r}\Gamma(a_i)}{\prod\limits_{i=1}^{s}\Gamma(b_i)}E_{\alpha,\beta;b_1,p_1;..;b_s,p_s}^{a_1,q_1;...;a_r,q_r}(z) = \frac{\prod\limits_{i=1}^{r}\Gamma(a_i)}{\prod\limits_{i=1}^{s}\Gamma(b_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\,z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)} = \sum_{n=0}^{\infty}\frac{(-1)^{-s}\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}$$

$$= \sum_{n=0}^{\infty}\lim_{s\to -n}\frac{\pi(s+n)}{\sin(s\pi)}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)} = \sum_{n=0}^{\infty}\operatorname*{Re\,s}_{s=-n}\left[\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}\right]$$

$$= \frac{1}{2\pi i}\int\limits_{L}\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}ds.$$

This is complete proof.

6.2 Beta Transform

Using (1.11)

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}E_{\alpha,\beta;b_1,p_1;...;b_s,p_s}^{a_1,q_1;...;a_r,q_r}\left(x^{\lambda}z\right)dx = \frac{\Gamma(\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma(n\lambda+\rho)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)}\frac{z^n}{n!}. \tag{6.2}$$

***Proof*:**

$$\text{L.H.S.} = \int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}\left(x^{\lambda}z\right)^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}dx$$

$$= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}\Gamma(n\lambda+\rho)\Gamma(\sigma)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}(b_i)_{np_i}} = \frac{\Gamma(\sigma)\Gamma(\rho)}{\Gamma(\sigma+\rho)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}(\rho)_{n\lambda}z^n}{\Gamma(\alpha n+\beta)(\rho+\sigma)_{n\lambda}\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \frac{\Gamma(\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma(n\lambda+\rho)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)}\frac{z^n}{n!} \qquad = \text{R.H.S. of (6.2).}$$

6.3 Laplace Transform

$$\int_0^{\infty} z^{\rho-1}e^{-sz}E_{\alpha,\beta;b_1,p_1;...;b_s,p_s}^{a_1,q_1;...;a_r,q_r}\left(xz^{\lambda}\right)dz = \frac{\Gamma(\rho)}{s^{\rho}}E_{\alpha,\beta;b_1,p_1;...;b_s,p_s}^{a_1,q_1;...;a_r,q_r;\rho,\lambda}\left(\frac{x}{s^{\lambda}}\right). \tag{6.3}$$

***Proof*:**

$$\text{L.H.S.} = \int_0^{\infty} z^{\rho-1}e^{-sz}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}\left(xz^{\lambda}\right)^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}dz$$

$$= \int_0^\infty z^{\lambda n+\rho-1} e^{-sz} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} x^n}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} dz = \sum_{n=0}^{\infty} \frac{x^n \prod_{i=1}^{r} (a_i)_{nq_i}}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} \frac{(\rho+\lambda n-1)!}{s^{\rho+\lambda n}}$$

$$= \frac{\Gamma(\rho)}{s^\rho} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} (\rho)_{\lambda n}}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} \left(\frac{x}{s^\lambda}\right)^n = \frac{\Gamma(\rho)}{s^\rho} E^{a_1,q_1;\ldots;a_r,q_r;\rho,\lambda}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}\left(\frac{x}{s^\lambda}\right) \qquad = \text{R.H.S. of (6.3).}$$

6.4 Mellin Transform

Using (1.13) we have

$$M\left[E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-wz);s\right] = f*(s) = \int_0^\infty z^{s-1} E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-wz)dz.$$

From equation (6.1) we get

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(-wz) = \frac{\prod_{i=1}^{s}\Gamma(b_i)}{2\pi i \prod_{i=1}^{r}\Gamma(a_i)} \int_L \frac{\Gamma(1-s)\Gamma(s)\prod_{i=1}^{r}\Gamma(a_i-sq_i) w^{-s} z^{-s}}{\Gamma(\beta-s\alpha)\prod_{i=1}^{s}\Gamma(b_i-sp_i)} ds,$$

$$E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-z) = M^{-1}\{f*(s);z\} = \frac{\prod_{i=1}^{s}\Gamma(b_i)}{2\pi i \prod_{i=1}^{r}\Gamma(a_i)} \int_L f*(s) z^{-s} ds,$$

$$f*(s) = \frac{\Gamma(1-s)\Gamma(s)\prod_{i=1}^{r}\Gamma(a_i-sq_i)(w)^{-s}}{\Gamma(\beta-s\alpha)\prod_{i=1}^{s}\Gamma(b_i-sp_i)}. \tag{6.4}$$

6.5 Whittaker Transform

Using (1.15)

$$\int_0^\infty e^{-\frac{\phi t}{2}} t^{\xi-1} W_{\lambda,\rho}(\phi t) E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(\omega t^\eta) dt = \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r}\Gamma(a_i)} {}_{r+3}\Psi_{s+2}\left[\begin{matrix}(a_i,q_i)_{(1,r)}; \left(\frac{1}{2}\pm\rho+\xi,\eta\right)(1,1); \\ (\beta,\alpha);(b_i,p_i)_{(1,s)}(1-\lambda+\xi,\eta);\end{matrix} \frac{\omega}{\phi^\eta}\right]. \tag{6.5}$$

***Proof*:** Substituting $\phi t = \nu$

$$\text{L.H.S. of (6.5)} = \int_0^\infty e^{-\frac{\nu}{2}} \left(\frac{\nu}{\phi}\right)^{\xi-1} W_{\lambda,\rho}(\nu) \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_{nq_i}(\omega)^n}{\Gamma(\alpha n+\beta)\prod_{i=1}^{s}(b_i)_{np_i}} \left(\frac{\nu}{\phi}\right)^{n\eta} \frac{d\nu}{\phi},$$

$$= \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r}\Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(a_i+nq_i)}{\Gamma(\alpha n+\beta)\prod_{i=1}^{s}\Gamma(b_i+np_i)} \left(\frac{\omega}{\phi^\eta}\right)^n \int_0^\infty e^{-\frac{\nu}{2}}(\nu)^{n\eta+\xi-1} W_{\lambda,\rho}(\nu) d\nu,$$

$$= \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r}\Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma\left(\frac{1}{2}\pm\rho+n\eta+\xi\right)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\prod_{i=1}^{s}\Gamma(b_i+np_i)\Gamma(1-\lambda+n\eta+\xi)} \frac{\left(\frac{\omega}{\phi^\eta}\right)^n}{n!},$$

$$= \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r}\Gamma(a_i)} {}_{r+3}\Psi_{s+2}\left[\begin{matrix}(a_i,q_i)_{(1,r)}; \left(\frac{1}{2}\pm\rho+\xi,\eta\right)(1,1); \\ (\beta,\alpha);(b_i,p_i)_{(1,s)}(1-\lambda+\xi,\eta);\end{matrix} \frac{\omega}{\phi^\eta}\right] \frac{\left(\frac{\omega}{\phi^\eta}\right)^n}{n!} \qquad = \text{R.H.S. of (6.5).}$$

7 Relationship with some known function

7.1 Relationship with Wright hypergeometric function

If the condition be satisfied, then (1.10) can be written as

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \frac{\Gamma(b_i)}{\Gamma(a_i)} {}_{r+1}\Psi_{s+1}\left[\begin{matrix}(q_i,a_i)_{(1,r)};(1,1); \\ (\beta,\alpha);(p_i,b_i)_{(1,s)};\end{matrix} z\right]. \tag{7.1}$$

***Proof*:**

$$\text{L.H.S. of (7.1)} \quad = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} z^n}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} = \frac{\Gamma(b_i)}{\Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i+nq_i)\Gamma(n+1) z^n}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} \Gamma(b_i+np_i) n!}$$

$$= \frac{\Gamma(b_i)}{\Gamma(a_i)} {}_{r+1}\Psi_{s+1} \left[\begin{matrix} (q_i,a_i)_{(1,r)};(1,1); \\ (\beta,\alpha);(p_i,b_i)_{(1,s)}; \end{matrix} z \right] \qquad = \text{R.H.S. of (7.1)}.$$

7.2 Relationship with Foxs H-function

Using (6.1) and (1.17), we establish the relationship

$$E_{\alpha,\beta;b_1,p_1;..;b_s,p_s}^{a_1,q_1;...;a_r,q_r}(z) = \frac{\prod_{i=1}^{s} \Gamma(b_i)}{2\pi i \prod_{i=1}^{r} \Gamma(a_i)} H_{r+1,s+2}^{1,r+1} \left[z \middle| \begin{matrix} (0,1);(1-a_i,q_i)_{(1,r)}; \\ (0,1);(1-\beta,\alpha)(1-b,p_i)_{(1,s)} \end{matrix} \right]. \tag{7.2}$$

8 Conclusion

The generalized Mittag-Leffler function can be used in many scientific problems because it can describe systems that have memory or exhibit slow and fast-varying behaviour. It naturally appears in the solutions of fractional differential equations, which are used to model real-life processes, such as anomalous diffusion in physics, viscoelastic materials, electrical circuits with non-ideal capacitors, and biological systems with long-term memory.

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LACUNARY POLYNOMIAL INTERPOLATION BY g-SPLINE: $(0, 1, 2, 5)$ CASE

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Abstract

The author focuses on Lacunary Polynomial Interpolation *(LPI)*. This study proves the existence and uniqueness of the g-spline of degree six for *LPI*, which interpolates $(0, 1, 2, 5)$ data. In addition to verifying the interpolation of data, error bounds associated with this method are derived and analyzed.
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Keywords and Phrases: LPI, Splines, g-splines, Error bounds.

1 Introduction

LPI is an extension of Hermite interpolation. It includes matching of values and derivatives at certain points but does not claim that these points to be successive. Thus, whenever observations provide scattered and irregular data about a function and its derivatives then *LPI* arises. Several innovative concepts, tools and applications have been developed for *LPI* [7]. The theory of spline plays a significant role in avoiding the complications occurred by polynomials of a higher degree and also to achieve accurate results. *LPI* is generalized by g-spline.

Schoenberg [17] initiated the study of g-splines for *LPI*. He presented that under certain conditions the interpolatory g-splines exist and are unique. Numerous authors presented global procedures for solution of *LPI* by splines [8, 9, 18, 20]. Fawzy [2, 3, 4, 5, 6] described certain local procedures to solve *LPI* by splines.

By applying these procedures some authors considered certain class of g-splines for solution of a particular *LPI* [1, 10, 19]. We Studied *LPI* in different directions [11, 12, 13, 14, 15, 16].

Here, the author focuses on a specialized form of *LPI* and deals with gaps in the data provided (knowing the value and first, second and fifth derivatives, but not the third and fourth derivatives). The research investigates a specific mathematical structure to solve this problem. This ensures that for a specific set of data points, there is one and only one polynomial curve of this type that fits perfectly. The study verifies that sixth-degree spline successfully interpolates the $(0, 1, 2, 5)$ data set. The paper provides a rigorous investigation into error bounds. By establishing these bounds, the author demonstrates the accuracy and stability of using high-degree g-splines for non-consecutive data points, providing a reliable framework for approximation in complex computational problems.

The considered g-spline is defined as follows:
$\Delta : 0 = x_0 < x_1 <, \ldots, < x_n = 1$, be any subdivision of $[0, 1]$ with $h_k = x_{(k+1)} - x_k; k = 0, 1, \ldots, n-1$ and S_Δ be the class of splines satisfying the following properties:

$$S_\Delta \in C^1[0, 1], \tag{1.1}$$

$$S_\Delta \in \Pi_6, \tag{1.2}$$

in each interval $\left[x_k, x_{(k+1)}\right], k = 0, 1, \ldots, n-1$.

In the next section, we investigate the existence and uniqueness of the above g-splines which interpolate $(0, 1, 2, 5)$ data and investigate the error bounds for considered spline when, $f(x) \in C^6[0, 1]$.

2 Existence, Uniqueness and Error Bounds

The spline interpolant for $(0,1,2,5)$- data and for $f(x) \in C^6[0,1]$, defined above is given by:

$$S_\Delta = \sum_{j=0}^{6} (x - x_k) \frac{S_k^{(j)}}{j!}; x \in [x_k, x_{k+1}],$$

where $k = 0, 1, \ldots, n-1$.
We constructed each $S_k^{(j)}$ as the following:

$$S_k = f_k, \tag{2.1}$$

$$S_k^{(1)} = f_k^{(1)}, \tag{2.2}$$

$$S_k^{(2)} = f_k^{(2)}, \tag{2.3}$$

$$S_k^{(5)} = f_k^{(5)}, \tag{2.4}$$

where $k = 0, 1, \ldots, n-1$.
Now

$$S_k = f_k + (x - x_k) f_k^{(1)} + \frac{(x-x_k)^2}{2!} f_k^{(2)} + \frac{(x-x_k)^3}{3!} S_k^{(3)} + \frac{(x-x_k)^4}{4!} S_k^{(4)} + \frac{(x-x_k)^5}{5!} f_k^{(5)} + \frac{(x-x_k)^6}{6!} S_k^{(6)}.$$

From the interpolatory conditions and continuity of S_Δ, we get

$$f_{k+1} = f_k + h_k f_k^{(1)} + \frac{h_k^2}{2!} f_k^{(2)} + \frac{h_k^3}{3!} S_k^{(3)} + \frac{h_k^4}{4!} S_k^{(4)} + \frac{h_k^5}{5!} f_k^{(5)} + \frac{h_k^6}{6!} S_k^{(6)}, \tag{2.5}$$

$$f_{k+1}^{(1)} = f_k^{(1)} + h_k f_k^{(2)} + \frac{h_k^2}{2!} S_k^{(3)} + \frac{h_k^3}{3!} S_k^{(4)} + \frac{h_k^4}{4!} f_k^{(5)} + \frac{h_k^5}{5!} S_k^{(6)}, \tag{2.6}$$

$$f_{k+1}^{(2)} = f_k^{(2)} + h_k S_k^{(3)} + \frac{h_k^2}{2!} S_k^{(4)} + \frac{h_k^3}{3!} f_k^{(5)} + \frac{h_k^4}{4!} S_k^{(6)}, \tag{2.7}$$

$$f_{k+1}^{(5)} = f_k^{(5)} + h_k S_k^{(6)}, \tag{2.8}$$

$$S_k^{(6)} = \frac{f_{k+1}^{(5)} - f_k^{(5)}}{h_k}. \tag{2.9}$$

From equations (2.5) and (2.6), we get

$$S_k^{(3)} = \frac{24}{h_k^3} \left[\mathbb{D}_k - \frac{h_k}{4} \mathbb{E}_k \right], \tag{2.10}$$

and

$$S_k^{(4)} = -\frac{72}{h_k^4} \left[\mathbb{D}_k - \frac{h_k}{3} \mathbb{E}_k \right], \tag{2.11}$$

where

$$\mathbb{D}_k = f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)}, \tag{2.12}$$

and

$$\mathbb{E}_k = f_{k+1}^{(1)} - f_k^{(1)} - h_k f_k^{(2)} - \frac{h_k^4}{4!} f_k^{(5)} - \frac{h_k^5}{5!} S_k^{(6)}. \tag{2.13}$$

The selection of $S_k^{(j)}; j = 3, 4$ agreed on the following basis:

$$D_{\mathcal{L}}^{(j)} S_k = D_{\mathcal{R}}^{(j)} S_k; j = 0, 1, 2, 5; k = 0, 1, \ldots, n-1,$$

where $D_{\mathcal{L}}$ and $D_{\mathcal{R}}$ are denote left derivative and right derivative respectively.

Thus

$$S_\Delta \in C^{(0,1,2,5)}\left[x_0, x_n\right] = \{f \in C\left[x_0, x_n\right]; D^{(j)}_{\mathcal{R}} f \in C\left[x_0, x_n\right] : j = 0, 1, 2, 5\}.$$

Hence there exists unique spline $S_\Delta(x)$ defined by equations (2.9) - (2.13) satisfying the interpolatory conditions (2.1) - (2.4).

Theorem 2.1. *If S_Δ be spline solving the $(0,1,2,5)$-LPI as defined above for which $f(x) \in C^6[0,1]$ then for all $x \in [0,1]$ the following inequality fulfilled:*

$$\left|S_k^{(j)} - f_k^{(j)}\right| \le c_{k,j} h^{6-j} w_6(f,h); j = 0, 1, \ldots, 6$$

where $w_6(f,h)$ is the modulus of continuity of $f(x)$ and the constants $c_{k,j}$; $j = 0, 1, \ldots, 6$ are given by followings:

$$\begin{gathered} c_{k,0} = \frac{1}{36}, \quad c_{k,1} = \frac{1}{10}, \\ c_{k,2} = \frac{11}{40}, \quad c_{k,3} = \frac{1}{12}, \\ c_{k,4} = \frac{3}{10}, \quad c_{k,5} = 1, \\ c_{k,6} = 1. \end{gathered}$$

Proof. From equation (2.9), we get

$$\left|S_k^{(6)} - f_k^{(6)}\right| \le w_6(f,h). \tag{2.14}$$

From equation (2.11), we derive

$$\left|S_k^{(4)} - f_k^{(4)}\right| = \left|-\frac{72}{h_k^4}\left[\mathbb{D}_k - \frac{h_k}{3}\mathbb{E}_k\right] - f_k^{(4)}\right|,$$

From equation (2.12) and (2.13), we obtain

$$\begin{aligned} |S_k^{(4)} - f_k^{(4)}| = | -\frac{72}{h_k^4}[f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)} \\ - \frac{h_k}{3}\{f_{k+1}^{(1)} - f_k^{(1)} - h_k f_k^{(2)} - \frac{h_k^4}{4!} f_k^{(5)} - \frac{h_k^5}{5!} S_k^{(6)}\}] - f_k^{(4)}|. \end{aligned} \tag{2.15}$$

Now

$$f_{k+1} = f_k + h_k f_k^{(1)} + \frac{h_k^2}{2!} f_k^{(2)} + \frac{h_k^3}{3!} f_k^{(3)} + \frac{h_k^4}{4!} f_k^{(4)} + \frac{h_k^5}{5!} f_k^{(5)} + \frac{h_k^6}{6!} f_k^{(6)}(t_{k,1}), \tag{2.16}$$

where $x_k < t_{k,1} < x_{k+1}$,

$$f_{k+1}^{(1)} = f_k^{(1)} + h_k f_k^{(2)} + \frac{h_k^2}{2!} f_k^{(3)} + \frac{h_k^3}{3!} f_k^{(4)} + \frac{h_k^4}{4!} f_k^{(5)} + \frac{h_k^5}{5!} f_k^{(6)}(t_{k,2}), \tag{2.17}$$

where $x_k < t_{k,2} < x_{k+1}$.
From equations (2.15), (2.16) and (2.17), we derive

$$\begin{aligned} \left|S_k^{(4)} - f_k^{(4)}\right| &\le \frac{1}{10} h_k^2 w_6(f, h_k) + \frac{2}{10} h_k^2 w_6(f, h_k), \\ &\le \frac{3}{10} h_k^2 w_6(f, h_k). \end{aligned} \tag{2.18}$$

From equation (2.10), we get

$$\left|S_k^{(3)} - f_k^{(3)}\right| = \left|\frac{24}{h_k^3}\left[\mathbb{D}_k - \frac{h_k}{4}\mathbb{E}_k\right] - f_k^{(3)}\right|.$$

From equation (2.12) and (2.13), we obtain

$$\left|S_k^{(3)} - f_k^{(3)}\right| = \left|\frac{24}{h_k^3}\left[f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)}\right.\right.$$

$$-\frac{h_k}{4}\{f_{k+1}^{(1)}-f_k^{(1)}-h_kf_k^{(2)}-\frac{h_k^4}{4!}f_k^{(5)}-\frac{h_k^5}{5!}S_k^{(6)}\}\Big]-f_k^{(3)}\Big|. \quad (2.19)$$

From equations (2.16), (2.17) and (**??**), we establish

$$\left|S_k^{(3)}-f_k^{(3)}\right|\leq\frac{1}{30}h_k^3w_6\left(f,h_k\right)+\frac{1}{20}h_k^3w_6\left(f,h_k\right),$$
$$\leq\frac{1}{12}h_k^3w_6\left(f,h_k\right). \quad (2.20)$$

Now

$$\left|S_k^{(5)}-f_k^{(5)}\right|=\left|f_k^{(5)}+h_kS_k^{(6)}-\{f_k^{(5)}+h_kf_k^{(6)}(t_{k,1})\}\right|,$$

where $x_k<t_{k,2}<x_{k+1}$.

$$\left|S_k^{(5)}-f_k^{(5)}\right|\leq h_kw_6\left(f,h_k\right).$$

$$\left|S_k^{(2)}-f_k^{(2)}\right|=\left|f_k^{(2)}+h_kS_k^{(3)}+\frac{h_k^2}{2!}S_k^{(4)}+\frac{h_k^3}{3!}f_k^{(5)}+\frac{h_k^4}{4!}S_k^{(6)}-\{f_k^{(2)}+h_kf_k^{(3)}+\frac{h_k^2}{2!}f_k^{(4)}+\frac{h_k^3}{3!}f_k^{(5)}+\frac{h_k^4}{4!}f_k^{(6)}(t_{k,1})\}\right|,$$

where $x_k<t_{k,1}<x_{k+1}$.
From equations (2.14), (2.18) and (2.20), we derive

$$\left|S_k^{(2)}-f_k^{(2)}\right|\leq\frac{11}{40}h_k^4w_6\left(f,h_k\right).$$

From equations (2.14), (2.18) and (2.20), we also get

$$\left|S_k^{(1)}-f_k^{(1)}\right|\leq\frac{1}{10}h_k^5w_6\left(f,h_k\right).$$

Applying same technique as above and from equations (2.14), (2.18) and (2.20), we finally establish

$$|S_k-f_k|\leq\frac{1}{36}h_k^6w_6\left(f,h_k\right),$$

where

$$h=\underset{k}{Max}\quad h_k.$$

□

3 Conclusion

A g-spline of degree six for $(0,1,2,5)$-type *LPI* exists uniquely. The study establishes the precise limits of potential error for the resulting g-spline, specifically evaluating these boundaries under the condition $f(x)\in C^6[0,1]$.

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RADII OF ANALYTIC FUNCTIONS SATISFYING SOME COEFFICIENT INEQUALITIES

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Abstract

Let $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is an analytic function defined on an open unit disc in complex plane with center at origin and radius 1. The a_2 coefficient is such that $|a_2| = 2b$, where $b \in [0, 1]$, for other coefficients $|a_n| \leq C/n^3 (C > 0)$ and $|a_n| \leq cn + d/n (c, d > 0)$. Optimum or extreme values of radius are evaluated for functions to have convexity of order $\alpha (0 \leq \alpha < 1)$ and starlikeness of order α, where $\alpha \in [0, 1[$. Radius values for few other properties are also evaluated.
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1 Introduction

Let $\mathcal{A}$ represents the collection of all normalised functions g analytic on the open unit disc $\mathbb{D} = \mathbb{D}(0, 1) = \{|z| : |z| < 1\}$ with center at origin and radius unity in complex plane. The power series representation of any function in $\mathcal{A}$ is of the form $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Thus, a normalised function takes zero value at origin and 1 for $g'(0)$. Let $\mathcal{S} \subseteq \mathcal{A}$, represents class of univalent functions. For $g \in \mathcal{S}$ it is well known through de Branges Theorem [4] that $|a_n| \leq n \ \forall \ n \geq 2$. However, the converse is not true, that is, coefficients of $g \in \mathcal{A}$ satifying the inequality $|a_n| \leq n, (n \geq 2)$ are not neccessarily univalent. The functions [2, 25] $g(z) = z - 2z^2 - 3z^3 - 4z^4 - \cdots = 2z - z/(1 - z)^2$ or $g(z) = z + 2z^2$ are not members of $\mathcal{S}$. This is the motivation for the complex analysts to obtain the radii of functions with predefined coefficient bounds.

Firstly, let us define the subclasses of $\mathcal{S}$ that will be needed in our investigation. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ represent well the known classes of $\mathcal{S}$ representing classes of starlike and convex functions of order α, respectively. These classes are collection of normalised functions which satisfy the following equivalences: $g \in \mathcal{S}^*(\alpha) \iff \operatorname{Re}(zg'(z)/g(z)) > \alpha$, and $g \in \mathcal{K}(\alpha) \iff \operatorname{Re}(1 + zg''(z)/g'(z)) > \alpha$. These classes were defined by Robertson [27] and $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{K}(0) := \mathcal{K}$ represent subclasses of starlike and convex functions.

Consider the subclasses $\mathcal{S}^*_\alpha$ and $\mathcal{K}_\alpha$ of $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ given by the following equivalences:

$$g \in \mathcal{S}^*_\alpha \iff \left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \alpha,$$

and

$$g \in \mathcal{K}_\alpha \iff \left|\frac{zg''(z)}{g'(z)}\right| < 1 - \alpha.$$

Goodman in [8] defined a new class of functions namely uniformly convex functions and denoted it by $\mathcal{UCV}$. A function $g \in \mathcal{A}$ is a uniformly convex function if it maps every circular arc in the unit disc $\mathbb{D}$ to a convex arc in $g(\mathbb{D})$. Equivalently,

$$g \in \mathcal{UCV} \iff \operatorname{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) > \left|\frac{zg''(z)}{g'(z)}\right| (z \in \mathbb{D}).$$

The above equivalence was proved independently by Ma and Minda [19] and Rønning [28] in the early nineties. Rønning [28] introduced the class of parabolic starlike functions $\mathcal{S}P$, consisting of functions $g \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > \left|\frac{zg'(z)}{g(z)} - 1\right|.$$

Observe that the class $\mathcal{S}P$ consists of functions $g = zG'$, where $G \in \mathcal{UCV}$. Also $\mathcal{K}_{1/2} \subset \mathcal{UCV}$, $\mathcal{S}^*_{1/2} \subseteq \mathcal{S}P$. Two other subclasses of analytic functions that we will investigate are $\mathcal{L}(\alpha, \beta)$ and $\mathcal{S}^*[A, B]$.

The class $\mathcal{L}(\alpha, \beta)$ is defined as

$$\mathcal{L}(\alpha, \beta) = \left\{ g \in \mathcal{A} : \alpha \frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \beta \in \mathbb{R} \setminus \{1\}, \alpha \geq 0 \right\}.$$

For different values of α and β, it reduces to known classes of analytic functions see [15, 16, 17, 23, 24, 39].

The other class that we will investigate is a widely studied class in univalent functions, known as Janowski [12] starlike functions represented by $\mathcal{S}^*[A, B]$ defined as,

$$\mathcal{S}^*[A, B] = \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1 \right\}.$$

For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] = \mathcal{S}^*(\beta)$.

The coefficient estimates have always been a matter of interest. Convex functions, starlike functions with respect to symmetric points, and starlike functions of order 1/2 [18, 29, 30] satisfy $|a_n| \leq 1, (n \geq 2)$. Singh and Singh [35] proved that if class of starlike functions with respect to symmetric point is subordinate to function $1 + \sin z$, then $|a_j| \leq 1/4 (j = 4, 5)$. they have worked on coefficient problems in [36, 37]. They generalised the results obtained in above paper in [38] . Uniformly convex functions satisfy $|a_n| \leq 1/n$[8]. Close-to-convex functions with argument β [10] satisfy $|a_n| \leq 1 + (n - 1)\beta$ if $f \in \mathcal{A}$ with $\operatorname{Re}\, g'(z) > 0 \Rightarrow |a_n| \leq 2/n$ [9]. In 1955 Reade [26] showed that $g(z)$ will be close-to-convex in $\mathbb{D}$ if $|a_n| \leq n|a_1|$. He further showed that $g(z)$ is close-to-starlike function for $|a_n| \leq n^2$. He proved that if $g \in \mathcal{A}$ is close-to-starlike function, there exist a univalent function $\psi(z)$, starlike with respect to $z = 0$ such that $\operatorname{Re}(g(z)/\psi(z)) > 0$ on $\mathbb{D}$. Close-to-starlike functions are not necessarily univalent. Note that the function $\psi(z) = z + n^2 z^n$ is close-to-starlike but not univalent. Silverman [31] in 1994 proved that for analytic function satisfying inequality $\operatorname{Re}(g'(z) + zg''(z)) > \alpha, (\alpha < 1), |a_n| \leq 2(1 - \alpha)/n^2$. These coefficient bounds were sharp and for $1 > \alpha \geq 0$, the class of $g \in \mathcal{A}$ was univalent. Coefficient estimates have been obtained by many, Gurmeet *et al.* [33, 34] have obtained estimates for initial coefficient of bivalent functions defined on unit disk. Bobalade and Sangle [3] have obtained coefficient bounds for a certain subclass of harmonic univalent functions defined using differential operations.

The radius of univalence of $\mathcal{A}$ is the largest $r \in (0, 1]$, such that $g \in \mathcal{A}$ satisfy univalence in $\mathbb{D}_r = \mathbb{D}(0, r) = \{z : |z| < r\}$. The largest $\mathbb{D}_r \subset \mathbb{D}$, whose images under g are starlike with respect to origin and are convex sets, are the starlike radius and the convexity radius. Similarly, the radii of other properties can be defined in similar fashion.

Gavrilov [6] in 1970 showed that the radius of univalence of functions satisfying the inequality $|a_n| \leq n$ is the real root of the equation $2(1 - r)^3 - (1 - r) = 0$, while for the functions whose coefficients satisfy $|a_n| \leq M$, the radius of univalence is $1 - \sqrt{M/(M + 1)}$. Yamashita[40] in 1982 showed that the radius of univalence obtained by Gavrilov is also the radius of starlikeness for the functions $f \in \mathcal{A}$ with coefficients satisfying $|a_n| \leq n$. The author in the same paper also obtained the lower bound for the convexity radius for functions $g \in \mathcal{A}$ that satisfies $|a_n| \leq n, (n \geq 2)$, which turned out to be the least real root $r_0 \approx 0.090$ of the equation $2(1 - r)^4 - (1 + 4r + r^2) = 0$ in $(0, 1)$. The author [40] calculated a lower bound of radius of convexity, for function $g \in \mathcal{A}$ satisfying $|a_n| \leq M$ and obtained as the least real number $(0, 1)$ satisfying equation $(M + 1)(1 - r)^3 - M(1 + r) = 0$.

Kalaj *et al.* [13] studied similar problems on harmonic functions. Graham *et al.* [11] in 2006 investigated radius problems of holomorphic functions on the unit disc $\mathbb{C}^n$. In 2013, radii of 'fully convex' and 'fully starlike functions' were obtained by Nagpal and Ravichandran [22] for harmonic mappings satisfying certain coefficient inequalities.

Let $\mathcal{A}_b$ denote the class of functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n, (0 \leq b \leq 1)$, with $|a_2| = 2b$ in $\mathbb{D}$. In [25], for $g \in \mathcal{A}_b$, Ravichandran determined the sharp radii of order α for starlikeness and convexity when the coefficients satisfy for $n \geq 3$ $|a_n| \leq n$; $|a_n| \leq M$ or $|a_n| \leq M/n$ $(M > 0)$. Radii for uniform convexity and parabolic starlikeness were also obtained by him in the same paper. In the same year, Ali *et al.* [2] obtained radii for the class $\mathcal{L}(\alpha,\beta)$ and $\mathcal{S}^*[A,B]$ for functions $g\in\mathcal{A}_b$ satisfying the coefficient inequalities $|a_n|\le cn+d$ or $|a_n|\le c/n$ $(c>0)$, and hence deduced the results of Ravichandran [25] and Yamashita [40] as particular cases.

In 2015, Mendiratta *et al.* [21] obtained sharp radii of starlikeness and convexity of order α, parabolic starlikeness, and uniform convexity for functions $g\in\mathcal{A}_b$ when $|a_n|\le M/n^2$ or $|a_n|\le Mn^2 (M>0)$. The authors have obtained sharp radii for Livingston problem for Sakaguchi functions with fixed second coefficients in [20] Kumar and Ravichandran [14] investigated analytic functions with some coefficient inequalities in 2017 and obtained sharp radii constants along with growth and distortion estimates.

There is a challenge in working on fixed second coefficient, as extremal functions are naturally removed. Prescribed coefficient bounds were further investigated by Aghalary and Mohammadin [1] in 2018. They determined radius for 'univalence', 'stable starlikeness', 'stable convexity', 'fully starlikeness' and 'convexity' of harmonic mappings satisfying some coefficient inequalities.

We will be using the following two lemmas to obtain results.

Lemma 1.1 ([17, 39])**.** *Let $\beta\in\mathbb{R}\setminus\{1\}$ and $\alpha\ge 0$. If $g\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n|\le|1-\beta|,$$

then $g\in\mathcal{L}(\alpha,\beta)$.

Lemma 1.2 ([7])**.** *If $g\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\left((1-B)n-(1-A)\right)|a_n|\le A-B,$$

then $g\in\mathcal{S}^*[A,B]$.

2 Radius Constants for $|a_n|\le C/n^3$

Normalised analytic functons with constant initial coefficient i.e. $|a_2|=2b$ will be studied in this section, for $n\ge 3$, coefficients will be satisfying the inequality $|a_n|\le C/n^3, C>0$. Chichra [5] showed that if $g\in\mathcal{A}$ satisfies $\operatorname{Re}\left(g'(z)+zg''(z)\right)>0, z\in\mathbb{D}$, then g is univalent i.e. $g\in\mathcal{S}$. Later, R. Singh and S. Singh [32] proved that if $\operatorname{Re}\left(g'(z)+zg''(z)\right)>0\Rightarrow g\in\mathcal{S}^*$.

Theorem 2.1. *Let $g\in\mathcal{A}_b$ be such that $|a_n|\le C/n^3, (C>0),\ n\ge 3$. The below mentioned results hold good;*

(i) g satisfies the inequality

$$\operatorname{Re}(g'(z)+zg''(z))>0 \text{ for } |z|<r_0,$$

where r_0 is the real number such that $0<r_0<1$ satisfying the equation:

$$(C-16b)r^2+2(C+1)r+2C\log(1-r)=0, \tag{2.1}$$

thus $g\in\mathcal{S}$ in $|z|<r_0$.

(ii) $g\in\mathcal{S}^*_\alpha$ for $|z|<r_1$. Where $r_1=r_1(\alpha)$ is the root lying between 0 and 1 of the equation

$$(16b-C)(2-\alpha)r^2-(8C+1)(1-\alpha)r+8C(\mathrm{Li}_2(r)-\alpha\mathrm{Li}_3(r))=0 \tag{2.2}$$

The value of r_1 for $\alpha=1/2$ will provide with the radius of parabolic starlikeness.

(iii) Function g will satisfy the inequality

$$\left|\frac{zg''(z)}{g'(z)}\right|\le 1-\alpha,\ |z|\le r_2,$$

for $r_2=r_2(\alpha)$, where $0<r_2<1$ is the root of the equation

$$(16b-C)(2-\alpha)r^2-4(C+1)(1-\alpha)r-4C\log(1-r)-4\alpha C\mathrm{Li}_2(r)=0$$

$r_2(\alpha)$ also represents the convexity radius of order α. For $\alpha=1/2$, $r_2(\alpha)$ reduces to the radius of uniform convexity.

Above radii are all extreme and cannot be improved.

Proof. **(i)** Assume that $|a_2| = 2b$ and $|a_n| \leq C/n^3, n \geq 3$. Then,

$$\begin{aligned}
g'(z) + zg''(z) &= 1 + \sum_{n=2}^{\infty} na_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n-1} \\
&= 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \\
&= 1 + 8bz + \sum_{n=3}^{\infty} n^2 a_n z^{n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{Re}(g'(z) + ng''(z)) &> 1 - 8br_0 - \sum_{n=3}^{\infty} \frac{Cr_0^{n-1}}{n} \\
&= 1 - 8br_0 - C\left(-\frac{\log(1-z_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\
&= 0,
\end{aligned}$$

where r_0 is the solution of equation (2.1) and is a real number lying in $(0,1)$. This is the best possible radius. As, for the function

$$g_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n.$$

$$zg_0''(z) + g_0'(z) = 1 - 8bz - \sum_{n=3}^{\infty} \frac{C}{n} z^{n-1}.$$

At $z = r_0$, the function g_0 satisfies $\mathrm{Re}(zg_0''(z) + g_0'(z)) = 0$.

Hence, the result.

(ii) It needs to be shown that the function $g(r_1 z)/r_1 \in \mathcal{S}_\alpha^*$. where r_1 is root of equation (2.2) lying in $(0,1)$. For $A = 1-\alpha$ and $B = 0$, from Lemma 1.2 it is sufficient to establish the inequality

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| r_1^{n-1} \leq 1 - \alpha.$$

Consider for $|a_n| \leq C/n^3$,

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-\alpha)|a_n| r_1^{n-1} &= 2(2-\alpha)br_1 + \sum_{n=3}^{\infty} (n-\alpha)|a_n| r_1^{n-1} \\
&\leq 2(2-\alpha)br_1 + \alpha C \sum_{n=3}^{\infty} \frac{r_1^{n-1}}{n^3} - \alpha C \sum_{n=3}^{\infty} \frac{r_1^{n-1}}{n^3} \\
&= 2(2-\alpha)br_1 + C\left(\frac{\mathrm{Li}_2(r_1)}{r_1} - 1 - \frac{r_1}{4}\right) - \frac{\alpha C}{r_1}\left(\mathrm{Li}_3(r_1) - r_1 - \frac{r_1^2}{8}\right).
\end{aligned}$$

We have used value of the polylogarithm functions of order 2 and 3. Polylogarithmic function of order s is

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots, \mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t}\, dt.$$

Thus,

$$\begin{aligned}
\sum_{n=2}^{\infty} (n-\alpha)|a_n| r_1^{n-1} &= 2(2-\alpha)br_1 + \frac{C}{r_1}\mathrm{Li}_2(r_1) - C - \frac{Cr_1}{4} \\
&\quad - \frac{\alpha C}{r_1}\mathrm{Li}_3(r_1) + \alpha C + \frac{\alpha C r_1}{8} = 1 - \alpha.
\end{aligned}$$

Hence, $g \in \mathcal{S}_\alpha^*$ for $|z| < r_1$, with $r_1 \in (0,1)$ being the root of equation (2.2).

As proved in the previous result, it can be easily shown here that

$$g_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n$$

is the required function for the best possible radius of starlikeness of order α. As the inequality

$$\left|\frac{zg_0'(z)}{g_0(z)}-1\right|\leq\frac{1}{2}$$

is sufficient for parabolic starlikeness, thus for $\alpha=\frac{1}{2}$ we must have

$$\left|\frac{zg_0'(z)}{g_0(z)}-1\right|=\frac{1}{2}=\mathrm{Re}\left(\frac{zg_0'(z)}{g_0(z)}\right),$$

Hence radius of parabolic starlikeness.

(iii) We are aware that to show $g\in\mathcal{K}_\alpha$, it is sufficient to show $zg'(z)\in\mathcal{S}^*_\alpha$. Thus, it is sufficient to establish the below inequality from Lemma 1.2:

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|r_2^{n-1}\leq 1-\alpha.$$

Note that,

$$\begin{aligned}\sum_{n=2}^{\infty}n(n-\alpha)|a_n|r_2^{n-1}&\leq 4(2-\alpha)br_2+\sum_{n=3}^{\infty}(k-\alpha)\frac{C}{k^2}r_2^{n-1}\\&=4(2-\alpha)br_2+C\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n}-\alpha C\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n^2}.\end{aligned}$$

Using standard series expansions, we obtain

$$\begin{aligned}&=4(2-\alpha)br_2+\frac{C}{r_2}\left(-\log(1-r_2)-r_2-\frac{r_2^2}{2}\right)-\frac{\alpha C}{r_2}\left(\mathrm{Li}_2(r_2)-r_2-\frac{r_2^2}{4}\right)\\&=1-\alpha.\end{aligned}$$

Hence the result follows. The smallest root in $(0,1)$ is the required radius. Sharpness is attained by the function

$$g_0(z)=z-2bz^2-C\sum_{n=3}^{\infty}\frac{z^n}{n^3},$$

which can be proved easily. □

Corollary 2.1. *(i)(a) For $g\in\mathcal{A}$ and $|a_n|\leq C/n^3,(n\geq 2)$, $\mathrm{Re}(zg''(z)+g'(z))>0$ where $r_0\in(0,1)$ is the solution of the equation*

$$2(C+1)r+2C\log(1-r)=0.$$

(i)(b) The starlikeness radius of normalised analytic functions with coefficients satisfying $a_2=0$ and $|a_n|\leq C/n^3, n\geq 3$, is the root $r_0\in(0,1)$ for the equation

$$Cr^2+2(C+1)r+2C\log(1-r)=0.$$

Corollary 2.2. *(ii)(a) The starlikeness radius of order α of functions with coefficients satisfying $|a_n|\leq C/n^3, n\geq 2$, is the real number $r_1\in(0,1)$ which is the solution of the equation*

$$(8C+1)(1-\alpha)r-8C(\mathrm{Li}_2(r)-\alpha\mathrm{Li}_3(r))=0$$

and starlikeness radius is given by the root of the equation

$$(8C+1)r-8C\mathrm{Li}_2(r)=0$$

(ii)(b)The starlikenss radius of order α of functions with coefficients $a_2=0$ and $|a_n|\leq C/n^3$ for $n\geq 3$, is the real number r_1 in $(0,1)$ satisfying the equation

$$C(\alpha-2)r^2+(8C+1)(\alpha-1)r+8C(\mathrm{Li}_2(r)-\alpha\mathrm{Li}_3(r))=0.$$

and radius of starlikness is the solution of the equation below

$$2Cr^2+(8C+1)r-8C\mathrm{Li}_2(r)=0$$

Corollary 2.3. *Let $g \in \mathcal{A}$ and $|a_n| \le C/n^3, (k \ge 2)$. Then the radius of convexity of order α is the real root r_2 in $(0,1)$ of the equation*

$$(C+1)(1-\alpha)r + C\log(1-r) + \alpha C \mathrm{Li}_2(r) = 0$$

For $\alpha = 1/2$, r_2 is the radius of uniform convexity. The results are best possible. For $\alpha = 0$ the radius of convexity can be obtained, it is solution of

$$(C+1)r + C\log(1-r) = 0$$

The radius for $g \in \mathcal{A}$ with $a_2 = 0$ can also be obtained easily.

We will now obtain sharp radii for the class of functions in $\mathcal{L}(\alpha,\beta)$ and $\mathcal{S}^*[A,B]$ for $g \in \mathcal{A}_b$ under the same bounds for coefficients $|a_n| \le C/n^3$.

Theorem 2.2. *Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \ge 0$. Then the radius of the class $\mathcal{L}(\alpha,\beta)$ for functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \le C/n^3, (n \ge 3)$, is $t_0 \in (0,1)$, where t_0 is the root of equation*

$$\begin{aligned} &C\beta t^3 + \big(8\big(4\alpha + 4 - 2\beta\big)b + 2C(\alpha-3)\big)\,t^2 - 8\big(2C - \alpha C - C\beta + |1-\beta|\big)t \\ &\qquad - 8C\alpha\log(1-t) + 8C(1-\alpha)\mathrm{Li}_2(t) - 8C\beta\,\mathrm{Li}_3(t) = 0. \end{aligned} \tag{2.4}$$

For $\beta < 1$, the number $\mathcal{L}_0(\alpha,\beta)$ $\Big(= \mathcal{S}^*_\beta = \mathcal{S}^*[1-\beta, 0]\Big)$ radius of $g \in \mathcal{A}_b$

$$\mathcal{L}_0(\alpha,\beta) = \{g \in \mathcal{A}_b : |\alpha \frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)} - 1| \le |1-\beta|\}$$

radius of $g \in \mathcal{A}_b$. The radius is best possible.

Proof. The real number $t_0 \in (0,1)$ is the $\mathcal{L}(\alpha,\beta)$ radius of the function $g \in \mathcal{A}_b$ if and only if $g(t_0 z)/t_0 \in \mathcal{L}(\alpha,\beta)$. Thus, from Lemma 1.1, it is sufficient to check for the inequality

$$\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| t_0^{n-1} \le |1-\beta|.$$

For $g \in \mathcal{A}_b$, we consider the left-hand side:

$$\begin{aligned} &\textstyle\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| t_0^{n-1} \\ &\le (4\alpha + 2(1-\alpha) - \beta)|a_2| t_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)\frac{C}{n^3} t_0^{n-1} \\ &= 2(2\alpha + 2 - \beta) b t_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)\frac{C}{n^3} t_0^{n-1} \\ &= 2(2\alpha + 2 - \beta) b t_0 + C\left[\sum_{n=3}^{\infty} \frac{\alpha t_0^{n-1}}{n} + (1-\alpha)\sum_{n=3}^{\infty} \frac{t_0^{n-1}}{n^2} - \beta \sum_{n=3}^{\infty} \frac{t_0^{n-1}}{n^3}\right] \\ &= 2(2\alpha + 2 - \beta) b t_0 + C\alpha\left(-\frac{\log(1-t_0)}{t_0} - 1 - \frac{t_0}{2}\right) \\ &\quad + (1-\alpha)C\left(\frac{\mathrm{Li}_2(t_0)}{t_0} - \frac{t_0}{4} - 1\right) - C\beta\left(\frac{\mathrm{Li}_3(t_0)}{t_0} - \frac{t_0^2}{8} - 1\right) = |1-\beta| \\ &\Rightarrow\ C\beta t_0^3 + ((32\alpha + 32 - 16\beta)b - 6C + 2C\alpha)t^2 + (-8C - 8C + 8\alpha C + 8C\beta - 8|1-\beta|)\,t_0 \\ &\quad -8C\alpha\log(1-t_0) + 8C(1-\alpha)\mathrm{Li}_2(t_0) - 8C\beta\mathrm{Li}_3(t_0) = 0 \\ &\Rightarrow\ C\beta t_0^3 + (8(4\alpha + 4 - 2\beta)b + 2C(\alpha-3))t_0^2 - 8(2C - C\alpha - M\beta + |1-\beta|)t_0 \\ &\quad -8C\alpha\log(1-t_0) + 8C(1-\alpha)\mathrm{Li}_2(t_0) - 8C\beta\mathrm{Li}_3(t_0) = 0. \end{aligned}$$

For $\beta < 1$, consider the normalised function as

$$g_0(z) = z - 2pz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n$$

at the point $z = t_0$, where $t_0 \in (0,1)$ is the root of equation (2.4), g_0 satisfies

$$\mathrm{Re}\left(\frac{\alpha z^2 g_0''(z)}{g_0(z)} + \frac{z g_0'(z)}{g_0(z)}\right) = \beta.$$

Hence it is best possible radius. For $\beta > 1$, the sharpness will hold for the function

$$g_0(z) = z + 2bz^2 + C\left(\mathrm{Li}_3(z) - z - \frac{z^2}{8}\right).$$

□

Corollary 2.4. *For $\alpha = 0$ and $\beta < 1$, the result of the previous Theorem 2.1 can be obtained. Other radii for $g \in \mathcal{A}$ and $a_2 = 0$ can also be determined.*

In the following result we will evaluate the sharp $\mathcal{S}^*[A,B]$ radius of $g \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$.

Theorem 2.3. *For $-1 \leq B < A \leq 1$. The $\mathcal{S}^*[A,B]$ radius for $g \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$ is the real root $r_0 \in (0,1)$ of the equation*

$$(16b - C)(1 - 2B + A)r^2 - 8(C+1)(A-B)r + 8C(\mathrm{Li}_2(r)(1-B) - \mathrm{Li}_3(r)(1-A)) = 0 \qquad (2.3)$$

The radius cannot be improved.

Proof. r_0 will be the $\mathcal{S}^*[A,B]$ radius of $g \in \mathcal{A}_b$ iff $g(r_0 z)/r_0 \in \mathcal{S}^*[A,B]$. Hence by Lemma 1.2 it suffices to show that

$$\sum_{n=2}^{\infty}\big((1-B)n - (1-A)\big)|a_n| r_0^{n-1} \leq A - B, \quad (-1 \leq B < A \leq 1),$$

where r_0 is the root in $(0,1)$ of eq.(2.3)

For function $g \in \mathcal{A}_b$, it follows that

$\sum_{n=2}^{\infty}\big((1-B)k - (1-A)\big)|a_n| r_0^{n-1}$

$\leq 2\big((1-B)2 - (1-A)\big)br_0 + \sum_{n=3}^{\infty}(1-B)k\frac{Cr_0^{n-1}}{n^3} - \sum_{n=3}^{\infty}(1-A)\frac{Cr_0^{n-1}}{n^3}$

$= 2\big(2(1-B) - (1-A)\big)br_0 + \frac{C(1-B)}{r_0}\left[\mathrm{Li}_2(r_0) - r_0 - \frac{r_0^2}{4}\right] - \frac{C(1-A)}{r_0}\left[\mathrm{Li}_3(r_0) - r_0 - \frac{r_0^2}{8}\right]$

$= 2(1 - 2B + A)br_0 - C(1 - B - 1 + A) - \frac{r_0}{8}M(1 - 2B + A)$

$+\frac{C}{r_0}(\mathrm{Li}_2(r_0)(1-B) - \mathrm{Li}_3(r_0)(1-A))$

$= A - B$

$$\Rightarrow (1 - 2B + A)r_0\left(2b - \frac{C}{8}\right) - (C-1)(A-B) + \frac{C}{r_0}\left((1-B)\,\mathrm{Li}_2(r_0) - (1-A)\,\mathrm{Li}_3(r_0)\right) = 0.$$

The function

$$g_0(z) = z - 2bz^2 - C\left(\mathrm{Li}_3(w) - w - \frac{w^2}{8}\right)$$

shows that the result is best possible and cannot be improved. □

Remark 2.1. *The $\mathcal{S}^*_\alpha$ radius for $g \in \mathcal{A}$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$ can be obtained by putting $A = 1 - 2\alpha; B = -1$ in the $\mathcal{S}^*[A,B]$ radius or $\alpha = 0$ in $\mathcal{L}(\alpha,\beta)$ radius respectively.*

3 Radius Constants for $|a_n| \le cn + d/n$.

We will investigate radii constants for functions $g \in \mathcal{A}_b$ with the coefficient inequality $|a_n| \le cn + d/n$ in the present section.

Theorem 3.1. *Let $|a_n| \le cn + d/n, (n \ge 3)$ for the function $g \in \mathcal{A}_b$. Then the following results hold:*

(i) $\mathrm{Re}(g'(z) + zg''(z)) > 0$, *where s_0 is the least real number in $(0,1)$ satisfying the equation*

$$[(1+c+d) + 2b(4c+d-4b)](1-s)^4 - d(1-s)^2 - c(1+4s+s^2) = 0. \tag{3.1}$$

The result cannot be improved and is sharp.

(ii) $g \in \mathcal{S}^*_\alpha$ where $s_1 = s_1(\alpha)$ is the root lying in $(0,1)$ of the equation

$$s\left(2(b-c)(2-\alpha) + \frac{\alpha d}{2}\right) + \alpha(1+c+d) - (1+c)$$
$$+\frac{c(1+s)}{(1-s)^3} - \frac{\alpha c}{(1-s)^2} + \frac{ds^2}{1-s} + \frac{\alpha d \log(1-s)}{s} = 0. \tag{3.2}$$

The root $s_1(\alpha)$ is also provides with starlikeness radius of order α, and $s_1(1/2)$ represents the radius for parabolic starlikeness of the function. The results are sharp.

(iii) $g \in \mathcal{A}_b$ and $|a_n| \le cn + d/n$ for $n \ge 3$, then the following condition will hold good for g,

$$\left|\frac{zg''(z)}{g'(z)}\right| \le 1-\alpha,$$

if $s_2 = s_2(\alpha) \in (0,1)$ is the root of the equation

$$\begin{aligned}&((1-\alpha)(c+1) + d - 2s(2(p-c)(2-\alpha)(1-s)4 - d))\\ &= d(1-\alpha s^2 + \alpha s^3)(1-s)^2 + c\big(s^2(1+\alpha) + 4s + 1 - \alpha\big).\end{aligned} \tag{3.3}$$

The number $s_2(\alpha)$ is also the radius of convexity of order α. The number $s_2(1/2)$ is the radius of uniform convexity of the given functions. The results are all sharp.

Proof. **(i).** For $g \in \mathcal{A}_b$, consider

$$g'(z) + zg''(z) = 1 + 8bz + c\sum_{n=3}^{\infty} n^3 z^{n-1} + n\sum_{n=3}^{\infty} nz^{n-1}.$$

$$\mathrm{Re}(g'(z) + zg''(z)) \ge 1 - 8bs - c\sum_{n=3}^{\infty} n^3 s^{n-1} - d\sum_{n=3}^{\infty} ns^{n-1}$$

$$= 1 - 8bs - c\left(\frac{1+4s+s^2}{(1-s)^4} - 1 - 8s\right) - d\left(\frac{1}{(1-s)^2} - 1 - 2s\right).$$

After simplification,

$$= (1+c+d) + s(8c+2d-8b) - \frac{d}{(1-s)^2} - \frac{c(1+4s+s^2)}{(1-s)^4}.$$

Let $s \in (0,1)$ be the root of

$$((1+c+d) + 2s(4c+d-4b))(1-s)^4 - d(1-s)^2 - c(1+4s+s^2) = 0.$$

Then $\mathrm{Re}(g'(z) + zg''(z)) \ge 0$, and this provides us with the result that g is a starlike function. Consider the function

$$g_0(z) = z - 2bz^2 - \sum_{k=3}^{\infty}\left(cn + \frac{d}{n}\right)z^n.$$

$$\begin{aligned} g_0(z) &= z - 2bz^2 - c\left(\frac{z}{(1-z)^2} - z - 2z^2\right) - d\left(-\log(1-z) - z - \frac{z^2}{2}\right)\\ &= z - 2bz^2 - \frac{cz}{(1-z)^2} + cz + 2cz^2 + d\log(1-z) + dz + \frac{dz^2}{2}\\ &= z(1+c+d) + z^2\left(2c - 2b + \frac{d}{2}\right) - \frac{cz}{(1-z)^2} + d\log(1-z)\end{aligned}$$

$$\Rightarrow g_0'(z) + zg_0''(z) = (1 + c + d) + z(8c - 8b + 2d) - \frac{c(1 + 4z + z^2)}{(1 - z)^4} - \frac{d}{(1 - z)^2}.$$

At the point $z = s_0$, where $s_0 \in (0, 1)$ is the root for the equation (3.1), then g_0 satisfies

$$\mathrm{Re}(zg_0''(z) + g_0'(z)) \geq 0.$$

Thus s_0 is the sharp radius for $g \in \mathcal{A}_b$.

(ii) For $g \in \mathcal{A}_b$ and $|a_n| \leq cn + d/n$, $n \geq 3$, we write

$$\begin{aligned} |zg'(z) - g(z)| - (1 - \alpha)|g(z)| &\leq \sum_{n=2}^{\infty}(n - 1)|a_n||z|^n - (1 - \alpha)\left(|z| - \sum_{n=2}^{\infty}|a_n||z|^n\right) \\ &= -(1 - \alpha)|z| + \sum_{n=2}^{\infty}(n - \alpha)|a_n||z|^n \\ &\leq -(1 - \alpha)s + \sum_{n=2}^{\infty}(n - \alpha)|a_n|s^n \leq 0. \end{aligned} \tag{3.4}$$

Using $|a_n| \leq cn + d/n$ and the identities mentioned below,

$$\sum_{n=3}^{\infty} k^2 s^{n-1} = \frac{1 + s}{(1 - s)^3} - 1 - 4s, \qquad \sum_{n=3}^{\infty} s^n = \frac{s^3}{1 - s},$$

$$\sum_{n=3}^{\infty} ns^{n-1} = \frac{1}{(1 - s)^2} - 1 - 2s, \qquad \sum_{n=3}^{\infty} \frac{s^n}{n} = -\log(1 - s) - s - \frac{s^2}{2},$$

inequality (3.4) gives

$$s_1\left(2(b - c)(2 - \alpha) + \frac{\alpha d}{2}\right) + (\alpha d - 1 + \alpha) - c(1 - \alpha) + \frac{c(1 + s_1)}{(1 - s_1)^3} + \frac{ds_1^2}{1 - s_1} - \frac{\alpha c}{(1 - s_1)^2} + \frac{\alpha d \log(1 - s_1)}{s_1} = 0,$$

which is the required radius $s_1 = s_1(\alpha)$.

This result is sharp for the function g_0 given by

$$\begin{aligned} g_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty}\left(cn + \frac{d}{n}\right) z^n \\ &= z(1 + c + d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1 - z) - \frac{cz}{(1 - z)^2}. \end{aligned}$$

A calculation shows

$$\frac{zg_0'(z)}{g_0(z)} - 1 = \frac{N(z)}{D(z)},$$

where

$$N(z) = z^2\left(2c - 2b + \frac{d}{2}\right) - \frac{dz}{1 - z} + \frac{cz}{(1 - z)^2} - \frac{c(1 + z)z}{(1 - z)^3} - d\log(1 - z)$$

and

$$D(z) = z(1 + c + d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1 - z) - \frac{cz}{(1 - z)^2}.$$

For $z = s_1$, where $s_1 \in (0, 1)$ is the root,

$$\left|\frac{zg_0'(z)}{g_0(z)} - 1\right| = 1 - \alpha.$$

Hence the result is sharp.

(iii) for $|a_2| = 2b$ and $|a_n| \leq cn + d/n$. Using the known series sum for $\sum_{n=2}^{\infty} n^i s^n (i = 0, 1, 2, 3)$ consider

$$\sum_{n=2}^{\infty} n(n - \alpha)|a_n||z|^{n-1} \leq \sum_{n=2}^{\infty} n(n - \alpha)|a_n|s^{n-1}$$

$$\begin{aligned}
&\leq 4b(2-\alpha)s + c\sum_{n=3}^{\infty} n^3 s^{n-1} - \alpha c\sum_{n=3}^{\infty} n^2 s^{n-1} + d\sum_{n=3}^{\infty} n s^{n-1} - \alpha d\sum_{n=2}^{\infty} s^{n-1}\\
&= 4b(2-\alpha)s + c\left(\frac{1+4s+s^2}{(1-s)^4} - 1 - 8s\right) - \alpha c\left(\frac{1+s}{(1-s)^3} - 1 - 4s\right)\\
&\quad + d\left(\frac{1}{(1-s)^2} - 1 - 2s\right) - \frac{\alpha d s^2}{1-s}\\
&= 1-\alpha.
\end{aligned}$$

The above equation (3.3) can be obtained. The result can not be improved further, as the function,

$$g_0(z) = z(1+c+d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1-z) - \frac{cz}{(1-z)^2}.$$

verifies required sharpness. □

Corollary 3.1. *(i)(a). The radius for starlikeness of functions with coefficients satisfying $a_2 = 0$ and $|a_n| \leq cn + d/n$ for all $n \geq 3$ is the real number in $(0,1)$ and root of the equation*

$$(1+c+d) + 2s(4c+d)(1-s)^4 = d(1-s)^2 + c(1+4s+s^2).$$

(i)(b). The starlikness radius s_1 of functions is such that $g(s_1 z)/s_1 \in \mathcal{A}$ such that $|a_n| \leq cn + d/n, (n \geq 2)$

$$(1+c+d)(1-s)^4 = d(1-s)^2 + c(1+4s+s^2).$$

Corollary 3.2. *If $g \in \mathcal{A}$, $|a_n| \leq cn + d/n, (n \geq 2)$, then the starlikeness radius s_1 of order α satisfies*

$$ds_1 + \alpha(1+c+d) - (1+c) + \frac{c(1+s_1)}{(1-s_1)^3} - \frac{\alpha c}{(1-s_1)^2} + \frac{ds_1^2}{1-s_1} + \frac{\alpha d\log(1-s_1)}{s_1} = 0.$$

For $\alpha = 0$, $g \in \mathcal{A}$, s_1 provides with radius of starlikeness as

$$\frac{ds_1}{1-s_1} - (1+c) + \frac{c(1+s_1)}{(1-s_1)^3} = 0.$$

Corollary 3.3. *For $g \in \mathcal{A}$ and $|a_n| \leq cn + d/n, n \geq 2$. The radius of convexity of order α is given by $s(\alpha)$, which is the least real root in $(0,1)$ satisfying the equation*

$$\big((1-\alpha)(1+c)d) + d\alpha s\big)(1-s)^4 = c\left(1 - \alpha s^2(1-s)\right)(1-s)^2 - c\big(s^2(1+\alpha) + 4s + 1 - \alpha\big).$$

$s(1/2)$ will be the radius of uniform convexity of the given function. The results are sharp. $s(0)$ will be the radius of convexity. The radii for the case $g \in \mathcal{A}$ and $a_2 = 0$ can be easily obtained from the above equation.

In the next theorem we obtain sharp $\mathcal{L}(\alpha,\beta)$ and $\mathcal{S}^*[A,B]$ radii of functions $g \in \mathcal{A}_b$ satisfying the above coefficient inequality.

Theorem 3.2. *Let $\beta \in \mathbb{R}\setminus\{1\}$ and $\alpha \geq 0$. The $\mathcal{L}(\alpha,\beta)$ radius of $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfy the coefficient inequality $|a_n| \leq ck + d/n, n \geq 3$, where $c, d \geq 0$ is the real number in $(0,1)$ satisfying the equation.*

$$\begin{aligned}
Pt^2(1-t)^4 + Qt(1-t)^4 + R(1-t)^4 + d(1-\alpha)t^3(1-t)^3 + (\alpha d - c\beta)t(1-t)^2&\\
+c(1-\alpha)t(1-t)^2 + c\alpha t(1+4t+t^2) = 0,& \qquad (3.4)
\end{aligned}$$

where P, Q, R are given as

$$\begin{aligned}
P &= 2\left(b(2\alpha+2-\beta) - 2c(\alpha+1) - \alpha d + c\beta\frac{\beta d}{4}\right)\\
Q &= \beta(c+d) - \alpha d - c - |1-\beta|\\
R &= \beta d\log(1-t).
\end{aligned}$$

For $\beta < 1$, this number is also $\mathcal{L}_0(\alpha,\beta)$radius of $g \in \mathcal{A}_b$. The radii's are all best possible and cannot be improved further.

Proof. Let t_0 be the $\mathcal{L}(\alpha,\beta)$radius of the function $g \in \mathcal{A}_b$. Then the inequality

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n|t^{n-1} \le |1-\beta|$$

holds for the root of equation (3.4) lying in $(0,1), g \in \mathcal{A}_b$ implies that

$$\begin{aligned}
\sum_{n=2}^{\infty}(\alpha n^2+(1-\alpha)k-\beta)|a_n|t^{n-1} &\le 2(2\alpha+2-\beta)bt+\sum_{n=3}^{\infty}(\alpha n^2+(1-\alpha)k-\beta)\Big(cn+\frac{d}{n}\Big)t^{n-1}\\
&= 2(2\alpha+2-\beta)bt + c\alpha\sum_{n=3}^{\infty}n^3t^{n-1}+c(1-\alpha)\sum_{n=3}^{\infty}k^2t^{n-1}-c\beta\sum_{n=3}^{\infty}nt^{n-1}\\
&\quad +\alpha d\sum_{n=3}^{\infty}kt^{n-1}+(1-\alpha)d\sum_{n=3}^{\infty}t^{n-1}-\beta d\sum_{n=3}^{\infty}\frac{t^{n-1}}{n}\\
&= 2(2\alpha+2-\beta)bt+c\alpha\Big(\frac{1+4t+t^2}{(1-t)^4}-1-8t\Big)+c(1-\alpha)\Big(\frac{1+t}{(1-t)^3}-1-4t\Big)\\
&\quad +(\alpha d-c\beta)\Big(\frac{1}{(1-t)^2}-1-2t\Big)+(1-\alpha)d\frac{t^2}{1-t}-\beta d\Big(\frac{-\log(1-t)}{t}-1-\frac{t}{2}\Big)
\end{aligned}$$

$$\begin{aligned}
&= 2t\Big(b(2\alpha+2-\beta)-4c\alpha-2c(1-\alpha)-(\alpha d-c\beta)+\frac{\beta d}{4}\Big)+(-c\alpha-c+c\alpha-\alpha d+c\beta+\beta d)\\
&\quad +\beta d\frac{\log(1-t)}{t}+d(1-\alpha)\frac{t^2}{1-t}+\frac{\alpha d-c\beta}{(1-t)^2}+\frac{c(1-\alpha)(1+t)}{(1-t)^3}+\frac{c\alpha(1+4t+t^2)}{(1-t)^4}=|1-\beta|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & 2t^2\big(b(2\alpha+2-\beta)-2c\alpha-2c-\alpha d+c\beta+\tfrac{\beta d}{4}\big)(1-t)^4+(\beta(c+d)-\alpha d-c)t(1-t)^4\\
&+\beta d(1-t)^4\log(1-t)+d(1-\alpha)t^3(1-t)^3+(\alpha d-c\beta)t(1-t)^2\\
&+c(1-\alpha)t(1-t^2)+c\alpha t(1+4t+t^2)=|1-\beta|\,t(1-t)^4.
\end{aligned}$$

For $\beta<1$, sharpness can be obtained for the function

$$\begin{aligned}
g_0(z) &= z-2bz^2-\sum_{k=3}^{\infty}\left(cn+\frac{d}{n}\right)z^n\\
&= z(1+c+d)+z^2\left(2c-2b+\frac{d}{2}\right)+d\log(1-z)-\frac{cz}{(1-z)^2}.
\end{aligned}$$

At $z=t_0$, where t_0 is root of eq (3.4) in $(0,1)$, g_0 satisfy

$$\mathrm{Re}\bigg(\alpha z^2\frac{g_0''(z)}{g_0(z)}+z\frac{g_0'(z)}{g_0(z)}\bigg)>0.$$

The $\mathcal{S}^*[A,B]$ radius can also be easily obtained and the required radius for Janowski class is given by

$$\begin{aligned}
& t\Big(2(c-b)(2B-A-1)+(1-A)\frac{d}{2}\Big)+\big((B+A)+d(1-A)\big)\\
& +\frac{(1-B)}{(1-t)^3}\Big(c(1+t)+dt^2(1-t^2)\Big)+\frac{(1-A)d\log(1-t)}{t}=0.
\end{aligned}$$

□

Remark 3.1. *For $c=0$, we have $|a_n|\le d/n$, and $d=0$ implies $|a_n|\le cn$. We are able to obtain $\mathcal{L}(\alpha,\beta)$ radius of functions in both cases which were obtained by Ali* et.al. *[2] for $|a_n|\le cn$ and $|a_n|\le c/n$, for which sharpness results were obtained by Ravichandran [25]. Thus the particular cases of these inequalities are obtained easily by proper substitution.*

Conclusion.

The purpose of this study was to detremine and verify the shard radii of univalence, starlikeness and convexity of analytic function defined on an open disc. Upon investigating the functions in class $\mathcal{A}_b$ with a fixed initial coefficient $|a_n| = 2b$ $(0 \leq b \leq 1)$, we have calculated the radius constants for $|a_n| \leq C/n^3$ and $|a_n| \leq cn+d/n$. Other classes like Janowsi starlike functions $\mathcal{S}^*[A, B]$ and the class $\mathcal{L}(\alpha, \beta)$ are also investigated. The reseachers can obtain the radius for other classes under the same coefficient bounds or can explore for other coefficient bounds as well.

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SET-VALUED OPTIMIZATION PROBLEMS WITH HIGHER-ORDER (α, β)-δ-CONE CONVEXITY

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Abstract

In this paper, we establish the higher-order sufficient Karush-Kuhn-Tucker *(KKT)* optimality conditions for set-valued optimization problems under the higher-order contingent epiderivative and higher-order (α, β)-δ-cone convexity assumptions. We also study the higher-order Mond-Weir (MWD) type of duality and establish the corresponding higher-order strong, weak and converse duality theorems in between the primal (P) and corresponding dual problem. An example is also developed to demonstrate that higher-order (α, β)-δ-cone convexity is more generalized than higher-order cone convexity.

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Keywords and Phrases: Convex cone; Set-valued map; Contingent epiderivative; Duality; Arcwisely connectedness.

1 Introduction

Convex analysis plays a vital role when investigating the vector optimization problems, or *VOPs* for short. Several generalized convexity concepts have been proposed to relax the convexity assumption. The notion of invexity in *VOPs* was first introduced by Craven [9], Ben-Israel and Mond [5], Hanson and Mond [19], and numerous others. Weir and Mond [32] and Weir and Jeyakumar [31] introduced preinvexity of scalar-valued maps as a generalization of invexity.

Zalmai [33] introduced the idea of δ-(η, θ)-invexity as a further generalization of invexity. Another type of additional development of invexity is the concept of (α, β)-invexity, which was introduced by Antczak [1, 2]. He presented the optimality conditions required for multiobjective programming problems and used the (α, β)-invexity assumptions to deduce the duality results. The idea of (α, β)-δ-(η, θ)-invexity in *VOPs* was introduced by Mandal and Nahak [26]. They employed the (α, β)-δ-(η, θ)-invexity assumption to establish the required optimality conditions and the Mond-Weir type duality results.

In recent years, a lot of authors have been interested in set-valued optimization problems (*SVOPs* for short), in which the functions related to constraints and the objective function are set-valued mappings (*SVMs* for short). Anbin and Frankowska [4] were the first to introduce the idea of contingent derivative of *SVMs*. Corley [8] applied contingent derivatives to develop the generalized necessary optimality conditions of Fritz John type for the existence of maximum elements of *SVOPs* in 1987. Additionally, he explored the assumption of cone concavity to construct the generalized Fritz John sufficient optimality conditions for *SVOPs*.

Subsequently, in 1991, Sach and Craven [28, 29] established Mond-Weir type duality theorems and developed the concept of invex *SVMs*. Luc and Malivert [25] coined the term "invexity" in set-valued optimization theory. The necessary and sufficient optimality conditions of *SVOPs* have also been investigated using contingent derivatives. The generalized Farkas-Minkowski type theorem for convex *SVMs* and the Mond-Weir and Wolfe types duality results were later derived by Lin [24]. Bhatia and Mehra [6] introduced the concept of preinvex *SVMs*. As an additional notion of differentiability of *SVMs*, Jahn and Rauh [20] proposed the concept of contingent epiderivative.

Sheng and Liu [30] used generalized contingent epiderivative assumptions and the concept of preinvexity to study the *KKT* conditions of *SVOPs*. The existence, uniqueness, and characteristics of contingent epiderivative were studied by Rodriguez-Marin and Sama [27]. Using a higher-order contingent derivative, Li *et al.* [22, 23] established the necessary and sufficient optimality conditions. They also investigated the duality theorems using convexity assumptions and developed the Mond-Weir dual of higher-order for *SVOP*.

In this paper, we propose the concept of higher-order (α, β)-δ-cone convexity of *SVMs* as a generalization of cone convex *SVMs* of higher-order. For $\alpha, \beta, \delta = 0$, we get the standard notion of higher-order cone convexity of *SVMs*. Additionally, we construct a higher-order (α, β)-δ-cone convex *SVM* example that is not cone convex of higher-order.

Our main goal is to determine the higher-order sufficient optimality conditions of the *SVOP* (P) in more generalized cases by applying higher-order generalized contingent epiderivative and higher-order (α, β)-δ-cone convexity assumptions on the constraints and objective functions. We also study the higher-order strong, weak and converse duality theorems of the Mond-Weir (MWD) type via higher-order (α, β)-δ-cone convexity assumption.

2 Definitions and preliminaries

Let S be a real normed space (in short, *RNS*) and $\emptyset \neq D \subseteq S$. Then D is said to be a cone if $\lambda b \in D$, for all $b \in D$ and $\lambda \geq 0$. Additionally, D is said to be proper if $D \neq S$, pointed if $D \cap (-D) = \{\theta_S\}$, solid if $\text{int}(D) \neq \emptyset$, closed if $\overline{D} = D$, and convex if $\lambda D + (1-\lambda)D \subseteq D$, for all $\lambda \in [0,1]$, where $\text{int}(D)$ and $\overline{D}$ represent the interior and closure of D, respectively and θ_S denotes the zero of S.

Let us consider the non-negative orthant $\mathbb{R}^j_+$ of j-dimensional Euclidean space $\mathbb{R}^j$ by

$$\mathbb{R}^j_+ = \{b = (b_1, ..., b_j) \in \mathbb{R}^j : b_i \geq 0, \forall i = 1, 2, ..., j\}.$$

Then $\mathbb{R}^j_+$ is a solid as well as pointed closed convex cone and $\text{int}(\mathbb{R}^j_+) \cup \{0_{\mathbb{R}^j}\}$ is a solid as well as pointed convex cone in $\mathbb{R}^j$, where $0_{\mathbb{R}^j}$ denotes the zero element of $\mathbb{R}^j$.

With respect to (w.r.t.) $\mathbb{R}^j_+$, there exist mainly two types of cone-orders in $\mathbb{R}^j$. For $b_1, b_2 \in \mathbb{R}^j$,

$$b_1 \leq b_2 \text{ if } b_2 - b_1 \in \mathbb{R}^j_+$$

and

$$b_1 < b_2 \text{ if } b_2 - b_1 \in \text{int}(\mathbb{R}^j_+).$$

We say $b_2 \geq b_1$, if $b_1 \leq b_2$ and $b_2 > b_1$, if $b_1 < b_2$. For $\emptyset \neq B, B' \subseteq \mathbb{R}^j$,

$$B \leq 0_{\mathbb{R}^j} \iff b \leq 0_{\mathbb{R}^j}, \forall b \in B,$$
$$B < 0_{\mathbb{R}^j} \iff b < 0_{\mathbb{R}^j}, \forall b \in B,$$
$$B \leq B' \iff b \leq b', \forall b \in B \text{ and } \forall b' \in B',$$

and

$$B < B' \iff b < b', \forall b \in B \text{ and } \forall b' \in B'.$$

For $\emptyset \neq B, B' \subseteq \mathbb{R}^j$ and $b^*, b'^* \in \mathbb{R}^j$, define

$$b^*B + b'^*B' = \bigcup_{\substack{b \in B \\ b' \in B'}} \{b^*b + b'^*b'\}.$$

The ordering of two subsets of $\mathbb{R}^j$ w.r.t. $\mathbb{R}^j_+$ is defined as

$$B \geq B' \iff b \geq b', \text{ for all } b \in B \text{ and } b' \in B'.$$

The most commonly used minimality notions w.r.t. a solid as well as pointed convex cone D in a *RNS* S are as follows.

Definition 2.1. *Let $\emptyset \neq B \subseteq S$. The weakly minimal, minimal, and ideal minimal points of B are defined as*

(i) *$b' \in B$ is said to be a weakly minimal point of B if there exists no $b \in B$, satisfying $b < b'$.*
(ii) *$b' \in B$ is said to be a minimal point of B if there exists no $b \in B \setminus \{b'\}$, satisfying $b \leq b'$.*
(iii) *$b' \in B$ is said to be an ideal minimal point of B if $b' \leq b$, for all $b \in B$.*

The sets of weakly minimal, minimal, and ideal minimal points of B as denoted by w-min(B), min(B), and I-min(B), respectively and are being described as

$$\text{w-min}(B) = \{b' \in B : (b' - \text{int}(D)) \cap B = \emptyset\},$$

$$\min(B) = \{b' \in B : (b' - D) \cap B = \{b'\}\},$$

and

$$\text{I-min}(B) = \{b' \in B : B \subseteq \{b'\} + D\}.$$

Let $b = (b_1, ..., b_j) \in \mathbb{R}^j$. The logarithm and exponential of b are defined w.r.t. $\mathbb{R}^j_+$ based on componentwise as

$$\log b = (\log b_1, ..., \log b_j)^T, \text{ for } b > 0_{\mathbb{R}^j}$$

and

$$e^b = (e^{b_1}, ..., e^{b_j})^T, \text{ for any } b.$$

Let $\emptyset \neq B \subseteq \mathbb{R}^j$. Additionally, let us define two sets $\log B$ and e^B w.r.t. $\mathbb{R}^j_+$ as

$$\log B = \{\log b : b \in B\}, \text{ for } B > 0_{\mathbb{R}^j}$$

and

$$e^B = \{e^b : b \in B\}, \text{ for any } B.$$

Likewise, we may define the element $b^{\frac{1}{\alpha}}$ and the set $B^{\frac{1}{\alpha}}$ for any nonzero real number α.

Aubin and Frankowska [3, 4] introduced the notion of higher-order contingent set in set-valued optimization theory.

Definition 2.2. ([3, 4]). *Suppose that S is a RNS and $\emptyset \neq B \subseteq S$, $b' \in \overline{B}$, $k \in \mathbb{N}$, in addition to $k \geq 2$, and $b_1, ..., b_{k-1} \in S$. Then the contingent set of k-th order $T^{(k)}(B, b', b_1, ..., b_{k-1})$ to B at $(b', b_1, ..., b_{k-1})$ is defined as*

$b \in T^{(k)}(B, b', b_1, ..., b_{k-1})$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$ and $\{b_n\}$ in S, in addition to $\lambda_n \to 0^+$ and $b_n \to b$, so that

$$b' + \lambda_n b_1 + ... + {\lambda_n}^{k-1} b_{k-1} + {\lambda_n}^k b_n \in B, \forall n \in \mathbb{N},$$

equivalently, $b \in T^{(k)}(B, b', b_1, ..., b_{k-1})$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$ and $\{b'_n\}$ in S, in addition to $\lambda_n \to 0^+$, $b'_n \in B$, and $b'_n \to b'$, so that

$$\frac{b'_n - b' - \lambda_n b_1 - ... - {\lambda_n}^{k-1} b_{k-1}}{{\lambda_n}^k} \to b, \text{ as } n \to \infty.$$

Let R, S be *RNSs*, 2^S be the set of all subsets of S, and D be a solid as well as pointed convex cone in S. Let $F : R \to 2^S$ be a *SVM* from R to S, i.e., $F(a) \subseteq S$, $\forall a \in R$. The domain, image, graph, and epigraph of F are defined by

$$\text{Dom}(F) = \{a \in R : F(a) \neq \emptyset\},$$

$$F(A) = \bigcup_{a \in A} \{F(a)\}, \text{ for any } \emptyset \neq A \subseteq R,$$

$$\text{Gr}(F) = \{(a, b) \in R \times S : b \in F(a)\},$$

and

$$\text{Epi}(F) = \{(a, b) \in R \times S : b \in F(a) + D\}.$$

The concept of higher-order contingent derivative of *SVMs* was developed by Aubin and Frankowska [4].

Definition 2.3. ([4]). *Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, R, S are RNSs, $F : R \to 2^S$ is a SVM along with* $\mathrm{Dom}(F) = R$, *$(a', b') \in \mathrm{Gr}(F)$, and $(a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$. Then the contingent derivative of k-th order $d^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})$ of F at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ is the SVM from R to S defined by*

$$\mathrm{Gr}(d^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})) \\ = T^{(k)}(\mathrm{Gr}(F), (a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1})).$$

Let R, S be *RNSs*, D be a solid as well as pointed convex cone in S, and $F : R \to 2^S$ be a *SVM*. Let us consider a *SVM* $F + D : R \to 2^S$ by

$$(F + D)(a) = F(a) + D, \forall a \in \mathrm{Dom}(F).$$

The concept of higher-order generalized contingent epi-derivative of *SVMs* was developed by Li and Chen [21].

Definition 2.4. ([21]). *Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, R, S are RNSs, $F : R \to 2^S$ is a SVM along with* $\mathrm{Dom}(F) = R$, *$(a', b') \in \mathrm{Gr}(F)$, and $(a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$. Then the generalized contingent epi-derivative of k-th order of F at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$, denoted by $\overrightarrow{d}_g^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})$, is the SVM from R to S defined by*

$$\overrightarrow{d}_g^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})(a) \\ = \min\{b \in B : (a, b) \in T^{(k)}(\mathrm{Epi}(F), (a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1}))\}, \\ a \in \mathrm{Dom}(d^{(k)}(F + D)(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})).$$

The concept of cone convexity of *SVMs* was developed by Borwein [7].

Definition 2.5. ([7]). *Let A be a nonempty convex subset of a RNS R. A SVM $F : R \to 2^S$, with $A \subseteq \mathrm{Dom}(F)$, is said to be D-convex on A if $\forall a_1, a_2 \in A$ and $\lambda \in [0, 1]$,*

$$\lambda F(a_1) + (1 - \lambda)F(a_2) \subseteq F(\lambda a_1 + (1 - \lambda)a_2) + D.$$

The following result was developed by Li *et al.* [23] in terms of the higher-order contingent derivative of *SVMs*.

Proposition 2.1. ([23]). *Suppose that R, S are RNSs and F is D-convex on a nonempty convex subset A of R, then $\forall a, a' \in A$ and $\forall b' \in F(a')$,*

$$F(a) - b' \subseteq d^{(k)}F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')(a - a'),$$

where $a_1, ..., a_{k-1} \in A$, $b_1 \in F(a_1) + D, ..., b_{k-1} \in F(a_{k-1}) + D$.

Das and Nahak [15] developed the notion of higher-order δ-cone convexity of *SVMs*.

Suppose that $F : R \to 2^S$ is a *SVM*, in addition to $A \subseteq \mathrm{Dom}(F)$. Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, $(a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$ in addition to $a', a_1, ..., a_{k-1} \in A$, $b' \in F(a')$, $b_1 \in F(a_1) + D, ..., b_{k-1} \in F(a_{k-1}) + D$.

Definition 2.6. ([15]). *Suppose that R, S are RNSs, $\emptyset \neq A \subseteq R$, D is a pointed solid convex cone of S, $\delta \in \mathbb{R}$, $e \in \mathrm{int}(D)$, and $F : R \to 2^S$ is a SVM, in addition to $A \subseteq \mathrm{Dom}(F)$. Suppose that F is generalized contingent epiderivable of k-th order at (a', b') for $(a_1 - a', b_1 - b'), ..., (a_{k-1} - a', b_{k-1} - b')$. Then F is said to be δ-D-convex of k-th order w.r.t. e at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ on A if*

$$F(a) - b' \subseteq \overrightarrow{d}_g^{(k)}F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\ (a - a') + \delta\|a - a'\|^2 e + D, \forall a \in A.$$

Das [10, 11] developed the notion of δ-cone convexity of *SVMs*.

Definition 2.7. ([10, 11]). *Let A be a convex subset of a RNS R, $e \in \mathrm{int}(D)$, and $F : R \to 2^S$ be a SVM, with $A \subseteq \mathrm{Dom}(F)$. Then F is said to be δ-D-convex w.r.t. e on A if there exists $\delta \in \mathbb{R}$, satisfying*

$$(1 - \lambda)F(a_1) + \lambda F(a_2) \subseteq F((1 - \lambda)a_1 + \lambda a_2) + \delta\lambda(1 - \lambda)\|a_1 - a_2\|^2 e + D, \\ \forall a_1, a_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

3 Higher-order (α,β)-δ-cone convexity

Das and Nahak [14, 15, 16, 17] and Das [10, 12] introduced the notion of δ-cone convexity of *SVMs*. They established sufficient *KKT* optimality conditions and used the contingent epiderivative and δ-cone convexity assumptions to formulate the duality results for various types of *SVOPs*. We get the standard conception of cone convex *SVMs* for $\delta = 0$, developed by Borwein [7].

Let A be a convex subset of $\mathbb{R}^i$, $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ be a *SVM* with $A \subseteq \mathrm{Dom}(F)$, and $(a',b') \in \mathrm{Gr}(F)$. In the paper, we make the assumption that $\mathbf{0} = (0,...,0) \in \mathbb{R}^i$, $\mathbf{0}' = (0,...,0) \in \mathbb{R}^j$, $\mathbf{0}'' = (0,...,0) \in \mathbb{R}^k$, $\mathbf{1} = (1,...,1) \in \mathbb{R}^i$, $\mathbf{1}' = (1,...,1) \in \mathbb{R}^j$, and $\mathbf{1}'' = (1,...,1) \in \mathbb{R}^k$.

We introduce the notion of higher-order (α,β)-δ-cone convex *SVMs*. For $\alpha = 0$ and $\beta = 0$, we have the usual notion of δ-cone convex *SVMs*.

Definition 3.1. *Let $k \in \mathbb{N}$, in addition to $k \geq 2$, A be a convex subset of $\mathbb{R}^i$, $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ be a SVM with $A \subseteq \mathrm{Dom}(F)$, $a, a' \in A$, and $(a',b') \in Gr(F)$. Let F be generalized contingent epiderivable of k-th order at (a',b') for $(a_1 - a', b_1 - b'), ..., (a_{k-1} - a', b_{k-1} - b')$ and $\alpha, \beta \in \mathbb{R}$. Then F is said to be (α,β)-δ-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a',b') for $(a_1,b_1),...,(a_{k-1},b_{k-1})$ on A if there exists $\delta \in \mathbb{R}$ with*

$$
\begin{aligned}
&(e^{\alpha(a-a')} - \mathbf{1})/\alpha \\
&\in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')), \\
&\qquad for\ \alpha \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&a - a' \in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')), \\
&\qquad for\ \alpha = 0,
\end{aligned}
$$

satisfying ,

$$
\begin{aligned}
(e^{\beta(F(a)-b')} - \mathbf{1}')/\beta &\subseteq \overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha \neq 0, \beta \neq 0), \\
F(a) - b' &\subseteq \overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, \\
&\quad (\alpha \neq 0, \beta = 0), \\
(e^{\beta(F(a)-b')} - \mathbf{1}')/\beta &\subseteq \overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad (a - a') + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha = 0, \beta \neq 0), \\
F(a) - b' &\subseteq \overrightarrow{d}_g^{(k)} F(a',b',a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad (a - a') + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha = 0, \beta = 0).
\end{aligned}
\tag{3.1}
$$

A higher-order (α,β)-δ-cone convex *SVM* example that is not cone convex is developed.

Example 3.1. Let us consider a subset A of $\mathbb{R}^2$ by

$$A = \left\{ a = (a_1, a_2) \mid a_1 + a_2 \geq \frac{1}{7},\ a_1 \geq 0,\ a_2 \geq 0 \right\}.$$

Here A is a convex subset of $\mathbb{R}^2$. Define a SVM $F : \mathbb{R}^2 \to 2^{\mathbb{R}}$ as follows:

$$F(a) = \begin{cases} [0,5], & \text{if } a_1 + a_2 \geq \frac{1}{7},\ a_1 \neq 5a_2, a = (a_1,a_2), \\ [6,8], & \text{otherwise.} \end{cases}$$

Here,

$$
\begin{aligned}
\mathrm{Epi}(F) = &\left\{ ((a_1,a_2),b) : a_1 + a_2 \geq \frac{1}{7},\ a_1 \neq 5a_2, a = (a_1,a_2), b \geq 0 \right\} \\
&\cup \{((a_1,a_2),b) : (a_1,a_2) \in \mathbb{R}^2, b \geq 6\}.
\end{aligned}
$$

Let $a' = (5,0)$ and $b' = 0$. So, $b' \in F(a')$. Let $k \in \mathbb{N}$, in addition to $k \geq 2$, $a'_1 = (1,0), ..., a'_{k-1} = (k-1, 0)$ and $b'_1 = ... = b'_{k-1} = 1$. As a result,

$$
\begin{aligned}
&T^{(k)}(\mathrm{Epi}(F), ((5,0),0), (a'_1 - a', b'_1 - b'), ..., (a'_{k-1} - a', b'_{k-1} - b')) \\
&= \left\{ ((a_1,a_2),b) : a_1 + a_2 + \frac{1}{7} \geq 0,\ a_1 \neq 5a_2, a = (a_1,a_2), b \geq 0 \right\} \\
&\qquad \cup \{((a_1,a_2),b) : (a_1,a_2) \in \mathbb{R}^2, b \geq 6\}.
\end{aligned}
$$

Hence,

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a_1,a_2)$$
$$= \text{I-min}\{b : ((a_1,a_2),b)$$
$$\in T^{(k)}(\text{Epi}(F),((5,0),0),(a_1'-a',b_1'-b'),...,(a_{k-1}'-a',b_{k-1}'-b'))\}.$$

Let $a=(0,1)$. Hence,

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$= \overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(5,-1) = 6.$$

Let $\lambda = \frac{1}{2}$. Then,

$$\frac{1}{2}a + \frac{1}{2}a' = \frac{1}{2}(0,1) + \frac{1}{2}(5,0) = \Big(\frac{5}{2},\frac{1}{2}\Big),$$

and

$$\frac{1}{2}F(0,1) + \frac{1}{2}F(5,0) = \frac{1}{2}[0,5] + \frac{1}{2}[0,5] = [0,5] \nsubseteq [6,8] + \mathbb{R}_+ = F\Big(\frac{5}{2},\frac{1}{2}\Big) + \mathbb{R}_+.$$

Hence, it is clear that F is not $\mathbb{R}_+$-convex w.r.t. 1. But on the other side, by choosing $\alpha=0$, $\beta=1$, and $\delta=-2$, we get that

$$(e^{\beta(F(a)-b')}-1)/\beta = e^{F(a)-b'}-1 = e^{[0,5]}-1.$$

and

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$+\delta\|a-a'\|^2 1 = 6-52=-46.$$

Hence,

$$(e^{\beta(F(a)-b')}-1)/\beta \subseteq \overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$+\delta\|a-a'\|^2 1 + \mathbb{R}_+ = -46 + \mathbb{R}_+.$$

As a consequence, F is $(5,0)$-(-2)-$\mathbb{R}_+$-convex of k-th order w.r.t. 1 at $((5,0),0)$ for the elements $(a_1',b_1'),...,(a_{k-1}',b_{k-1}')$ on A.

4 Higher-order optimality conditions

Let A be a nonempty subset of $\mathbb{R}^i$. Suppose that $F:\mathbb{R}^i \to 2^{\mathbb{R}^j}$ and $G:\mathbb{R}^i \to 2^{\mathbb{R}^k}$ are two *SVMs*, with

$$A \subseteq \text{Dom}(F) \cap \text{Dom}(G).$$

We consider a *SVOP* (P) as follows

$$\begin{aligned} &\underset{a\in A}{\text{minimize}} \quad F(a), \\ &\text{subject to} \quad G(a)\cap(-\mathbb{R}_+^k) \neq \emptyset. \end{aligned} \tag{P}$$

The feasible set of (P) is provided by

$$X = \{a \in A : G(a)\cap(-\mathbb{R}_+^k) \neq \emptyset\}.$$

Definition 4.1. *A point $(a_0,b_0)\in\mathbb{R}^i\times\mathbb{R}^j$, along with $a_0\in X$ and $b_0\in F(a_0)$, is said to be a feasible point of (P).*

Definition 4.2. *A point $(a_0,b_0)\in\mathbb{R}^i\times\mathbb{R}^j$, with $a_0\in X$ and $b_0\in F(a_0)$, is said to be a minimizer of the problem (P) if for all $(a,b)\in\mathbb{R}^i\times\mathbb{R}^j$, with $a\in X$ and $b\in F(a)$,*

$$b-b_0 \notin (-\mathbb{R}_+^j)\setminus\{0_{\mathbb{R}^j}\}.$$

Definition 4.3. *A point $(a_0,b_0)\in\mathbb{R}^i\times\mathbb{R}^j$, with $a_0\in X$ and $b_0\in F(a_0)$, is said to be a weak minimizer of the problem (P) if for all $(a,b)\in\mathbb{R}^i\times\mathbb{R}^j$, with $a\in X$ and $b\in F(a)$,*

$$b-b_0 \notin (-\text{int}(\mathbb{R}_+^j)).$$

Suppose that $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ and $G : \mathbb{R}^i \to 2^{\mathbb{R}^k}$ are two *SVMs*, with

$$A \subseteq \mathrm{Dom}(F) \cap \mathrm{Dom}(G).$$

Suppose that $(a_0, b_0), (a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^j$ with $a_0, a_1, ..., a_{k-1} \in A$, $b_0 \in F(a_0)$, $b_1 \in F(a_1) + \mathbb{R}^j_+, ..., b_{k-1} \in F(a_{k-1}) + \mathbb{R}^j_+$.

Assume that $(a_0, c_0), (a_1, c_1), ..., (a_{k-1}, c_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^k$ in addition to $c_0 \in G(a_0)$, $c_1 \in G(a_1) + \mathbb{R}^k_+, ..., c_{k-1} \in G(a_{k-1}) + \mathbb{R}^k_+$.

Das and Nahak [16] established that the Fritz John optimality conditions are sufficient under the (α, β)-δ-(η, θ)-invexity assumption. The following theorem employs higher-order (α, β)-δ-cone convexity assumption to establish the higher-order sufficient *KKT* optimality conditions of the problem (P).

Theorem 4.1. ***(Higher-order sufficient optimality conditions)*** *Let A be a convex subset of $\mathbb{R}^i$, (a_0, b_0) be a feasible point of the problem (P), and $c_0 \in G(a_0) \cap (-\mathbb{R}^k_+)$. Suppose that F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a_0, b_0) for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a_0, c_0) for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A and*

$$\delta_1(b^* \mathbf{1}') + \delta_2(c^* \mathbf{1}'') \geq 0.$$

Suppose that there exists $(b^*, c^*) \in \mathbb{R}^j_+ \times \mathbb{R}^k_+$, with $b^* \neq 0_{\mathbb{R}^j}$, satisfying

$$\begin{aligned}
&b^* \overrightarrow{d}^{(k)}_g F(a_0, b_0, a_1 - a_0, b_1 - b_0, ..., a_{k-1} - a_0, b_{k-1} - b_0)((e^{\alpha(a_0 - a')} - \mathbf{1})/\alpha) \\
&+ c^* \overrightarrow{d}^{(k)}_g G(a_0, c_0, a_1 - a_0, c_1 - c_0, ..., a_{k-1} - a_0, c_{k-1} - c_0)((e^{\alpha(a_0 - a')} - \mathbf{1})/\alpha) \\
&\geq 0, \forall a \in X, (for\ \alpha \neq 0), \\
&b^* \overrightarrow{d}^{(k)}_g F(a_0, b_0, a_1 - a_0, b_1 - b_0, ..., a_{k-1} - a_0, b_{k-1} - b_0)(a_0 - a') \\
&+ c^* \overrightarrow{d}^{(k)}_g G(a_0, c_0, a_1 - a_0, c_1 - c_0, ..., a_{k-1} - a_0, c_{k-1} - c_0)(a_0 - a') \\
&\geq 0, \forall a \in X, (for\ \alpha = 0),
\end{aligned} \tag{4.1}$$

and

$$c^* c_0 = 0. \tag{4.2}$$

Then (a_0, b_0) is a weak minimizer of the problem (P).

Proof. By using the contradiction approach, we prove the theorem for the instance when $\alpha \neq 0$. For $\alpha = 0$, we can prove likewise. Suppose that (a_0, b_0) is not a weak minimizer of the problem (P). Then,

$$(b_0 - \mathrm{int}(\mathbb{R}^j_+)) \cap F(X) \neq \emptyset.$$

As a result, there exist $a \in X$, $b \in F(a)$ satisfying

$$b < b_0.$$

Hence,

$$e^b < e^{b_0} \Rightarrow \frac{1}{\beta} e^{\beta b} < \frac{1}{\beta} e^{\beta b - 0} \Rightarrow (e^{\beta(b - b_0)} - \mathbf{1}')/\beta < \mathbf{0}.$$

As $b^* \neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b - b_0)} - \mathbf{1}')/\beta < 0.$$

As $a \in X$, there exists an element $c \in G(a) \cap (-\mathbb{R}^k_+)$. Let $c^* = (c^*_1, ..., c^*_k)$, $c = (c_1, ..., c_k)$, and $c_0 = ({c_1}', ..., {c_k}')$. As $c \in -\mathbb{R}^k_+$, we have

$$c \leq \mathbf{0}'' \Rightarrow e^c \leq \mathbf{1}'' \Rightarrow \frac{1}{\beta} e^{\beta c} \leq \frac{1}{\beta} \mathbf{1}'' \Rightarrow (e^{\beta c} - \mathbf{1}'')/\beta \leq 0.$$

Now, $c^* c_0 = 0$, $c_0 \in -\mathbb{R}^k_+$, and $c^* \in \mathbb{R}^k_+$. As a result, if ${c_n}' < 0$ for some n, where $1 \leq n \leq k$, then ${c_n}^* = 0$. So,

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = 0.$$

Again, if ${c_n}' = 0$ for some n, where $1 \leq n \leq k$, then

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = {c_n}^*(e^{\beta c_n} - 1)/\beta \leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c - c_0)} - \mathbf{1}'')/\beta \leq 0.$$

Hence, we have

$$b^*(e^{\beta(b-b_0)} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta < 0. \tag{4.3}$$

As F is (α,β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a_0,b_0) for (a_1,b_1),...,(a_{k-1},b_{k-1}) and G is (α,β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a_0,c_0) for (a_1,c_1),...,(a_{k-1},c_{k-1}) on A, we have

$$\begin{aligned}(e^{\beta(b-b_0)} - \mathbf{1}')/\beta \in \overrightarrow{d}_g^{(k)} F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)\\ ((e^{\alpha(a_0-a')} - \mathbf{1})/\alpha) + \delta_1\|a-a_0\|^2\mathbf{1}' + \mathbb{R}^j_+\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta \in \overrightarrow{d}_g^{(k)} G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)\\ ((e^{\alpha(a_0-a')} - \mathbf{1})/\alpha) + \delta_2\|a-a_0\|^2\mathbf{1}'' + \mathbb{R}^k_+.\end{aligned} \tag{4.5}$$

As a result, from (4.1), (4.4), (4.5), and the condition

$$\delta_1(b^*\mathbf{1}') + \delta_2(c^*\mathbf{1}'') \geq 0,$$

we have

$$b^*(e^{\beta(b-b_0)} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta \geq 0.$$

It contradicts (4.3). Hence, (a_0,b_0) is a weak minimizer of the problem (P). □

5 Higher-order Mond-Weir type duality

We prove the results of higher-order duality under higher-order (α,β)-δ-cone convexity assumptions. Let $a' \in \mathbb{R}^i$, $b' \in F(a')$, and $c' \in G(a')$.

Suppose that $(a',b'),(a_1,b_1),...,(a_{k-1},b_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^j$ with $a',a_1,...,a_{k-1} \in A$, $b' \in F(a')$, $b_1 \in F(a_1) + \mathbb{R}^j_+,...,b_{k-1} \in F(a_{k-1}) + \mathbb{R}^j_+$.

Also, assume that $(a',c'),(a_1,c_1),...,(a_{k-1},c_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^k$ in addition to $c' \in G(a')$, $c_1 \in G(a_1) + \mathbb{R}^k_+,...,c_{k-1} \in G(a_{k-1}) + \mathbb{R}^k_+$.

We suppose that F is generalized contingent epiderivable of k-th order at (a',b') for $(a_1-a',b_1-b'),...,(a_{k-1}-a',b_{k-1}-b')$ and G is generalized contingent epiderivable of k-th order at (a',c') for $(a_1-a',c_1-c'),...,(a_{k-1}-a',c_{k-1}-c')$ with,

$$\begin{aligned}&((e^{\alpha(a-a')} - \mathbf{1})/\alpha)\\ &\in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')))\\ &\cap\mathrm{Dom}(\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')),\\ &for\ \alpha \neq 0, a \in \mathbb{R}^i\end{aligned}$$

and

$$\begin{aligned}a'-a \in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b'))\\ \cap\mathrm{Dom}(\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')),\\ for\ \alpha = 0, a \in \mathbb{R}^i.\end{aligned}$$

A higher-order dual problem of the Mond-Weir type, (MWD), is under consideration.

$$\begin{aligned}&\text{maximize } b' && (MWD)\\ &\text{subject to } b^*\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')\\ &\qquad((e^{\alpha(a-a')} - \mathbf{1})/\alpha)\\ &\qquad + c^*\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')\\ &\qquad((e^{\alpha(a-a')} - \mathbf{1})/\alpha) \geq 0, \forall a \in X, (for\ \alpha \neq 0),\\ &\qquad b^*\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')\\ &\qquad(a-a')\\ &\qquad + c^*\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')\end{aligned}$$

$$(a - a') \geq 0, \forall a \in X, (for\ \alpha = 0),$$
$$c^* c' \geq 0,$$
$$b^* \mathbf{1}' = 1,\ b^* \in \mathbb{R}^j_+ \setminus \{0_{\mathbb{R}^j}\},\ \text{and } c^* \in \mathbb{R}^k_+.$$

Let us define the feasible set W of the problem (MWD) by

$$W = \{b' : (a', b', c', b^*, c^*) \text{ is a feasible point of } (MWD)\}.$$

Definition 5.1. *A feasible point (a', b', c', b^*, c^*) of the problem (MWD) is said to be a weak maximizer of (MWD), if*

$$(b' + \operatorname{int}(\mathbb{R}^j_+)) \cap W = \emptyset.$$

In the following theorems, we formulate higher-order duals of the Mond-Weir (MWD) type and prove the higher-order strong, weak and converse duality theorems using higher-order (α, β)-δ-cone convexity assumption.

Theorem 5.1. ***(Higher-order weak duality)*** *Let A be a convex subset of $\mathbb{R}^i$. Let (a_0, b_0) and (a', b', c', b^*, c^*) be feasible points for the problems (P) and (MWD), respectively, with $c' \in G(a') \cap (-\mathbb{R}^k_+)$. Suppose that F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A and*

$$\delta_1 + \delta_2(c^* \mathbf{1}'') \geq 0. \tag{5.1}$$

Then,

$$b_0 \not< b'.$$

Proof. Using the contradiction approach, we demonstrate the theorem. Suppose that

$$b_0 < b'.$$

Hence,

$$e^{b_0} < e^{b'} \Rightarrow \frac{1}{\beta} e^{\beta b_0} < \frac{1}{\beta} e^{\beta b'} \Rightarrow (e^{\beta(b_0 - b')} - \mathbf{1}')/\beta < \mathbf{0}.$$

As $b^* \neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b_0 - b')} - \mathbf{1}')/\beta < 0.$$

As $a_0 \in X$, there exists an element

$$c_0 \in G(a_0) \cap (-\mathbb{R}^k_+).$$

Let $c^* = (c^*_1, ..., c^*_k)$, $c_0 = (c_1, ..., c_k)$, and $c' = (c_1', ..., c_k')$. As $c_0 \in -\mathbb{R}^k_+$, we have

$$c_0 \leq \mathbf{0}'' \Rightarrow e^{c_0} \leq \mathbf{1}'' \Rightarrow \frac{1}{\beta} e^{\beta c_0} \leq \frac{1}{\beta} \mathbf{1}'' \Rightarrow (e^{\beta c_0} - \mathbf{1}'')/\beta \leq 0.$$

As $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$, we have

$$c^* c' \leq 0.$$

From duality constraints,

$$c^* c' \geq 0.$$

As a result,

$$c^* c' = 0.$$

Now, $c^* c' = 0$, $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$. As a consequence, if $c_n' < 0$ for some n, where $1 \leq n \leq k$, then $c_n{}^* = 0$. So,

$$c_n{}^*(e^{\beta(c_n - c_n')} - 1)/\beta = 0.$$

Again, if $c_n' = 0$ for some n, where $1 \leq n \leq k$, then

$$c_n{}^*(e^{\beta(c_n - c_n')} - 1)/\beta = c_n{}^*(e^{\beta c_n} - 1)/\beta \leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c_0 - c')} - \mathbf{1}'')/\beta \leq 0.$$

So, we have

$$b^*(e^{\beta(b_0-b')}-\mathbf{1}')/\beta+c^*(e^{\beta(c_0-c')}-\mathbf{1}'')/\beta<0. \tag{5.2}$$

As F is (α,β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a',b') for $(a_1,b_1),...,(a_{k-1},b_{k-1})$ and G is (α,β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a',c') for $(a_1,c_1),...,(a_{k-1},c_{k-1})$ on A, we have

$$(e^{\beta(b-b')}-\mathbf{1}')/\beta\in \overrightarrow{d}_g^{(k)}F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')$$
$$((e^{\alpha(a-a')}-\mathbf{1})/\alpha)+\delta_1\|a-a'\|^2\mathbf{1}'+\mathbb{R}^j_+ \tag{5.3}$$

and

$$(e^{\beta(c-c')}-\mathbf{1}'')/\beta\in \overrightarrow{d}_g^{(k)}G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')$$
$$((e^{\alpha(a-a')}-\mathbf{1})/\alpha)+\delta_2\|a-a'\|^2\mathbf{1}''+\mathbb{R}^k_+. \tag{5.4}$$

Hence, from (5.3), (5.4), and the conditions $b^*\mathbf{1}'=1$ and $\delta_1+\delta_2(c^*\mathbf{1}'')\geq 0$, we have

$$b^*(e^{\beta(b_0-b')}-\mathbf{1}')/\beta+c^*(e^{\beta(c_0-c')}-\mathbf{1}'')/\beta\geq 0.$$

It contradicts (5.2). Subsequently,

$$b_0\not< b'.$$

□

Theorem 5.2 (**Higher-order strong duality).** *Let A be a convex subset of $\mathbb{R}^i$ and (a_0,b_0) be a weak minimizer of (P). Suppose that for some $(b^*,c^*)\in\mathbb{R}^j_+\times\mathbb{R}^k_+$, with $b^*\mathbf{1}'=1$, Eqs.* (4.1) *and* (4.2) *are satisfied for some element $c_0\in G(a_0)\cap(-\mathbb{R}^k_+)$. Then (a_0,b_0,c_0,b^*,c^*) is a feasible solution for (MWD). At this point if Theorem* 5.1 *between (P) and (MWD) is satisfied, then (a_0,b_0,c_0,b^*,c^*) is a weak maximizer of (MWD).*

Proof. As the Eqs. (4.1) and (4.2) are satisfied, we have

$$\begin{aligned}
&b^*\overrightarrow{d}_g^{(k)}F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)\\
&((e^{\alpha(a_0-a')}-\mathbf{1})/\alpha)\\
&+c^*\overrightarrow{d}_g^{(k)}G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)\\
&((e^{\alpha(a_0-a')}-\mathbf{1})/\alpha)\geq 0,\forall a\in X,(for\ \alpha\neq 0),\\
&b^*\overrightarrow{d}_g^{(k)}F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)\\
&(a_0-a')\\
&+c^*\overrightarrow{d}_g^{(k)}G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)\\
&(a_0-a')\geq 0,\forall a\in X,(for\ \alpha=0),
\end{aligned}$$

and

$$c^*c_0=0.$$

Hence, (a_0,b_0,c_0,b^*,c^*) is a feasible solution for (MWD). Next, we show that

$$(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W=\emptyset.$$

Let

$$b'\in(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W.$$

As a result,

$$b'-b_0\in\mathrm{int}(\mathbb{R}^j_+)\Rightarrow b_0<b'.$$

It contradicts Theorem 5.1 between (P) and (MWD). As a result,

$$(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W=\emptyset.$$

Hence, (a_0,b_0,c_0,b^*,c^*) is a weak maximizer for (MWD). □

Theorem 5.3 (**Higher-order converse duality).** *Let A be a convex subset of $\mathbb{R}^i$ and (a', b', c', b^*, c^*) be a weak maximizer of the problem (MWD) with*

$$c' \in G(a') \cap (-\mathbb{R}^k_+).$$

Suppose that F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A satisfying (5.1). Then (a', b') is a weak minimizer of (P).

Proof. Here (a', b') is a feasible solution of the problem (P). Let (a', b') be not a weak minimizer of the problem (P). Then,

$$(b' - \operatorname{int}(\mathbb{R}^j_+)) \cap F(X) \neq \emptyset.$$

So, there exist $a \in X$ and $b \in F(a)$, satisfying

$$b' - b \in \operatorname{int}(\mathbb{R}^j_+) \text{ i.e., } b < b'.$$

Hence,

$$e^b < e^{b'} \Rightarrow \frac{1}{\beta} e^{\beta b} < \frac{1}{\beta} e^{\beta b'} \Rightarrow (e^{\beta(b-b')} - \mathbf{1}')/\beta < \mathbf{0}'.$$

As $b^* \neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta < 0.$$

As $a \in X$, there exists an element

$$c \in G(a) \cap (-\mathbb{R}^k_+).$$

Let $c^* = (c_1^*, ..., c_k^*)$, $c = (c_1, ..., c_k)$, and $c' = ({c_1}', ..., {c_k}')$. As $c \in -\mathbb{R}^k_+$, we have

$$c \leq \mathbf{0}'' \Rightarrow e^c \leq \mathbf{1}'' \Rightarrow \frac{1}{\beta} e^{\beta c} \leq \frac{1}{\beta} \mathbf{1}'' \Rightarrow (e^{\beta c} - \mathbf{1}'')/\beta \leq 0.$$

As $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$, we have

$$c^* c' \leq 0.$$

Further, from duality constraints, we have

$$c^* c' \geq 0.$$

As a result,

$$c^* c' = 0.$$

Now, $c^* c' = 0$, $c' \in -\mathbb{R}^k_+$, and $c^* \in \mathbb{R}^k_+$. As a result, if ${c_n}' < 0$ for some n, where $1 \leq n \leq k$, then ${c_n}^* = 0$. So,

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = 0.$$

Again, if ${c_n}' = 0$ for some n, where $1 \leq n \leq k$, then

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = {c_n}^*(e^{\beta c_n} - 1)/\beta \leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta \leq 0.$$

As a result, we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta < 0. \tag{5.5}$$

As F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A, we have

$$\begin{aligned}(e^{\beta(b-b')} - \mathbf{1}')/\beta \in \overrightarrow{d}^{(k)}_g F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')& \\ ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta_1 \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+&\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}(e^{\beta(c-c')} - \mathbf{1}'')/\beta \in \overrightarrow{d}^{(k)}_g G(a', c', a_1 - a', c_1 - c', ..., a_{k-1} - a', c_{k-1} - c')& \\ ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta_2 \|a - a'\|^2 \mathbf{1}'' + \mathbb{R}^k_+.&\end{aligned} \tag{5.7}$$

Hence, from (5.6), (5.7), and the conditions $b^* \mathbf{1}' = 1$ and

$$\delta_1 + \delta_2(c^* \mathbf{1}'') \geq 0,$$

we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta \geq 0.$$

It contradicts (5.5). Hence, (a', b') is a weak minimizer of (P). □

6 Conclusion

In this study, we introduce the notion of higher-order (α, β)-δ-cone convexity in set-valued optimization problems. We establish the sufficient optimality conditions for $SVOP$ (P) under the assumption of higher-order (α, β)-δ-cone convexity. With regard to the problem (P), we formulate the higher-order duals of the Mond-Weir (MWD) type. Under the assumed higher-order (α, β)-δ-cone convexity, we additionally establish the higher-order duality results of Mond-Weir type. In order to prove that the notion of higher-order (α, β)-δ-cone convexity is more generalized than higher-order cone convexity, an example is developed.

Future research may explore several promising directions, including the incorporation of fractional-order convexity to further generalize the model, the study of fuzzy set-valued optimization to address uncertainty, and stability analysis of solutions under perturbations. These extensions may enhance both the theoretical depth and practical applicability of the present work.

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HYBRID FIXED POINT THEORY FOR SUM OF TWO OPERATORS IN A LATTICE ORDERED BANACH SPACE WITH APPLICATIONS TO NONLINEAR DISCONTINUOUS INTEGRAL EQUATIONS

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Abstract

We prove a hybrid fixed point theorem for sum of two operators in a lattice ordered Banach space and apply to nonlinear discontinuous hybrid integral equations of mixed type for proving the existence of maximal and minimal integrable solutions under certain mixed Lipschitz and monotonicity conditions of the nonlinear functions. Our main existence result is illustrated with a numerical example as well as with an application to initial value problems of nonlinear first and second order discontinuous hybrid linearly perturbed ordinary differential equations for proving the existence of maximal and minimal solutions.
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Keywords and Phrases: Lattice ordered Banach space; Hybrid fixed point principle; Nonlinear hybrid integral equation; Existence of extremal integrable solutions.

1 Introduction

Most of the problems of universe are nonlinear in nature and so governed by dynamic systems of nonlinear equations. Analysis of such problems for different aspects of their solutions is the nonlinear analysis and so its importance in the subject of mathematical analysis. The existence theory of nonlinear problems constitutes the major and core part of the stream of nonlinear analysis. There are several approaches to deal with nonlinear problems for existence and other aspects of their study. One of these approaches is the operator theoretic tools from the subject of nonlinear functional analysis. Therefore, several operatorial techniques are developed and used from the subject of nonlinear operator theory to accomplish the purpose. The fixed point theory is a part of nonlinear operator theory which provides most powerful tools for resolving the existence theory of nonlinear equations quantitatively and qualitatively. The applications of fixed point theorems to nonlinear problems require a little skill and a clever selection of fixed point theorem gives the cleverer results.

The existence theorems for both continuous and discontinuous standard nonlinear, that is, nonlinearly perturbed linear differential and integral equations are available in the literature and obtained under the continuity, Caratheódory and Chandrabhan type conditions by using the classical analytic, topological and algebraic fixed point principles respectively. See Granas and Dugundji [25], Hekkilä and Lakshmikatham [26], Dhage [7, 8, 13], Dhage *et al.* [16, 17, 18, 19] and references therein. But the case with nonlinear hybrid differential equations is different, where the existence results have been obtained only for continuous hybrid differential and integral equations by using the hybrid fixed point principles on the lines of Krasnoselskii [27] and Dhage [7, 8, 10, 11, 12, 13, 14, 15] and no existence result is so far proved for nonlinear discontinuous hybrid differential and integral equations. Therefore, it is of interest to obtain the existence results for discontinuous hybrid differential and integral equations which is the main motivation of the present work. In the present discussion, we establish a hybrid fixed point theorem for sum of the two operators in a lattice ordered Banach space and apply to nonlinear discontinuous linearly perturbed hybrid integral equations for proving the existence of maximal and minimal integrable solutions. Before going to the main hybrid fixed point theorems, we give some preliminaries which are needed in what follows.

2 Preliminaries

Let $(E, \|\cdot\|)$ denote a Banach space with norm $\|\cdot\|$. We introduce a lattice structure in E with the help of a partial order $\preceq$ in E which is a reflexive, antisymmetric and transitive binary relation on E into itself. See Lipschitz [28] and Maddux [29]. Then $(E, \|\cdot\|, \preceq)$ becomes a partially ordered Banach space provided the partial order is compatible with the linearity in E, that is, (i) if $x_1 \leq y_1$ and $x_2 \preceq y_2$, then $x_1 + x_2 \preceq y_1 + y_2$, and (ii) if $x_1 \preceq y_1$, then $\lambda x_1 \preceq \lambda y_1$ for all $x_1, x_2, y_1, y_2 \in E$ and $\lambda \in \mathbb{R}_+$. We assume that $(E, \preceq)$ is a lattice. Note that a partially ordered set (in short poset) $(E, \preceq)$ is called a **lattice** if $\wedge\{a, b\}$ and $\vee\{a, b\}$ exist for all $a, b \in E$, where $\wedge$ and $\vee$ denote respectively the **meet** and **join** lattice operations in E. Next, the lattice $(E, \preceq)$ is called **complete lattice** if $\wedge S$ and $\vee S$ exist for every subset S of E. The details of the lattice structure of the Banach space appear in Birkkoff [3]. Now, the triplet $(E, \|\cdot\|, \preceq)$ is called a **partially lattice ordered Banach space**. If the order relation $\preceq$ is defined by the cone K in the partially lattice ordered Banach space E, then the triplet $(E, \|\cdot\|, K)$ is simply called a **lattice ordered Banach space**. Note that a non-empty closed convex subset K of the Banach algebra is called a **cone** if it satisfies (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$ with $\lambda > 0$, and (iii) $\{-K\} \cap K = \{0\}$. Furthermore, if the norm $\|\cdot\|$ satisfies the property that $\|x\| \leq \|y\|$ whenever $|x| \preceq |y|$, where $|x| = \max\{x\,,\, -x\}$, then $(E, \|\cdot\|, \preceq)$ is called a **Banach lattice**. In particular, $(E, \|\cdot\|, K)$ is a Banach lattice if the norm $\|\cdot\|$ is monotone on K, that is, if $x, y \in K$ and $x \preceq y$, then $\|x\| \leq \|y\|$. It is known that every Banach lattice is lattice ordered Banach space but the converse is not necessarily true. Below we state a couple of classical fixed point theorems which we need in the sequel. To state the first analytical fixed point theorem, we need the following definitions in what follows.

Definition 2.1. *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a **$\mathcal{D}$-function** if it is upper semi-continuous and nondecreasing satisfying $\psi(0) = 0$. The set of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.*

Definition 2.2. *An operator $\mathcal{T} : E \to E$ is said to be a **$\mathcal{D}$-Lipschitzian** if there exists a $\mathcal{D}$-function $\psi \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

*for all $x, y \in E$. If $\psi(r) = k\,r$, then $\mathcal{T}$ is called a Lipschitz operator on E with **Lipschitz** constant k. Again, if $0 \leq k < 1$, $\mathcal{T}$ is called a **contraction** operator on E with contraction constant k. Furthermore, if $\psi(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a **nonlinear $\mathcal{D}$-contraction** on E with contraction $\mathcal{D}$-function ψ.*

Our first analytical or geometrical fixed point theorem is as follows.

Theorem 2.1 (Dhage [10, 11])**.** *Let E be a Banach space and let $\mathcal{A} : E \to E$ be a nonlinear $\mathcal{D}$-contraction. Then $\mathcal{A}$ has a unique fixed point ξ and the sequence $\{\mathcal{A}^n(x)\}$ of successive iterations converges to ξ for each $x \in E$.*

To state the second algebraic fixed point theorem, we need the following definition.

Definition 2.3. *A mapping $\mathcal{T}$ on a lattice $(L, \preceq)$ into itself is called **isotone increasing** if it preserve the order relation $\preceq$, that is, if $x, y \in L$ with $x \preceq y$, then $\mathcal{T}x \preceq \mathcal{T}y$.*

Theorem 2.2 (Tarski [30])**.** *Let $(L, \preceq)$ be a partially ordered set and let $\mathcal{T} : L \to L$ be a mapping. Suppose that*

(a) $\mathcal{T}$ is isotone increasing,
(b) $(L, \preceq)$ is a complete lattice, and
(c) $F_{\mathcal{T}} = \{u \in L \mid \mathcal{T}u = u\}$.

Then $F_{\mathcal{T}} \neq \emptyset$ and $\big(F_{\mathcal{T}}, \preceq\big)$ is a complete lattice.

In the present paper, we combine Theorems 2.1 and 2.2 and prove a hybrid fixed point theorem involving the sum of two operators satisfying mixed contraction and isotone increasing conditions in a partially lattice ordered Banach space.

3 Hybrid Fixed Point Principles

In this section we discuss the hybrid fixed point principles for sum of the two operators in a partially lattice ordered Banach space. Throughout rest of the paper unless otherwise mentioned, let E denote the partially lattice ordered Banach space. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two operators and consider the operator equation

$$\mathcal{A}x + \mathcal{B}x = x. \tag{3.1}$$

Note that the fixed point of the operator $\mathcal{T} = \mathcal{A} + \mathcal{B}$ in E is the solution of the operator equation (3.1) and vice versa. If E is a complete lattice and $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, then by Theorem 2.2, the operator equation (3.1) has a solution and the set of all solutions is a complete lattice. Similarly, a hybrid fixed point theorem for sum of two discontinuous operators on a non-empty subset of a partially lattice ordered Banach space may be stated as follows.

Theorem 3.1. *Let S be a non-empty subset of a partially lattice ordered Banach space $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators such that*
(a) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing,
(b) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and
(c) $(S, \preceq)$ is complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on S by
$$\mathcal{T}x = \mathcal{A}x + \mathcal{B}x. \tag{3.2}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, the operator $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is also a complete lattice in view if hypothesis (c), the desired conclusion follows by an application of Theorem 2.2. □

Corollary 3.1. *Let S be a non-empty subset of a lattice ordered Banach space $(E, \|\cdot\|, K)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators such that*
(a) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing,
(b) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and
(c) $(S, \preceq)$ is complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

The conclusion of above hybrid fixed point results does not remain true if the operators $\mathcal{A}$ and $\mathcal{B}$ are not isotone increasing on E or in particular on S . Therefore, it is of interest to prove the solution of the operator equation (3.1) when both the operators $\mathcal{A}$ and $\mathcal{B}$ are not continuous as well as isotone increasing on E. In the following we prove some results in this direction.

Theorem 3.2. *Let $(E, \|\cdot\|, \preceq)$ be a partially lattice ordered Banach space and let $\mathcal{A}, \mathcal{B} : E \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,
(c) $(E, \preceq)$ is a complete lattice, and
(d) $F_{(\mathcal{A}+\mathcal{B})} = \{u \in E \mid \mathcal{A}u + \mathcal{B}u = u\}$.
Then $F_{(\mathcal{A}+\mathcal{B})} \neq \emptyset$ and $\left(F_{(\mathcal{A}+\mathcal{B})}, \preceq\right)$ is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on E into itself by
$$\mathcal{T} = (I - \mathcal{A})^{-1}\mathcal{B}. \tag{3.3}$$
We show that $\mathcal{T}$ is well defined operator and maps E into itself. Let $y \in E$ be fixed element and define an operator A_y on E by
$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.4}$$

Let $x_1, x_2 \in E$ be arbitrary. Then from (3.3) it follows that
$$\|\mathcal{A}_y(x_1) - \mathcal{A}_y(x_2)\| = \|\mathcal{A}x_1 - \mathcal{A}x_2\| \leq \psi(\|x_1 - x_2\|),$$
where ψ is a $\mathcal{D}$-function satisfying $\psi(r) < r$, $r > 0$. This shows that $\mathcal{A}_y$ is a nonlinear $\mathcal{D}$-contraction operator on E. Now by application of Theorem 2.1, there is a unique point $y' \in E$ such that
$$\mathcal{A}_y(y') = y',$$
which by definition of $\mathcal{A}_y$ implies that
$$\mathcal{A}y' + \mathcal{B}y = y'$$
or
$$(I - \mathcal{A})y' = \mathcal{B}y. \tag{3.5}$$

Now,

$$(I-\mathcal{A})^{-1} = I + A + A^2 + \cdots + \mathcal{A}^n + ...$$
$$= \sum_{n=0}^{\infty} \mathcal{A}^n,$$

which does exists in view of hypothesis (a). Moreover, $(I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on E in view of hypothesis (b). Operating $(I-\mathcal{A})^{-1}$ on both sides of (3.5) we obtain $\mathcal{T}y = y'$. Therefore, the mapping $\mathcal{T}$ is well defined and maps E into itself. By hypotheses (b) and (c), $\mathcal{T}$ is isotone increasing on the complete lattice E. Now the desired conclusion follows by an application of Theorem 2.2. □

Remark 3.1. *We note that the operator $\mathcal{T} = (I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on E if $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on E, however the converse may not be true.*

Next we prove a hybrid fixed point theorem for sum of two operators in a subset of the lattice ordered Banach space which is applicable to nonlinear integral equations of mixed type for proving various aspects of the integrable solutions.

Theorem 3.3. *Let S be a non-empty subset of a partially lattice ordered Banach space $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}: E \to E$ and $\mathcal{B}: S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,
(b) $(I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,
(c) $x = \mathcal{A}x + \mathcal{B}y \Longrightarrow x \in S$ for all $y \in S$, and
(d) $(S, \preceq)$ is a complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on S by (3.3). We show that $\mathcal{T}$ is well defined and maps S into itself. Let $y \in S$ be fixed and consider the operator $\mathcal{A}_y$ on E defined by

$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.6}$$

Clearly, $\mathcal{A}$ defines the mapping $\mathcal{A}_y : E \to E$. Then proceeding as in the proof of Theorem 3.1 it is proved that there exists a unique element $y' \in E$ such that

$$\mathcal{A}_y(y') = \mathcal{A}y' + \mathcal{B}y = y'. \tag{3.7}$$

Now, by hypothesis (c), we have that $y' \in S$. Further from (3.6) it follows that

$$(I-\mathcal{A})^{-1}\mathcal{B}y = y' \quad \text{or} \quad \mathcal{T}y = y'.$$

This shows that $\mathcal{T}$ is well defined and maps S into itself. Next, by hypothesis (c), $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is a complete lattice in view of hypothesis (d). Hence the desired conclusion follows by an application of Theorem 2.2. □

Definition 3.1. *Let $u, v \in (E, \|\cdot\|, \preceq)$ be any two elements with $u \preceq v$, then the closed order interval $[u, v]$ is a set of points in $(E, \|\cdot\|, \preceq)$ defined by*

$$[u, v] = \{x \in E : u \preceq x \preceq v\}.$$

The order interval $[u, v]$ is also called a lattice interval in the Banach lattice $(E, \|\cdot\|, \preceq)$.

Lemma 3.1. *A non-empty, closed lattice interval $[u, v]$ of the complete Banach lattice $(E, \|\cdot\|, K)$ is norm-closed and norm-bounded subset of E which is again a complete lattice.*

Proof. To prove $[u, v]$ is norm-closed, we must show that the partial order $\preceq$ is compatible with the norm topology. In any partially ordered Banach space E, the positive cone

$$K = \{x \in E \mid x \succeq 0\}$$

is norm-closed subset of E. Now, we can express the lattice interval $[u, v]$ using the translations of positive cone as

$$[u, v] = \{x \in E \mid u \preceq x \preceq v\} = (u + K) \cap (v - K).$$

Since K is a norm-closed subset of E, the translations $u + K$ and $v - K$ are norm-closed. As $[u, v]$ is he intersection of two norm-closed sets in E, it is norm-closed.

Next, we prove $[u, v]$ is norm-bounded subset of E. Now, in a partially orderd banach space (such as Banach lattice), the norm $\|\cdot\|$ is monotone with respect to the partial ordereing $\preceq$ in E. This means there exists a constant $M > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. Now for any element $x \in [u, v]$, we have $u \preceq x \preceq v \Longrightarrow x - u \preceq v - u$. Applying monotonicity of the norm $\|\cdot\|$, we obtain

$$\|x - u\| \leq M\|v - u\|.$$

Using the tiangle inequality property of the nrom, we have

$$\|x\| \leq \|x - u + u\| \leq \|x - u\| + \|u\| \leq M\|v - u\| + \|u\|.$$

Since the bound $M\|v - u\| + \|u\|$ is an upper bound independent of x, $[u, v]$ is a norm-bounded subset of E.

Finally, we show that $[u, v]$ is a complete lattice. Let A be a non-empty subset of the lattice interval $[u, v]$. We must prove that A possesses both $\wedge A$ and $\vee A$ inside $[u, v]$. Since E is given a complete lattice, A has $s_0 = \vee A$ and $t_0 = \wedge A$ evaluated over the entire space E. Because every element $a \in A$ belongs to $[u, v]$, we have $u \preceq a \preceq v$ for all $a \in A$. Since v is an upper bound for A, the least upper bound must satisfy $s_0 \preceq v$. Since A is non-empty, choose nny $a \in A$. Then we have $u \preceq a \preceq s_0$. Hence we must have $u \preceq s_0 \preceq v$ and $s_0 \in [u, v]$. Similarly, by dual reasoning we obtain $t_0 \in [u, v]$. Hence $[u, v]$ is a complete lattice. □

As a consequence of Theorem 3.2 and Lemma 3.1 we obtain the following applicable hybrid fixed point result as corollary.

Corollary 3.2. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete Banach lattice $(E, \|\cdot\|, K)$ and let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two operators. Suppose that*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, and

(b) $x = \mathcal{A}x + \mathcal{B}y \Longrightarrow x \in S$ for all $y \in S$.

Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing operators on the closed and bounded subset S of E, the operator $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on S in view of Remark 3.1. Furthermore, the lattice completeness of S follows from Lemma 3.1. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2. Hence the operator equation (3.1) has a solution in S and the set of all solutions is a complete lattice. □

The above corollary 3.2 is an improvement of the Burton [4] type hybrid fixed point theorem involving the sum of two discontinuous operators in a lattice ordered Banach space. If we restrict the domain of the operator $\mathcal{A}$ to the subset S of the lattice ordered Banach space E, then the hybrid fixed result, Theorem 3.2 also remains true under a stronger hypothesis than hypothesis (c) as shown below.

Theorem 3.4. *Let S be a non-empty closed subset of a complete Banach lattice $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators. Suppose that*

(a) $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction,,

(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,

(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$, and

(d) $(S, \preceq)$ is a complete lattice.

Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Proof. Define an operator $\mathcal{T}$ on S by (3.2). We show that $\mathcal{T}$ is well defined and maps S into itself. Let $y \in S$ be fixed and consider the operator $\mathcal{A}_y$ on S defined by

$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.8}$$

Clearly, $\mathcal{A}_y$ defines the mapping $\mathcal{A}_y : S \to S$ in view of hypothesis (c). Then proceeding as in the proof of Theorem 3.1, it is proved that there exists a unique element $y' \in S$ such that

$$\mathcal{A}_y(y') = \mathcal{A}y' + \mathcal{B}y = y'. \tag{3.9}$$

Now, from (3.6) it follows that

$$(I - \mathcal{A})^{-1}\mathcal{B}y = y' \quad \text{or} \quad \mathcal{T}y = y'.$$

This shows that $\mathcal{T}$ is well defined and maps S into itself. Next, by hypothesis (c), $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is a complete lattice in view of hypothesis (d). Hence the desired conclusion follows by an application of Theorem 2.2. □

As a consequence of above hybrid fixed point theorem, we obtain a few more results of hybrid fixed point theory in view of Lemma 3.1. Since the proofs of these results are obvious, we omit the details.

Corollary 3.3. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete lattice partially ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is a contraction,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E, and
(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$.
Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Corollary 3.4. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete Banach lattice $(E, \|\cdot\|, K)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is a contraction,,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E, and
(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$.
Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Remark 3.2. *If we compare all the above Corollaries 3.2 to 3.4, we observe that the hypothesis (a) of Corollary 3.2 is more general than than hypothesis (a) of Corollary 3.4, whereas the hypothesis (b) of Corollary 3.4 is is more general than hypothesis (b) of Corollary 3.2. Again hypothesis (c) of Corollary 3.2 is weaker than hypothesis (c) of Corollary 3.4. However, all the above hybrid fixed point results are applicable to nonlinear hybrid differential and integral equations under suitable conditions depending upon the prevailing situation and the circumstances of the problems for proving the existence of extremal solutions.*

4 Nonlinear Hybrid Integral Equations

In this section we present the statement of the proposed nonlinear hybrid integral equation of this paper, preliminary definitions, abstract spaces and some fundamental results which are useful in the subsequent development of the paper.

Given a closed and bounded interval $J = [a, b]$ in $\mathbb{R}$, the set of real numbers, we consider the nonlinear hybrid integral equation (in short *HIE*) of mixed type,

$$x(t) = q(t) + \lambda \int_a^t v(t, s) f(s, x(s))\, ds + \mu \int_a^b k(t, s) g(s, x(s))\, ds \tag{4.1}$$

for all $t \in J$, where $\lambda, \mu \in \mathbb{R}$, $\lambda > 0$, $\mu > 0$ and the functions $q : J \to \mathbb{R}$, $v, k : J \times J \to \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain hybrid conditions of integrability, Lipschitz, Carathéodory and monotonicity on their respective domains of definition to be specified later.

Definition 4.1. *A function $x \in L^1(J, \mathbb{R})$ is said to be an integrable solution of the HIE* (4.1) *if it satisfies the equation* (4.1) *on J, where $L^1(J, \mathbb{R})$ is the space of Lebesgue integrable real-valued functions defined on J.*

The *HIE* (4.1) is well-known and includes both nonlinear Volterra and Fredholm integral equations given respectively by

$$x(t) = q(t) + \lambda \int_a^t v(t, s) f(s, x(s))\, ds, \;\; t \in J, \tag{4.2}$$

and

$$x(t) = q(t) + \mu \int_a^b k(t, s) g(s, x(s))\, ds, \;\; t \in J, \tag{4.3}$$

as the particular cases. All the nonlinear integral equations given above have already been discussed in the literature extensively for various aspects of the integrable solutions. See Banas [1], Banas and Sayed [2] Emmanuel [23] and the references therein. The existence of continuous solution of the *HIE* (4.1) can be proved by using hybrid fixed point theorem of Krasnoselskii or its improvement given by Dhage (see [13] and references therein). Here we discuss the *HIE* (4.1) for extremity aspects of integrable solutions on J. In particular, we discuss the maximal and minimal integrable solutions via analytic and lattice theoretic hybrid fixed point theorem developed in this paper.

We place the problem of FrIE (4.1) in the function space $L^\infty(J,\overline{\mathbb{R}})$ of essentially bounded Lebesgue measurable extended real-valued functions defined on J, where $\overline{\mathbb{R}}=[-\infty,+\infty]$. It is clear that every essentially bounded and Lebesgue measurable function is Lebesgue integrable, so $L^1(J,\mathbb{R})\supset L^\infty(J,\overline{\mathbb{R}})$. We define a standard norm $\|\cdot\|_\infty$ in $L^\infty(J,\overline{\mathbb{R}})$ by

$$\begin{aligned}\|x\|_\infty &= \operatorname*{ess\,sup}_{t\in J}|x(t)| \\ &= \inf\{C>0 : |x(t)|\leq C \text{ a.e. } t\in J\} \\ &= \inf\{C>0 : -C\leq x(t)\leq C \text{ a.e. } t\in J\}.\end{aligned} \tag{4.4}$$

Clearly, $L^\infty(J,\overline{\mathbb{R}})$ becomes a Banach space w.r.t. the norm $\|\cdot\|_\infty$ defined above. Next, we introduce a partial order relation $\preceq$ in $L^\infty(J,\overline{\mathbb{R}})$ by the order cone K given by

$$K=\{x\in L^\infty(J,\overline{\mathbb{R}})\mid x(t)\geq 0 \ \text{ a. e. } \ t\in J\}. \tag{4.5}$$

Thus,

$$x\preceq y \iff y-x\in K,$$

or equivalently,

$$x\preceq y \iff x(t)\leq y(t) \ \text{ a. e. } \ t\in J. \tag{4.6}$$

The few details of order cones and their properties appear in the monographs of Guo and Lakshmikathham [24] and Granas and Dugundji [25]. We need the following results from lattice theory in what follows. See Dhage [10] and references therein.

Lemma 4.1 (Birkhoff [3])**.** *The partially ordered Banach space $\big(L^\infty(J,\overline{\mathbb{R}}),\|\cdot\|_\infty,\preceq\big)$ is a complete Banach lattice.*

Lemma 4.2. *A non-empty, norm-closed and norm-bounded subset of the form*

$$S=\{x\in L^\infty(J,\overline{\mathbb{R}})\mid \|x\|_\infty\leq r\}$$

for some real number $r>0$, is a closed lattice interval in $L^\infty(J,\overline{\mathbb{R}})$.

Proof. Let S be a non-empty, norm-closed and norm-bounded subset of the complete Banach lattice $L^\infty(J,\overline{\mathbb{R}})$ defined by

$$S=\{x\in L^\infty(J,\overline{\mathbb{R}})\mid \|x\|_\infty\leq r\}$$

for some real number $r>0$. Then, we have

$$\begin{aligned}S&=\{x\in L^\infty(J,\overline{\mathbb{R}})\ \mid\ |x(t)|\leq r \ \text{ a. e. } t\in J\}\\ &=\{x\in L^\infty(J,\overline{\mathbb{R}})\ \mid\ -r\leq x(t)|\leq r \ \text{ a. e. } t\in J\}\\ &=[-r,r].\end{aligned}$$

If we let $u_0(t)=-r$ and $v_0(t)=r$ for almost all $t\in J$, then u_0 and v_0 are constant real-valued functions defined on J. So $u_0,v_0\in L^\infty(J,\overline{\mathbb{R}})$ and $u_0\preceq v_0$. As a result, the lattice interval $[u_0,v_0]$ is well defined and $S=[u_0,v_0]$. The proof of the lemma is complete. □

In the following section we prove our main existence result for the *HIE* (4.1) under suitable conditions on the functions involved in it.

5 Existence of Extremal Integrable Solutions

We consider the following definitions in the sequel.

Definition 5.1. *A function $f:J\times\mathbb{R}\to\mathbb{R}$ is said to be* ***Chandrabhan***

(i) the map $t\mapsto f(t,x)$ is measurable for each $x\in\mathbb{R}$, and

(ii) the map $x\mapsto f(t,x)$ is monotone nondecreasing for almost every $t\in J$.

*Furthermore, a Chandrabhan function $f(t,x)$ is called L^1_r-**Chandrabhan** if*

(iii) there exists a function $h_r\in L^1(J,\mathbb{R})$ such that

$$|f(t,x)|\leq h_r(t) \ \ \text{a. e.} \ \ t\in J,$$

for all $x\in\mathbb{R}$ with $|x|\leq r$.

*Again, a Chandrabhan function $f(t,x)$ is called $L^1_{\mathbb{R}}$-**Chandrabhan** if* *(iv) there exists a function $h \in L^1(J, \mathbb{R})$ such that*

$$|f(t,x)| \leq h(t) \quad a.\,e. \quad t \in J,$$

for all $x \in \mathbb{R}$.

*Finally, a Chandrabhan function $f(t,x)$ is called $\mathbb{R}$-**Chandrabhan** if*

(iv) there exists a number $M_f \in \mathbb{R}_+$ such that

$$|f(t,x)| \leq M_f \quad a.\,e. \quad t \in J,$$

for all $x \in \mathbb{R}$.

Remark 5.1. *Note that the relation between the above types of functions goes as follows:*

" $\mathbb{R}$-Chandrabhan $\Rightarrow$ $L^1_{\mathbb{R}}$-Chandrabhan $\Rightarrow$ L^1_r-Chandrabhan,"

however, the reverse implication may not hold.

Similarly, we have the following definition.

Definition 5.2. *A function $k : J \times J \to \mathbb{R}$ is said to satisfy **integrability condition** if*

(i) the map $(t,s) \mapsto k(t,s)$ is jointly measurable, and

(iii) there exists a function $\gamma_k \in L^1(J, \mathbb{R})$ such that

$$|k(t,s)| \leq \gamma_k(s) \quad a.\,e.\ s \in J,$$

for all $t \in J$.

Definition 5.3. *A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if*

(i) the map $t \mapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto f(t,x)$ is continuous for almost everywhere $t \in J$.

Further, a Carathéodory function is called L^1-Carathéodory if

(iii) there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|f(t,x)| \leq h(t) \ a.\,e.\ t \in J,$$

for all $x \in \mathbb{R}$.

Remark 5.2. *It is clear that every continuous real-valued function $k(t,s)$ on $J \times J$ is L^1-Carathéodory and every such L^1-Carathéodory function k satisfies the integrability condition on $J \times J$, however the converse may not be true. Moreover, if a real-valued function k satisfies the integrability condition on $J \times J$, then it also satisfies the integrability condition on every subset $J' \times J'$ of $J \times J$ for some $J' \subset J$. The details of L^1-Carathéodory functions appear in Banas [1], Banas and El-Sayed [2], Dhage [10] and Heikkilä and Lakshmikantham [26].*

Proposition 5.1. *Let $f(t,x)$ be a Chandrabhan function on $J \times \mathbb{R}$. Then the superposition operator $\mathcal{F}$ given by $\mathcal{F}x(t) = f(t, x(t))$, $t \in J$, monotonically maps the space $L^1(J, \mathbb{R})$ into itself if and only if f satisfies the growth condition*

$$|f(t,x)| \leq a(t) + b|x| \tag{5.1}$$

for almost all $t \in J$ and $x \in \mathbb{R}$, where $a \in L^1(J, \mathbb{R})$ and $b \geq 0$ is a real number.

Proof. The proof of the proposition is similar to a result that given in Krasnoselskii [27] for Caratheódory functions f on $J \times \mathbb{R}$ into $\mathbb{R}$. But for the sake of completeness we give the details of proof.

Part I (Sufficiency): Assume first that the growth condition (5.1) hold. We first prove the measurability of the superposition operator $\mathcal{F}$. Let $x \in L^1(J, \mathbb{R})$ be a measurable function. Since f is Chandrabhan function, $t \mapsto f(t,x)$ is measurable and $x \mapsto f(t,x)$ is nondecreasing on $\mathbb{R}$. It is known that every monotonic function is almost everywhere continuous and has at most countably many jump discontinuities J with finite values. Therefore, the composition function $\mathcal{F}x(t) = f(t, x(t))$ remains a measurable function on J. Taking the Lebesgue integration on both sides of the inequality (5.1), we obtain

$$\|\mathcal{F}x\|_{L^1} = \int_J |f(t, x(t))|\,dt \leq \int_J \big(a(t) + b|x(t)|\big)\,dt = \|a\|_{L^1} + b\|x\|_{L^1} < \infty.$$

As a result, we have that $\mathcal{F}x \in L^!(J,\mathbb{R})$. Next, let $x,y \in L^1(J,\mathbb{R})$ be such that $x(t) \le y(t)$ for almost every $t \in J$. As $f(t,\cdot)$is nondecreasing for almost all $t \in J$, it follows that

$$\mathcal{F}x(t) = f(t,x(t)) \le f(t,y(t)) = \mathcal{F}y(t)$$

for almost everywhere $t \in J$. Therefore $\mathcal{F}$ is a monotone nondecreasing operator on $L^1(J,\mathbb{R})$ into itself.

Part II (Necessity): Assume that $\mathcal{F}$ monotonically maps the space $L^1(J,\mathbb{R})$ into itself. We show that f must satisfy the growth condition (5.1. Suppose not and for each $n \in \mathbb{N}$, there exist elements $x_n \in \mathbb{R}$ and a subset $J_n \subset J$ of positive measure such that

$$|f(t,x_n)| > a(t) + n|x_n|$$

for $t \in J_n$. Now, choose a disjoint sequence of subintervals/subsets $J_n \subset J$ and construct a function $x^*(t)$ defined by piecewise values from x_n on J_n ensuring that

$$\int_{J_n} |x_n|\, dt < \frac{1}{2^n}.$$

This ensures that $x^*(t) = \sum_{n=1}^{\infty} x_n \chi_{J_n}(t)$ which is an element of $L^1(J,\mathbb{R})$. However, when evaluating the superposition operator $\mathcal{F}$ on this function, we obtain

$$\|\mathcal{F}x^*\|_{L^1} = \int_J |f(t,x^*(t))|\, dt = \sum_{n=1}^{\infty} \int_{J_n} |f(t,x_n)|\, dt > \sum \int_{J_n} \big(a(t) + n|x_n|\big)\, dt = \infty.$$

This contradicts to our assumptions that $\mathcal{F}x \in L^1(J,\mathbb{R})$. Hence the growth condition (5.1) must hold. This completes the proof. □

In the special case when $b = 0$ in above Proposition 5.1, we obtain the following useful result for our further discussion.

Lemma 5.1 (Dhage [9, 10]). *If $f(t,x)$ is Chandrabhan, then the function $t \mapsto f(t,x(t))$ is measurable for each $x \in L^1(J,,\mathbb{R})$. Moreover, if $f(t,x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then $f(\cdot,x(\cdot))$ is Lebesgue integrable on J for each $x \in L^1(J,\mathbb{R})$.*

Definition 5.4. *An integrable solution $x_M \in L^1(J,\mathbb{R})$ of the HIE* (4.1) *is said to be maximal if x is any other integrable solution, then $x(t) \le x_M(t)$ for almost every $t \in J$. Similarly, a minimal integrable solution x_m of the HIE* (4.1) *is defined on J.*

We need the following hypotheses in what follows.

(H_1) The function $q : J \to \mathbb{R}$ is Lebesgue measurable and bounded.
(H_2) The functions v and k are nonnegative and satisfy integrability conditions on $J \times J$.
(H_3) There exists a constant $\alpha > 0$ such that

$$|f(t,x) - f(t,y)| \le \alpha\, |x - y| \text{ a. e. t} \in \text{J},$$

for all $x,y \in \mathbb{R}$. Moreover, $\lambda\alpha(b-a) < 1$.
(H_4) The function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$.
(H_5) The function g are L^1_r-Chandrabhan on $J \times \mathbb{R}$.

Theorem 5.1. *Assume that hypotheses (H_1) through (H_5) hold. Then the HIE* (4.1) *has a maximal and a minimal integrable solution defined on J.*

Proof. Here set $E = L^\infty(J,\overline{\mathbb{R}})$. Define a subset S of the complete lattice $\big(L^\infty(J,\overline{\mathbb{R}}), \preceq\big)$ by

$$S = \big\{x \in L^\infty(J,\overline{\mathbb{R}}) \,\big|\, \|x\|_\infty \le r\big\}, \tag{5.2}$$

where,

$$r = \|q\|_\infty + \big[\lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}\big](b-a),$$

and

$$H_{vf}(t) = \gamma_v(t)h(t), \quad H_{kg}(t) = \gamma_k(t)h_r(t),\ t \in J.$$

By Lemmas 3.1, 4.1 and 4.2, $(S,\preceq)$ is a complete Banach lattice. Define two operators $\mathcal{A}$ on $L^\infty(J,\overline{\mathbb{R}})$ and $\mathcal{B}$ on S by

$$\mathcal{A}x(t) = q(t) + \lambda \int_a^t v(t,s)f(s,x(s))\, ds,\ \ t \in J, \tag{5.3}$$

and

$$\mathcal{B}x(t) = \mu \int_a^b k(t,s)g(s,x(s))\,ds, \ \ t \in J. \tag{5.4}$$

Then the *HIE* (4.1) is transformed into the hybrid operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ \ t \in J. \tag{5.5}$$

We show that $\mathcal{A}$ defines a mapping $\mathcal{A} : L^\infty(J,\overline{\mathbb{R}}) \to L^\infty(J,\overline{\mathbb{R}})$. Now by hypothesis ($\mathrm{H}_4$), the function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$ and by hypothesis (H_2), the function v satisfies the integrability condition on $J \times J$. Therefore, the integral on right hand of the equation (5.3) exists. Moreover, the integral is continuous and hence Lebesgue integrable. Again the sum of two Lebesgue integrable functions is again Lebesgue integrable on J. Hence $\mathcal{A}x \in L^\infty(J,\overline{\mathbb{R}})$. Similarly, we have $\mathcal{B}x \in L^\infty(J,\overline{\mathbb{R}})$ for each $x \in S$. Next, we show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on their respective domains of definition. Let $x, y \in L^\infty(J,\overline{\mathbb{R}})$ be such that $x \preceq y$. Then we have

$$\begin{aligned}
\mathcal{A}x(t) &= q(t) + \lambda \int_a^t v(t,s)f(s,x(s))\,ds \\
&\leq q(t) + \lambda \int_a^t v(t,s)f(s,y(s))\,ds \\
&= \mathcal{A}y(t)
\end{aligned}$$

for almost every $t \in J$. This shows that $\mathcal{A}x \preceq \mathcal{A}y$ on J. Consequently, $\mathcal{A}$ is an isotone increasing operator on E. Similarly it can be proved that the operator $\mathcal{B}$ is an isotone increasing on S.

Next, we show that $\mathcal{A}$ is a contraction on $L^\infty(J,\overline{\mathbb{R}})$. Let $x, y \in L^\infty(J,\overline{\mathbb{R}})$. Then, by hypothesis ($\mathrm{H}_3$), we get

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \lambda \int_a^t |f(s,x(s)) - f(s,y(s))|\,ds \\
&\leq \lambda \int_a^t \alpha\,|x(s) - y(s)|\,ds \\
&\leq \lambda\,\alpha \int_a^b |x(s) - y(s)|\,ds \\
&= \lambda\,\alpha\,(b-a)\|x-y\|_\infty.
\end{aligned}$$

Taking the essential supremum on both sides of the above inequality over t, we get

$$\|\mathcal{A}x - \mathcal{A}y\|_\infty \leq \lambda\alpha(b-a)\|x-y\|_\infty$$

for all $x, y \in S$, where $0 \leq \lambda\alpha(b-a) < 1$. This shows that $\mathcal{A}$ is a contraction on $L^\infty(J,\overline{\mathbb{R}})$ and so satisfies the condition (a) of Theorem 3.4. Next we show that condition (b) of Theorem 3.4 is satisfied. Let $y \in S$ be arbitrary and consider the operator equation $x = \mathcal{A}x + \mathcal{B}y$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathcal{A}x(t)| + |\mathcal{B}y(t)| \\
&\leq |q(t)| + \lambda \int_0^t |v(t,s)|\,|f(s,x(s))|\,ds + \mu \int_a^b |k(t,s)|\,|g(s,y(s))|\,ds \\
&\leq |q(t)| + \lambda \int_a^t \gamma_v(s)h(s)\,ds + \mu \int_a^b \gamma_k(s)h_r(s)\,ds \\
&\leq |q(t)| + \lambda \int_a^b H_{vf}(s)\,ds + \mu \int_a^b H_{kg}(s)\,ds \\
&\leq \|q\|_\infty + \lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}.
\end{aligned}$$

Taking the essential supremum on both sides of the above inequality over t, we obtain

$$\|x\|_\infty \leq \|q\|_\infty + (b-a)\big[\lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}\big] \leq r$$

which shows that $x \in S$. Thus condition (b) of Theorem 3.4 is satisfied. Thus all the conditions of Theorem 3.4 hold, whence the operator equations $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S and the set $F_{\mathcal{A}+\mathcal{B}}$ of all solutions is a complete lattice. Consequently $x_m = \wedge F_{\mathcal{A}+\mathcal{B}}$ and $x_M = \vee F_{\mathcal{A}+\mathcal{B}}$ both exist and are respectively the minimal and maximal integrable solutions of the *HIE* (4.1) defined on J. This completes the proof. □

Remark 5.3. *The existence theorem, Theorem 5.1 may be extended under similar hypotheses to generalized nonlinear discontinuous hybrid functional integral equations of the type,*

$$x(t) = q(t) + \lambda \int_a^{\alpha(t)} v(t,s) f(s,x(s))\,ds + \mu \int_a^{\beta(t)} k(t,s) g(s,x(s))\,ds, \;\; t \in J, \tag{5.6}$$

with appropriate modifications, where $\alpha, \beta : J \to J$ are continuous functions.

Remark 5.4. *Notice that Theorem 5.1 also remains true if we replace the function $v(t,s)$ in hypothesis (H_2) by a weaker condition that $v(t,s)$ is nonnegative and satisfies the integrablity condition on the closed subinterval $[a,t]$ for every $t \in J$. As t is a variable in the interval $[a,b]$, we have to check the validity of this condition for every $t \in [a,b]$. To avoid this cumbersome job, we have assumed that the function $v(t,s)$ satisfies the nonnegativity and integrability condition on whole of the interval $[a,b] \times [a,b]$. The situation is useful while dealing with the functional integral equation of the type* (5.6) *defined on J.*

Example 5.1. Let $J = [0,1] \subset \mathbb{R}$ and consider the nonlinear HIE of mixed type,

$$x(t) = t^2 + \frac{1}{2}\int_0^t (t-s)\tan^{-1} x(s)\,ds + \int_0^1 (t+s)\tanh x(s)\,ds, \;\; t \in [0,1]. \tag{5.7}$$

Here, $q(t) = t^2$ which is a continuous and hence Lebesgue measurable and bounded function on $[0,1]$. Again, $v(t,s) = t - s$, $t \geq s$, and $k(t,s) = t + s$ for $t, s \in [0,1]$. Obviously, v and k are continuous and nonnegative real-valued functions defined on $[0,1] \times [0,1]$, so they satisfy the nonnegativity condition on the domain $[0,1] \times [0,t]$ and $[0,1] \times [0,1]$ respectively. The integrability condition also holds for the functions v and k on their domain of definition in view of Remark 5.2. Next, $f(t,x) = \tan^{-1} x$. Clearly, f is a $L^1_{\mathbb{R}}$-Chandrabhan function on $[0,1] \times \mathbb{R}$ with $h(t) = \frac{\pi}{2}$ for all $t \in [0,1]$ and satisfies the hypothesis (H_3) with with Lipschitz constant $\alpha = \frac{1}{2}$. Finally, $g(t,x) = \tanh x$ which is also is a $L^1_{\mathbb{R}}$-Chandrabhan function on $[0,1] \times \mathbb{R}$ with $h(t) = 1$ for all $t \in [0,1]$. Thus, all the conditions (H_1) through (H_5) of Theorem 5.1 are satisfied for $t \in [0,1]$ and $x \in \mathbb{R}$. Therefore, the HIE (5.7) has minimal and maximal integrable solutions defined on the interval $[0,1]$.

6 The Applications

In this section we apply our main existence theorem of the previous section to a couple of the IVPs of nonlinear hybrid first and second order ordinary linearly perturbed differential equations of first kind for proving the existence of maximal and minimal solutions under suitable conditions. Note that such hybrid perturbed differential equations are well-known and have already been discussed for existence of solution via fixed point theorems of Krasnoselskii [27] and Dhage [14] and for existence of extremal solutions via hybrid fixed point theorems due to Dhage [12]. The existence of both lower and upper solutions is assumed and the solution is obtained under continuity and discontinuity of Caratheodory type conditions. Here, we discuss the HPDE (6.1) for existence of maximal and minimal solutions without assuming the existence of both lower and upper solutions as well as relaxing the continuity conditions of any type.

6.1 First order hybrid differential equations

Let $J = [0,T]$ be a closed and bounded interval in $\mathbb{R}$ and consider the hybrid perturbed differential equation (in short *HPDE*) of the type

$$\left.\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t)) \;\text{ a.e. } t \in J, \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{6.1}$$

where the functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain Lipschitz and Chandrabhan type hybrid conditions repectively.

Definition 6.1. *A function $x \in AC(J,\mathbb{R})$ is said to be a solution of the HPDE* (6.1) *if x satisfies the equations in* (6.1)*, where $AC(J,\mathbb{R})$ is the space of absolutely continuous real-valued functions defined on J.*

Lemma 6.1. *If $h \in L^1(J,\mathbb{R})$, then the differential equation*

$$\left.\begin{aligned} x'(t) &= h(t) \;\; a.\,e.\; t \in J, \\ x(0) &= x_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{6.2}$$

is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t h(s)\,ds, \;\; t \in J. \tag{6.3}$$

Theorem 6.1. *Assume that hypotheses (H_1), (H_3), (H_4) and (H_5) hold. If $\alpha < 1$, then the HPDE* (6.1) *has maximal and minimal solutions defined on J.*

Proof. By Lemma 6.1, the *HPDE* (6.1) is equivalent to the nonlinear hybrid perturbed integral equation (in short *HPIE*)

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds, \quad t \in J. \tag{6.4}$$

Set $E = L^\infty(J, \overline{\mathbb{R}})$. Let $q(t) = x_0$. Then q is a constant real function which is Lebesgue measurable and bounded on J. Now the desired conclusion follows by an application of Theorem 5.1 together with Remark 5.3 satisfying $v \equiv 1 \equiv k$ on $J \times J$ and $\alpha(t) = t = \beta(t)$ for all $t \in J$. □

Example 6.1. Given a closed interval $J = [0, 1]$ of the real line $\mathbb{R}$, consider the IVP of ordinary first order nonlinear HPDE,

$$\left.\begin{aligned} x'(t) &= \frac{1}{2}\tan^{-1} x + \tanh x \ \text{ a. e. } t \in J, \\ x(0) &= 1. \end{aligned}\right\} \tag{6.5}$$

It is easy to verify that the functions $f(t, x) = \tan^{-1} x$ and $g(t, x) = \tanh x$ satisfy all hypotheses (H_1), (H_3), (H_4) and (H_5) of Theorem 6.1 and hence the *HPDE* (6.5) has maximal and minimal solutions defined on J.

6.2 Second order hybrid differential equations

Let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$ and consider the hybrid perturbed differential equation (in short *HPDE*) of the type

$$\left.\begin{aligned} x''(t) &= f(t, x(t)) + g(t, x(t)) \ \text{ a. e. } t \in J, \\ x(0) &= x_0, \ \ x'(0) = x_1, \end{aligned}\right\} \tag{6.6}$$

where $x_0, x_1 \in \mathbb{R}$ and the functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain Lipschitz and Chandrabhan type hybrid conditions receptively.

Definition 6.2. *A function $x \in AC^1(J, \mathbb{R})$ is said to be a solution of the HPDE* (6.6) *if x satisfies the equations in* (6.1)*, where $AC^1(J, \mathbb{R})$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J.*

Lemma 6.2. *If $h \in L^1(J, \mathbb{R})$, then the differential equation*

$$\left.\begin{aligned} x''(t) &= h(t) \ \text{ a. e. } t \in J, \\ x(0) &= x_0, \ \ x'(0) = x_1, \end{aligned}\right\} \tag{6.7}$$

is equivalent to the integral equation

$$x(t) = x_0 + x_1 t + \int_0^t (t - s)\, h(s)\, ds, \quad t \in J. \tag{6.8}$$

Theorem 6.2. *Assume that hypotheses (H_1), (H_3), (H_4) and (H_5) hold. If $\alpha < 1$, then the HPDE* (6.6) *has maximal and minimal solutions defined on J.*

Proof. By Lemma 6.1, the *HPDE* (6.1) is equivalent to the nonlinear hybrid perturbed integral equation (in short HPIE)

$$x(t) = x_0 + x_1 t + \int_0^t (t - s)\, f(s, x(s))\, ds + \int_0^t (t - s)\, g(s, x(s))\, ds, \quad t \in J. \tag{6.9}$$

Set $E = L^\infty(J, \overline{\mathbb{R}})$. Let $q(t) = x_0 + x_1 t$. Then q is a continuous real function which is Lebesgue measurablea and bounded on J. Now the desired conclusion follows by an application of Theorem 5.1 together with Remark 5.3 satisfying $v \equiv (t - s) \equiv k$ on $J \times J$ and $\alpha(t) = t = \beta(t)$ for all $t \in J$. □

Example 6.2. Given a closed interval $J = [0, 1]$ of the real line $\mathbb{R}$, consider the IVP of ordinary second order nonlinear HPDE,

$$\left.\begin{aligned} x''(t) &= f_1(t, x(t)) + \tanh x(t) \ \text{ a.e. } t \in J, \\ x(0) &= 1, \ \ x'(0) = 2, \end{aligned}\right\} \tag{6.10}$$

where

$$f_1(t, x) = \begin{cases} 1 & \text{if } \ x \le 0, \\ \dfrac{1}{2} \cdot \dfrac{x}{1 + x} & \text{if } \ x > 0. \end{cases}$$

It is easy to verify that the functions $f(t, x) = f_1(t, x)$ and $g(t, x) = \tanh x$ satisfy all hypotheses (H_1), (H_3), (H_4) and (H_5) of Theorem 6.2 and hence the HPDE (6.10) has maximal and minimal solutions defined on J.

Remark 6.1. *The study of the nonlinear first and second order discontinuous HPDEs* (6.1) *and* (6.6) *of this section may be extended to the higher n^{th} order nonlinear ordinary HPDEs in an analogous way under similar conditions with appropriate modifications for proving the existence of maximal and minimal solutions. Some of the results in this direction will be reported elsewhere.*

Open Problem

Finally, while concluding, we mention an open problem: whether the hybrid fixed point results, Theorems 3.1 and 3.2 of this paper can be extended with appropriate modifications to the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ involving the product of two continuous and discontinuous nonlinear operators $\mathcal{A}$ and $\mathcal{B}$ on a lattice ordered Banach algebra E into itself. A few results along this line under some stringent conditions recently appear in Dhage *et al.* [17], however other related problems are still open.

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RATIONAL-TYPE CONTRACTIONS IN DOUBLE CONTROLLED METRIC TYPE SPACES WITH APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

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Abstract
In this paper, we introduce several classes of rational-type contractive mappings in double controlled metric type spaces, namely rational, quadratic rational, maximum rational, interpolative rational, and mixed rational contractions. Sufficient conditions are established for the existence of fixed points and the convergence of Picard iterative sequences. Existence and uniqueness results are obtained for the rational, quadratic rational, maximum rational, and mixed rational contractions, while existence together with Picard convergence is proved for the interpolative rational contraction. Several examples are presented to illustrate the applicability of the proposed results. Furthermore, an application to a nonlinear Fredholm integral equation is included to demonstrate the effectiveness of the developed theory.
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1 Introduction

Fixed point techniques constitute a central tool in nonlinear analysis, appearing in integral equations, equilibrium problems, and dynamical systems. Classical results such as the Banach contraction principle rely on metric structures, where the geometry is governed by a uniform triangle inequality. However, many nonlinear models evolve in settings where such rigid assumptions fail, motivating the development of generalized distance frameworks.

Among the most flexible of these are *double controlled metric type spaces*, where the standard triangle inequality is replaced by a two-parameter inequality. In this setting, for any $\xi, \eta, \zeta \in \mathscr{X}$,

$$\mathscr{D}(\xi, \zeta) \le \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

The functions α and β adapt the geometry of $\mathscr{X}$ to the local behaviour of sequences, thereby unifying a broad class of generalized metric constructions [1, 19].

Another major advance in fixed point theory is the use of *rational contractive conditions*. Unlike classical linear contractions, rational inequalities incorporate nonlinear terms involving the defect quantities $\mathscr{D}(\xi, \mathscr{T}\xi)$ and $\mathscr{D}(\eta, \mathscr{T}\eta)$. This approach, initiated by Dass and Gupta and later refined by Jaggi and Arshad, yields convergence even in cases where standard Lipschitz conditions are not available [4, 10, 13].

The objective of this article is to combine these two directions by establishing a new rational contraction principle in the framework of double controlled metric type spaces. Our contractive scheme simultaneously reduces the mutual distance $\mathscr{D}(\xi, \eta)$ and controls the defect terms through a rational correction. Under natural orbit-dependent bounds on the control functions α and β, we prove the existence, uniqueness, and convergence of fixed points. We also derive a common fixed point theorem, a stability result for perturbed operators, and an application to a nonlinear integral equation.

For comparison, let $(\mathcal{M}, d)$ be a complete metric space and let $T : \mathcal{M} \to \mathcal{M}$.

Theorem 1.1 (Banach [6]). *If there exists $\kappa \in (0,1)$ such that*

$$d(T\xi, T\eta) \le \kappa\, d(\xi,\eta), \qquad \forall\, \xi,\eta \in \mathcal{M},$$

then T admits a unique fixed point ξ^*, and the Picard sequence $\xi_{n+1} = T\xi_n$ converges to ξ^* for every initial point $\xi_0 \in \mathcal{M}$.

Theorem 1.2 (Kannan [16]). *If there exists $\kappa \in \left(0, \frac{1}{2}\right)$ such that*

$$d(T\xi, T\eta) \le \kappa \left[d(\xi, T\xi) + d(\eta, T\eta)\right], \qquad \forall\, \xi,\eta \in \mathcal{M},$$

then T has a unique fixed point.

Theorem 1.3 (Chatterjea [8]). *If there exists $\kappa \in \left(0, \frac{1}{2}\right)$ such that*

$$d(T\xi, T\eta) \le \kappa \left[d(\xi, T\eta) + d(\eta, T\xi)\right], \qquad \forall\, \xi,\eta \in \mathcal{M},$$

then T has a unique fixed point.

Theorem 1.4 (Rational Type [10, 13]). *If there exist $\lambda, \mu \ge 0$ with $\lambda + \mu < 1$ such that*

$$d(T\xi, T\eta) \le \lambda d(\xi,\eta) + \mu \frac{d(\xi, T\xi)\, d(\eta, T\eta)}{1 + d(\xi,\eta)},$$

for all $\xi, \eta \in \mathcal{M}$, then T has a unique fixed point.

Theorem 1.5 (Double Controlled Metric Type [1]). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete double controlled metric type space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. If there exists $\kappa \in (0,1)$ such that*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \le \kappa\, \mathscr{D}(\xi,\eta), \qquad \forall\, \xi,\eta \in \mathscr{X},$$

and the control functions α and β are bounded along the Picard orbit, then $\mathscr{T}$ has a unique fixed point in $\mathscr{X}$.

2 Preliminaries

This section recalls the basic concepts and results on double controlled metric type spaces *(DCMTS)* [1,2,18,23] used throughout the paper.

Definition 2.1 (Double controlled metric type space [1]). *Let $\mathscr{X} \neq \emptyset$ and let*

$$\mathscr{D} : \mathscr{X} \times \mathscr{X} \to [0,\infty), \qquad \alpha, \beta : \mathscr{X} \times \mathscr{X} \to [1,\infty).$$

The quadruple $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is called a double controlled metric type space *(DCMTS) if, for all $\xi, \eta, \zeta \in \mathscr{X}$,*

(DC1) $\mathscr{D}(\xi,\eta) = 0$ *if and only if* $\xi = \eta$;
(DC2) $\mathscr{D}(\xi,\eta) = \mathscr{D}(\eta,\xi)$;
(DC3)

$$\mathscr{D}(\xi,\zeta) \le \alpha(\xi,\eta)\mathscr{D}(\xi,\eta) + \beta(\eta,\zeta)\mathscr{D}(\eta,\zeta).$$

Remark 2.1. *If $\alpha(\xi,\eta) = \beta(\xi,\eta) = 1$ for all $\xi,\eta \in \mathscr{X}$, then $(\mathscr{X}, \mathscr{D})$ is a metric space. If $\alpha = \beta \equiv b \ge 1$, then $(\mathscr{X}, \mathscr{D})$ is a b-metric space* [9]*. Thus, DCMTS generalizes metric, b-metric, and controlled metric spaces.*

Remark 2.2. *The control functions need not be globally bounded. In many fixed point results, boundedness along the Picard orbit is sufficient* [1, 2].

Definition 2.2 (Convergence and completeness). *A sequence $\{\xi_n\} \subseteq \mathscr{X}$ converges to $\xi \in \mathscr{X}$ if*

$$\mathscr{D}(\xi_n, \xi) \to 0 \qquad (n \to \infty).$$

It is called Cauchy if

$$\mathscr{D}(\xi_n, \xi_m) \to 0 \qquad (n, m \to \infty).$$

A DCMTS is complete if every Cauchy sequence converges in $\mathscr{X}$.

Lemma 2.1 (Orbit telescoping estimate). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\{\xi_n\} \subseteq \mathscr{X}$. Then, for every $m > n$,*

$$\mathscr{D}(\xi_n, \xi_m) \le \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1}) \mathscr{D}(\xi_k, \xi_{k+1}),$$

where the empty product is interpreted as 1.

Proof. For $r = n, \ldots, m$, let

$$u_r = \mathscr{D}(\xi_r, \xi_m),$$

so that $u_m = 0$. Applying (DC3) to the triple $(\xi_r, \xi_{r+1}, \xi_m)$ yields

$$u_r \le \beta(\xi_r, \xi_{r+1})\mathscr{D}(\xi_r, \xi_{r+1}) + \alpha(\xi_r, \xi_{r+1})u_{r+1}.$$

Repeated substitution, together with $u_m = 0$, gives

$$\mathscr{D}(\xi_n, \xi_m) \le \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1})\mathscr{D}(\xi_k, \xi_{k+1}),$$

where the empty product equals 1. This completes the proof. □

Comparison of Generalized Metric Structures

Structure	**Generalized Triangle Inequality**	**Control**	**Source**
Metric space	$d(\xi, \zeta) \le d(\xi, \eta) + d(\eta, \zeta)$	None	[6]
b-metric space	$d(\xi, \zeta) \le b[d(\xi, \eta) + d(\eta, \zeta)]$	Constant $b \ge 1$	[9]
Controlled metric type	$d(\xi, \zeta) \le \alpha(\xi, \eta)d(\xi, \eta) + d(\eta, \zeta)$	One function α	[19]
Double controlled metric type	$\mathscr{D}(\xi, \zeta) \le \alpha(\xi, \eta)\mathscr{D}(\xi, \eta)+\beta(\eta, \zeta)\mathscr{D}(\eta, \zeta)$	Two functions α, β	[1]

Example 2.1. Let $\mathscr{X} = [0, \infty)$ and define

$$\mathscr{D}(\xi, \eta) = (\sqrt{\xi} - \sqrt{\eta})^2,$$

with

$$\alpha(\xi, \eta) = 1 + \sqrt{\xi} + \sqrt{\eta}, \qquad \beta(\eta, \zeta) = 1 + \sqrt{\eta} + \sqrt{\zeta}.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are immediate. Let $u = \sqrt{\xi}$, $v = \sqrt{\eta}$, and $w = \sqrt{\zeta}$. Then

$$(u - w)^2 \le 2(u - v)^2 + 2(v - w)^2.$$

Since

$$\alpha(\xi, \eta) \ge 2, \qquad \beta(\eta, \zeta) \ge 2,$$

we obtain

$$\mathscr{D}(\xi, \zeta) \le \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

Hence, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

Example 2.2. Let $\mathscr{X} = (0, \infty)$ and define

$$\mathscr{D}(\xi, \eta) = (\ln \xi - \ln \eta)^2,$$

with

$$\alpha(\xi, \eta) = 1 + |\ln \xi| + |\ln \eta|, \qquad \beta(\eta, \zeta) = 1 + |\ln \eta| + |\ln \zeta|.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are obvious. Setting $a = \ln \xi$, $b = \ln \eta$, and $c = \ln \zeta$, we have

$$(a - c)^2 \le 2(a - b)^2 + 2(b - c)^2.$$

Since

$$\alpha(\xi, \eta) \ge 2, \qquad \beta(\eta, \zeta) \ge 2,$$

it follows that

$$\mathscr{D}(\xi, \zeta) \le \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

Hence, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

Example 2.3. Let $\mathscr{X} = \mathbb{N}$ and define

$$L(n) = \lfloor \log_2 n \rfloor, \qquad \mathscr{D}(m,n) = |L(m) - L(n)| + \mathbf{1}_{\{m \neq n\}},$$

with

$$\alpha(m,n) = 2 + |L(m) - L(n)|, \qquad \beta(n,k) = 2 + |L(n) - L(k)|.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are immediate. Moreover,

$$|L(m) - L(k)| \leq |L(m) - L(n)| + |L(n) - L(k)|,$$

and

$$\mathbf{1}_{\{m \neq k\}} \leq \mathbf{1}_{\{m \neq n\}} + \mathbf{1}_{\{n \neq k\}}.$$

Hence,

$$\mathscr{D}(m,k) \leq \mathscr{D}(m,n) + \mathscr{D}(n,k).$$

Since

$$\alpha(m,n) \geq 2, \qquad \beta(n,k) \geq 2,$$

we obtain

$$\mathscr{D}(m,k) \leq \alpha(m,n)\mathscr{D}(m,n) + \beta(n,k)\mathscr{D}(n,k),$$

which proves (DC3). Therefore, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

3 Rational Contractions in *DCMTS*

In this section, we introduce several rational-type contractive conditions in the framework of double controlled metric type spaces. These definitions extend the classical Dass–Gupta scheme and provide flexible tools for treating nonlinear operators whose behaviour is not captured by standard Lipschitz inequalities.

Definition 3.1 (*DC*–rational contraction)**.** *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is a (λ, μ)* double-controlled rational contraction *if there exist $\lambda, \mu \in [0, 1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)\mathscr{D}(\eta, \mathscr{T}\eta)}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.1}$$

Definition 3.2 (*DC*–quadratic rational contraction)**.** *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is called a (λ, μ) DC*–quadratic rational contraction *if there exist $\lambda, \mu \in [0, 1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)^2 + \mathscr{D}(\eta, \mathscr{T}\eta)^2}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.2}$$

Definition 3.3 (*DC*–max rational contraction)**.** *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is called a (λ, μ) DC*–max rational contraction *if there exist $\lambda, \mu \in [0, 1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\max\{\mathscr{D}(\xi, \mathscr{T}\xi), \mathscr{D}(\eta, \mathscr{T}\eta)\}}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.3}$$

Definition 3.4 (*DC*–interpolative rational contraction)**.** *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Then $\mathscr{T}$ is called a (λ, μ, θ) DC*–interpolative rational contraction *if there exist $\lambda, \mu \in (0, 1)$ and $\theta \in (0, 1)$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta)^{\theta} + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)^{1-\theta}\mathscr{D}(\eta, \mathscr{T}\eta)^{1-\theta}}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.4}$$

Definition 3.5 (*DC*–mixed rational contraction)**.** *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Then $\mathscr{T}$ is called a (λ, μ) DC*–mixed rational contraction *if there exist $\lambda, \mu \in [0, 1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \max\{\mathscr{D}(\xi, \eta), \mathscr{D}(\xi, \mathscr{T}\xi)\} + \mu \frac{\mathscr{D}(\eta, \mathscr{T}\eta)}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.5}$$

Remark 3.1. *Inequalities* (3.2)–(3.5) *generalize the basic rational contraction model* (3.1) *by modifying the defect terms through quadratic, maximum, interpolative, and mixed rational structures. Each of these contractive conditions reduces to the Banach contraction whenever the rational term vanishes, and all are compatible with the main fixed point theorem under suitable parameter bounds.*

4 Main Results

Throughout this section, let $\xi_0 \in \mathscr{X}$ be an arbitrary initial point, and let $\{\xi_n\}_{n\geq 0}$ denote the corresponding Picard sequence defined by

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

For convenience, we write

$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}), \qquad n \geq 0.$$

Throughout this section, we assume the following orbit-control boundedness condition:

$$\alpha pha^* := \sup_{n\geq 0} \alpha pha(\xi_n, \xi_{n+1}) < \infty, \qquad \beta^* := \sup_{n\geq 0} \beta(\xi_n, \xi_{n+1}) < \infty. \tag{4.1}$$

We also assume that, for every $\omega \in \mathscr{X}$,

$$\sup_{n\geq 0} \alpha pha(\omega, \xi_n) < \infty, \qquad \sup_{n\geq 0} \beta(\xi_n, \omega) < \infty. \tag{4.2}$$

Theorem 4.1. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy the DC–rational contractive condition* (3.1)*. Fix $\xi_0 \in \mathscr{X}$, and define the corresponding Picard sequence $\{\xi_n\}_{n\geq 0}$ by*

$$\xi_{n+1} = \mathscr{T}\xi_n, n \geq 0.$$

Assume that the conditions (4.1) *and* (4.2) *hold. Suppose, in addition, that*

$$q_{\mathrm{r}} := \frac{\lambda}{1-\mu} \in [0, 1), \qquad \beta^* q_{\mathrm{r}} < 1. \tag{4.3}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and the Picard sequence $\{\xi_n\}$ converges to $\xi^\star$.

Proof. For each $n \geq 0$, set

$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}).$$

Applying the DC–rational contractive condition (3.1) to the pair (ξ_n, ξ_{n-1}), where $n \geq 1$, and using

$$\mathscr{T}\xi_n = \xi_{n+1}, \qquad \mathscr{T}\xi_{n-1} = \xi_n,$$

we obtain

$$\begin{aligned} \delta_n &= \mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_{n-1}) \\ &\leq \lambda\mathscr{D}(\xi_n, \xi_{n-1}) + \mu\frac{\mathscr{D}(\xi_n, \mathscr{T}\xi_n)\mathscr{D}(\xi_{n-1}, \mathscr{T}\xi_{n-1})}{1 + \mathscr{D}(\xi_n, \xi_{n-1})} \\ &= \lambda\delta_{n-1} + \mu\frac{\delta_n\delta_{n-1}}{1+\delta_{n-1}}. \end{aligned} \tag{4.4}$$

Since

$$\frac{\delta_{n-1}}{1+\delta_{n-1}} \leq 1,$$

inequality (4.4) yields

$$\delta_n \leq \lambda\delta_{n-1} + \mu\delta_n.$$

Consequently,

$$(1-\mu)\delta_n \leq \lambda\delta_{n-1},$$

and hence, by (4.3),

$$\delta_n \leq q_{\mathrm{r}}\delta_{n-1} \leq q_{\mathrm{r}}^n\delta_0, \qquad n \geq 1. \tag{4.5}$$

Since $q_{\mathrm{r}} \in [0, 1)$, it follows that

$$\delta_n \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Let $m > n \geq 0$. By Lemma 2.1,

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1}\left(\prod_{j=n}^{k-1}\beta(\xi_j, \xi_{j+1})\right)\alpha(\xi_k, \xi_{k+1})\delta_k, \tag{4.6}$$

where an empty product is understood to be equal to 1.

Using (4.1) and (4.5), we obtain

$$\begin{aligned}
\mathscr{D}(\xi_n, \xi_m) &\leq \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n} \delta_k \\
&\leq \alpha^* \delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{r}}^k \\
&= \alpha^* \delta_0 q_{\mathrm{r}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{r}})^r \\
&\leq \frac{\alpha^* \delta_0 q_{\mathrm{r}}^n}{1 - \beta^* q_{\mathrm{r}}}. \qquad (4.7)
\end{aligned}$$

Since $\beta^* q_{\mathrm{r}} < 1$, the right-hand side of (4.7) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Thus,

$$\mathscr{D}(\xi_n, \xi_m) \longrightarrow 0 \qquad \text{as } m, n \to \infty.$$

Therefore, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$. Since $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is complete, there exists $\xi^\star \in \mathscr{X}$ such that

$$\xi_n \longrightarrow \xi^\star \qquad \text{as } n \to \infty.$$

It remains to prove that $\xi^\star$ is a fixed point of $\mathscr{T}$. By (DC3), with ξ_{n+1} as the intermediate point, we have

$$\begin{aligned}
\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) \leq\ & \alpha(\mathscr{T}\xi^\star, \xi_{n+1}) \mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) \\
&+ \beta(\xi_{n+1}, \xi^\star) \mathscr{D}(\xi_{n+1}, \xi^\star). \qquad (4.8)
\end{aligned}$$

Since $\xi_{n+1} = \mathscr{T}\xi_n$, we have

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star, \mathscr{T}\xi_n).$$

Applying (3.1) to the pair $(\xi^\star, \xi_n)$ gives

$$\begin{aligned}
\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) &\leq \lambda \mathscr{D}(\xi^\star, \xi_n) \\
&\quad + \mu \frac{\mathscr{D}(\xi^\star, \mathscr{T}\xi^\star) \mathscr{D}(\xi_n, \mathscr{T}\xi_n)}{1 + \mathscr{D}(\xi^\star, \xi_n)} \\
&= \lambda \mathscr{D}(\xi^\star, \xi_n) + \mu \frac{\mathscr{D}(\xi^\star, \mathscr{T}\xi^\star) \delta_n}{1 + \mathscr{D}(\xi^\star, \xi_n)}. \qquad (4.9)
\end{aligned}$$

Because

$$\mathscr{D}(\xi^\star, \xi_n) \longrightarrow 0 \qquad \text{and} \qquad \delta_n \longrightarrow 0,$$

inequality (4.9) implies that

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

By (4.2), applied with $\omega = \mathscr{T}\xi^\star$ and $\omega = \xi^\star$, respectively, the sequences

$$\big\{\alpha(\mathscr{T}\xi^\star, \xi_{n+1})\big\} \qquad \text{and} \qquad \big\{\beta(\xi_{n+1}, \xi^\star)\big\}$$

are bounded. Letting $n \to \infty$ in (4.8), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) = 0.$$

By (DC1),

$$\mathscr{T}\xi^\star = \xi^\star.$$

Hence, $\xi^\star$ is a fixed point of $\mathscr{T}$.

To establish uniqueness, suppose that $\upsilon, \varpi \in \mathscr{X}$ are two fixed points of $\mathscr{T}$. Applying (3.1) to the pair (υ, ϖ) yields

$$\begin{aligned}
\mathscr{D}(\upsilon, \varpi) &= \mathscr{D}(\mathscr{T}\upsilon, \mathscr{T}\varpi) \\
&\leq \lambda \mathscr{D}(\upsilon, \varpi) + \mu \frac{\mathscr{D}(\upsilon, \mathscr{T}\upsilon) \mathscr{D}(\varpi, \mathscr{T}\varpi)}{1 + \mathscr{D}(\upsilon, \varpi)} \\
&= \lambda \mathscr{D}(\upsilon, \varpi).
\end{aligned}$$

Therefore,

$$(1 - \lambda) \mathscr{D}(\upsilon, \varpi) \leq 0.$$

Since $q_{\mathrm{r}} < 1$, we have
$$\frac{\lambda}{1-\mu} < 1,$$
and hence
$$\lambda < 1 - \mu \le 1.$$
Thus, $\lambda < 1$. Since $\mathscr{D}(\upsilon, \varpi) \ge 0$, it follows that
$$\mathscr{D}(\upsilon, \varpi) = 0.$$
By (DC1), we obtain $\upsilon = \varpi$.

Therefore, $\mathscr{T}$ admits a unique fixed point $\xi^\star \in \mathscr{X}$, and the Picard sequence $\{\xi_n\}$ converges to $\xi^\star$. $\square$

Theorem 4.2. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be sequentially continuous and satisfy the DC–quadratic rational contractive condition* (3.2). *For $\xi_0 \in \mathscr{X}$, define the Picard sequence by*
$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \ge 0,$$
and set
$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}), \qquad n \ge 0.$$
Assume that (4.1) *and* (4.2) *hold, and that*
$$\Delta^* := \sup_{n \ge 0} \delta_n < \infty. \tag{4.10}$$
Suppose, in addition, that
$$\mu\Delta^* < 1, \qquad q_{\mathrm{q}} := \frac{\lambda + \mu\Delta^*}{1 - \mu\Delta^*} \in [0, 1), \qquad \beta^* q_{\mathrm{q}} < 1. \tag{4.11}$$
Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and $\xi_n \to \xi^\star$ as $n \to \infty$.

Proof. For $n \ge 1$, applying (3.2) to (ξ_n, ξ_{n-1}) gives
$$\begin{aligned}\delta_n &= \mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_{n-1}) \\ &\le \lambda\delta_{n-1} + \mu\frac{\delta_n^2 + \delta_{n-1}^2}{1 + \delta_{n-1}}.\end{aligned} \tag{4.12}$$
By (4.10),
$$\delta_n^2 \le \Delta^*\delta_n, \qquad \delta_{n-1}^2 \le \Delta^*\delta_{n-1}.$$
Since $1 + \delta_{n-1} \ge 1$, inequality (4.12) yields
$$\delta_n \le \lambda\delta_{n-1} + \mu\Delta^*\big(\delta_n + \delta_{n-1}\big).$$
Hence,
$$(1 - \mu\Delta^*)\delta_n \le (\lambda + \mu\Delta^*)\delta_{n-1}.$$
Therefore, by (4.11),
$$\delta_n \le q_{\mathrm{q}}\delta_{n-1} \le q_{\mathrm{q}}^n\delta_0, \qquad n \ge 1. \tag{4.13}$$
In particular, $\delta_n \to 0$ as $n \to \infty$.

For $m > n \ge 0$, Lemma 2.1, together with (4.1) and (4.13), gives
$$\begin{aligned}\mathscr{D}(\xi_n, \xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\le \alpha^*\delta_0 q_{\mathrm{q}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{q}})^r \\ &\le \frac{\alpha^*\delta_0 q_{\mathrm{q}}^n}{1 - \beta^* q_{\mathrm{q}}}.\end{aligned} \tag{4.14}$$
Since $\beta^* q_{\mathrm{q}} < 1$, the right-hand side tends to zero as $n \to \infty$, uniformly for $m > n$. Thus, $\{\xi_n\}$ is a Cauchy sequence. By completeness, there exists $\xi^\star \in \mathscr{X}$ such that
$$\xi_n \longrightarrow \xi^\star.$$

Sequential continuity of $\mathscr{T}$ gives

$$\mathscr{T}\xi_n \longrightarrow \mathscr{T}\xi^\star.$$

Moreover, $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$. By (DC3),

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) \le \alpha(\mathscr{T}\xi^\star, \xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) \\ + \beta(\xi_{n+1}, \xi^\star)\mathscr{D}(\xi_{n+1}, \xi^\star).\end{aligned} \tag{4.15}$$

The control sequences in (4.15) are bounded by (4.2), while

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star, \mathscr{T}\xi_n) \longrightarrow 0$$

and

$$\mathscr{D}(\xi_{n+1}, \xi^\star) \longrightarrow 0.$$

Letting $n \to \infty$ in (4.15), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) = 0.$$

Thus, (DC1) implies

$$\mathscr{T}\xi^\star = \xi^\star.$$

For uniqueness, let υ and ϖ be fixed points of $\mathscr{T}$. Then (3.2) gives

$$\mathscr{D}(\upsilon, \varpi) \le \lambda\mathscr{D}(\upsilon, \varpi),$$

since

$$\mathscr{D}(\upsilon, \mathscr{T}\upsilon) = \mathscr{D}(\varpi, \mathscr{T}\varpi) = 0.$$

Furthermore, $q_{\mathrm{q}} < 1$ implies

$$\lambda + \mu\Delta^* < 1 - \mu\Delta^*,$$

and hence $\lambda < 1$. Therefore,

$$\mathscr{D}(\upsilon, \varpi) = 0,$$

so $\upsilon = \varpi$ by (DC1). This completes the proof. ☐

Theorem 4.3. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be sequentially continuous and satisfy the DC–max rational contractive condition* (3.3)*. For $\xi_0 \in \mathscr{X}$, define the Picard sequence by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \ge 0.$$

Assume that (4.1) *and* (4.2) *hold. Suppose, in addition, that*

$$q_{\mathrm{m}} := \lambda + \mu \in [0, 1), \qquad \beta^* q_{\mathrm{m}} < 1. \tag{4.16}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and $\xi_n \to \xi^\star$ as $n \to \infty$.

Proof. For $n \ge 1$, applying (3.3) to (ξ_n, ξ_{n-1}) gives

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\max\{\delta_n, \delta_{n-1}\}}{1 + \delta_{n-1}}. \tag{4.17}$$

If $\delta_n \le \delta_{n-1}$, then

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\delta_{n-1}}{1 + \delta_{n-1}} \le (\lambda + \mu)\delta_{n-1}.$$

If $\delta_n > \delta_{n-1}$, then

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\delta_n}{1 + \delta_{n-1}} \le \lambda\delta_{n-1} + \mu\delta_n,$$

and hence

$$\delta_n \le \frac{\lambda}{1 - \mu}\delta_{n-1}.$$

Since $\lambda + \mu < 1$,

$$\frac{\lambda}{1 - \mu} \le \lambda + \mu = q_{\mathrm{m}}.$$

Thus, in either case,

$$\delta_n \le q_{\mathrm{m}}\delta_{n-1} \le q_{\mathrm{m}}^n\delta_0, \qquad n \ge 1. \tag{4.18}$$

Consequently, $\delta_n \to 0$.

For $m > n \geq 0$, Lemma 2.1, together with (4.1) and (4.18), yields

$$\begin{aligned}
\mathscr{D}(\xi_n, \xi_m) &\leq \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n} \delta_k \\
&\leq \alpha^* \delta_0 q_{\mathrm{m}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{m}})^r \\
&\leq \frac{\alpha^* \delta_0 q_{\mathrm{m}}^n}{1 - \beta^* q_{\mathrm{m}}}.
\end{aligned} \tag{4.19}$$

Since $\beta^* q_{\mathrm{m}} < 1$, the right-hand side tends to zero as $n \to \infty$, uniformly for $m > n$. Hence, $\{\xi_n\}$ is Cauchy. By completeness, there exists $\xi^\star \in \mathscr{X}$ such that

$$\xi_n \longrightarrow \xi^\star.$$

Sequential continuity of $\mathscr{T}$ implies

$$\mathscr{T}\xi_n \longrightarrow \mathscr{T}\xi^\star.$$

Since $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$, it remains to show that these limits coincide. By (DC3),

$$\begin{aligned}
\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) \leq{}& \alpha(\mathscr{T}\xi^\star, \xi_{n+1}) \mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) \\
&+ \beta(\xi_{n+1}, \xi^\star) \mathscr{D}(\xi_{n+1}, \xi^\star).
\end{aligned} \tag{4.20}$$

The control sequences in (4.20) are bounded by (4.2), whereas

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star, \mathscr{T}\xi_n) \longrightarrow 0$$

and

$$\mathscr{D}(\xi_{n+1}, \xi^\star) \longrightarrow 0.$$

Letting $n \to \infty$ in (4.20), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) = 0.$$

Therefore, (DC1) gives

$$\mathscr{T}\xi^\star = \xi^\star.$$

For uniqueness, let υ and ϖ be fixed points of $\mathscr{T}$. Then (3.3) gives

$$\mathscr{D}(\upsilon, \varpi) \leq \lambda \mathscr{D}(\upsilon, \varpi),$$

because

$$\mathscr{D}(\upsilon, \mathscr{T}\upsilon) = \mathscr{D}(\varpi, \mathscr{T}\varpi) = 0.$$

Since (4.16) implies $\lambda < 1$, we obtain

$$\mathscr{D}(\upsilon, \varpi) = 0.$$

Thus, $\upsilon = \varpi$ by (DC1), which completes the proof. □

Theorem 4.4. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy the DC–interpolative rational contractive condition (3.4). For an arbitrary $\xi_0 \in \mathscr{X}$, let $\{\xi_n\}_{n \geq 0}$ be the corresponding Picard sequence.*

Assume that (4.1) and (4.2) hold. Suppose that there exists $q_{\mathrm{i}} \in [0, 1)$ such that

$$\delta_n \leq q_{\mathrm{i}} \delta_{n-1}, \qquad n \geq 1, \qquad \beta^* q_{\mathrm{i}} < 1. \tag{4.21}$$

Then $\{\xi_n\}$ converges to a fixed point $\xi^\star \in \mathscr{X}$ of $\mathscr{T}$.

Proof. Iterating the first inequality in (4.21), we obtain

$$\delta_n \leq q_{\mathrm{i}}{}^n \delta_0, \qquad n \geq 0. \tag{4.22}$$

Since $q_{\mathrm{i}} \in [0, 1)$, it follows that $\delta_n \to 0$ as $n \to \infty$.

Let $m > n$. By Lemma 2.1,

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1}) \delta_k. \tag{4.23}$$

Using (4.1) and (4.22), we obtain

$$\begin{aligned}\mathscr{D}(\xi_n,\xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\le \alpha^*\delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{i}}^{\,k} \\ &= \alpha^*\delta_0 q_{\mathrm{i}}^{\,n} \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{i}})^r \\ &\le \frac{\alpha^*\delta_0 q_{\mathrm{i}}^{\,n}}{1-\beta^* q_{\mathrm{i}}}. \end{aligned} \tag{4.24}$$

Since $\beta^* q_{\mathrm{i}} < 1$, the right-hand side of (4.24) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Hence, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$.

By completeness, there exists $\xi^\star \in \mathscr{X}$ such that $\xi_n \to \xi^\star$. We now verify that $\xi^\star$ is a fixed point of $\mathscr{T}$.

By (DC3), using ξ_{n+1} as an intermediate point, we have

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) \le{}& \alpha(\mathscr{T}\xi^\star,\xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \\ &+ \beta(\xi_{n+1},\xi^\star)\mathscr{D}(\xi_{n+1},\xi^\star). \end{aligned} \tag{4.25}$$

Since $\xi_{n+1} = \mathscr{T}\xi_n$, applying (3.4) to $(\xi^\star,\xi_n)$ gives

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) &\le \lambda\mathscr{D}(\xi^\star,\xi_n)^\theta \\ &\quad + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)^{1-\theta}\mathscr{D}(\xi_n,\mathscr{T}\xi_n)^{1-\theta}}{1+\mathscr{D}(\xi^\star,\xi_n)} \\ &= \lambda\mathscr{D}(\xi^\star,\xi_n)^\theta + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)^{1-\theta}\delta_n^{1-\theta}}{1+\mathscr{D}(\xi^\star,\xi_n)}. \end{aligned} \tag{4.26}$$

Because $\mathscr{D}(\xi^\star,\xi_n) \to 0$ and $\delta_n \to 0$, relation (4.26) implies that

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

By (4.2), the sequences $\{\alpha(\mathscr{T}\xi^\star,\xi_{n+1})\}$ and $\{\beta(\xi_{n+1},\xi^\star)\}$ are bounded. Passing to the limit in (4.25), we obtain $\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) = 0$. Therefore, (DC1) yields $\mathscr{T}\xi^\star = \xi^\star$.

Hence, the Picard sequence $\{\xi_n\}$ converges to a fixed point of $\mathscr{T}$. □

Remark 4.1. *Condition* (3.4) *alone does not guarantee uniqueness. Indeed, if υ and ϖ are distinct fixed points of $\mathscr{T}$, then*

$$\mathscr{D}(\upsilon,\varpi) \le \lambda\mathscr{D}(\upsilon,\varpi)^\theta,$$

and consequently

$$\mathscr{D}(\upsilon,\varpi)^{1-\theta} \le \lambda.$$

This inequality does not imply $\mathscr{D}(\upsilon,\varpi) = 0$. Thus, an additional separation condition is required to establish uniqueness.

Theorem 4.5. *Let $(\mathscr{X},\mathscr{D};\alpha,\beta)$ be a complete DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be a sequentially continuous mapping satisfying the DC–mixed rational contractive condition* (3.5). *For an arbitrary $\xi_0 \in \mathscr{X}$, let $\{\xi_n\}_{n\ge 0}$ be the corresponding Picard sequence.*

Assume that (4.1) *and* (4.2) *hold. Suppose that there exists $q_{\mathrm{x}} \in [0,1)$ such that*

$$\delta_n \le q_{\mathrm{x}}\delta_{n-1}, \qquad n \ge 1, \qquad \beta^* q_{\mathrm{x}} < 1. \tag{4.27}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and $\{\xi_n\}$ converges to $\xi^\star$.

Proof. Iterating the first inequality in (4.27), we obtain

$$\delta_n \le q_{\mathrm{x}}^{\,n}\delta_0, \qquad n \ge 0. \tag{4.28}$$

Hence, $\delta_n \to 0$ as $n \to \infty$.

Let $m > n$. By Lemma 2.1,

$$\mathscr{D}(\xi_n,\xi_m) \le \sum_{k=n}^{m-1}\left(\prod_{j=n}^{k-1}\beta(\xi_j,\xi_{j+1})\right)\alpha(\xi_k,\xi_{k+1})\delta_k. \tag{4.29}$$

Using (4.1) and (4.28), we get

$$\begin{aligned}
\mathscr{D}(\xi_n,\xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\
&\le \alpha^*\delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{x}}^k \\
&= \alpha^*\delta_0 q_{\mathrm{x}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{x}})^r \\
&\le \frac{\alpha^*\delta_0 q_{\mathrm{x}}^n}{1-\beta^* q_{\mathrm{x}}}. \qquad (4.30)
\end{aligned}$$

Since $\beta^* q_{\mathrm{x}} < 1$, the right-hand side of (4.30) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Thus, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$.

By completeness, there exists $\xi^\star \in \mathscr{X}$ such that $\xi_n \to \xi^\star$. Sequential continuity of $\mathscr{T}$ yields $\mathscr{T}\xi_n \to \mathscr{T}\xi^\star$, whereas $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$.

To identify these limits, apply (DC3) with ξ_{n+1} as the intermediate point:

$$\begin{aligned}
\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) \le\ & \alpha(\mathscr{T}\xi^\star,\xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \\
&+ \beta(\xi_{n+1},\xi^\star)\mathscr{D}(\xi_{n+1},\xi^\star). \qquad (4.31)
\end{aligned}$$

By (4.2), the sequences $\{\alpha(\mathscr{T}\xi^\star,\xi_{n+1})\}$ and $\{\beta(\xi_{n+1},\xi^\star)\}$ are bounded. Moreover,

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star,\mathscr{T}\xi_n) \longrightarrow 0,$$

while $\mathscr{D}(\xi_{n+1},\xi^\star) \to 0$. Passing to the limit in (4.31), we obtain $\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) = 0$. Hence, by (DC1), $\mathscr{T}\xi^\star = \xi^\star$.

For uniqueness, let $\upsilon,\varpi \in \mathscr{X}$ be fixed points of $\mathscr{T}$. Applying (3.5) to (υ,ϖ) gives

$$\begin{aligned}
\mathscr{D}(\upsilon,\varpi) &= \mathscr{D}(\mathscr{T}\upsilon,\mathscr{T}\varpi) \\
&\le \lambda \max\{\mathscr{D}(\upsilon,\varpi),\mathscr{D}(\upsilon,\mathscr{T}\upsilon)\} + \mu\frac{\mathscr{D}(\varpi,\mathscr{T}\varpi)}{1+\mathscr{D}(\upsilon,\varpi)} \\
&= \lambda\mathscr{D}(\upsilon,\varpi).
\end{aligned}$$

Therefore,

$$(1-\lambda)\mathscr{D}(\upsilon,\varpi) \le 0.$$

Since $\lambda < 1$ and $\mathscr{D}(\upsilon,\varpi) \ge 0$, it follows that $\mathscr{D}(\upsilon,\varpi) = 0$. Thus, (DC1) yields $\upsilon = \varpi$.

Hence, $\mathscr{T}$ has a unique fixed point and the Picard sequence converges to it. $\square$

5 Examples

In this section, we present examples illustrating the applicability of the main results established in the previous section.

Example 5.1. Let

$$\mathscr{X} = [0,\infty), \qquad \mathscr{D}(\xi,\eta) = |\xi-\eta|^2,$$

and define the control functions by

$$\alpha(\xi,\eta) \equiv 2, \qquad \beta(\xi,\eta) \equiv 2.$$

As verified above, $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \ln\left(1+\frac{\xi}{2}\right), \qquad \xi \ge 0.$$

Since

$$\mathscr{T}'(\xi) = \frac{1}{2+\xi} \le \frac{1}{2},$$

the Mean Value Theorem gives

$$|\mathscr{T}(\xi)-\mathscr{T}(\eta)| \le \frac{1}{2}|\xi-\eta|,$$

and hence

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \le \frac{1}{4}\,\mathscr{D}(\xi, \eta). \tag{5.1}$$

Therefore, the DC–rational contractive condition (3.1) holds with

$$\lambda = \frac{1}{4}, \qquad \mu = 0.$$

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$, (5.1) yields

$$\delta_n \le \frac{1}{4}\,\delta_{n-1},$$

so that

$$\delta_n \le \left(\frac{1}{4}\right)^n \delta_0.$$

Moreover,

$$\alpha^* = \beta^* = 2, \qquad q_{\mathrm{r}} = \frac{1}{4}, \qquad \beta^* q_{\mathrm{r}} = \frac{1}{2} < 1.$$

Thus, all assumptions of Theorem 4.1 are satisfied. Hence, $\mathscr{T}$ possesses a unique fixed point. Since

$$\mathscr{T}(0) = 0,$$

the unique fixed point is

$$\xi^\star = 0,$$

and every Picard iteration converges to $\xi^\star$.

Example 5.2. Let

$$\mathscr{X} = [0, 1], \qquad \mathscr{D}(\xi, \eta) = |\xi - \eta|,$$

and define

$$\alpha(\xi, \eta) \equiv 1, \qquad \beta(\xi, \eta) \equiv 1.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \frac{\xi}{2}, \qquad \xi \in [0, 1].$$

Since

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{1}{2}\,\mathscr{D}(\xi, \eta),$$

the DC–quadratic rational contractive condition (3.2) holds with

$$\lambda = \frac{1}{2}, \qquad \mu = 0.$$

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$, we have

$$\xi_n = \frac{\xi_0}{2^n}, \qquad \delta_n = \mathscr{D}(\xi_n, \xi_{n+1}) = \frac{\xi_0}{2^{n+1}} = \frac{1}{2}\,\delta_{n-1}.$$

Hence,

$$\delta_n = \left(\frac{1}{2}\right)^n \delta_0.$$

Moreover,

$$\Delta^* = \delta_0 \le \frac{1}{2}, \qquad \alpha^* = \beta^* = 1, \qquad q_{\mathrm{q}} = \frac{\lambda + \mu\Delta^*}{1 - \mu\Delta^*} = \frac{1}{2},$$

and

$$\beta^* q_{\mathrm{q}} = \frac{1}{2} < 1.$$

Therefore, all assumptions of Theorem 4.2 are satisfied.

Since

$$\mathscr{T}(0) = 0,$$

the mapping $\mathscr{T}$ has the unique fixed point

$$\xi^\star = 0,$$

and every Picard sequence converges to $\xi^\star$.

Example 5.3. Let
$$\mathscr{X} = [0,3], \qquad \mathscr{D}(\xi,\eta) = |\xi - \eta|,$$
and define
$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$
Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define
$$\mathscr{T}(\xi) = \min\left\{1, \frac{\xi}{2}\right\}, \qquad \xi \in [0,3].$$
Since both $\xi \mapsto \xi/2$ and $t \mapsto \min\{1,t\}$ are Lipschitz continuous with constants $1/2$ and 1, respectively,
$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta) = |\mathscr{T}\xi - \mathscr{T}\eta| \le \frac{1}{2}\,\mathscr{D}(\xi,\eta).$$
Hence, the *DC*–max rational contractive condition (3.3) holds with
$$\lambda = \frac{1}{2}, \qquad \mu = \frac{1}{4}.$$
For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,
$$\delta_n = \mathscr{D}(\xi_n,\xi_{n+1}) \le \frac{1}{2}\,\delta_{n-1},$$
and therefore
$$\delta_n \le \left(\frac{1}{2}\right)^n \delta_0.$$
Moreover,
$$\alpha^* = \beta^* = 1, \qquad q_{\mathrm{m}} = \lambda + \mu = \frac{3}{4}, \qquad \beta^* q_{\mathrm{m}} = \frac{3}{4} < 1.$$
Thus, all assumptions of Theorem 4.3 are satisfied.

Finally, if $\xi = \mathscr{T}(\xi)$, then
$$\xi = \min\left\{1, \frac{\xi}{2}\right\}.$$
If $0 \le \xi \le 2$, then $\xi = \xi/2$, giving $\xi = 0$; whereas $\xi > 2$ would imply $\xi = 1$, which is impossible. Hence,
$$\xi^\star = 0$$
is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

Example 5.4. Let
$$\mathscr{X} = [0,1], \qquad \mathscr{D}(\xi,\eta) = |\xi - \eta|,$$
and define
$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$
Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define
$$\mathscr{T}(\xi) = \frac{\xi}{4}, \qquad \xi \in [0,1].$$
Choose
$$\theta = \frac{1}{2}, \qquad \lambda = \frac{1}{4}, \qquad \mu = \frac{1}{4}.$$
Since $0 \le |\xi - \eta| \le 1$,
$$|\xi - \eta| \le |\xi - \eta|^{1/2},$$
and therefore
$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta) = \frac{1}{4}\,\mathscr{D}(\xi,\eta) \le \lambda \mathscr{D}(\xi,\eta)^\theta.$$
Hence, $\mathscr{T}$ satisfies the *DC*–interpolative rational contractive condition (3.4).

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,
$$\xi_n = \frac{\xi_0}{4^n}, \qquad \delta_n = \frac{1}{4}\,\delta_{n-1},$$

so that (4.21) holds with

$$q_{\mathrm{i}} = \frac{1}{4}.$$

Moreover,

$$\alpha^* = \beta^* = 1, \qquad \beta^* q_{\mathrm{i}} = \frac{1}{4} < 1.$$

Hence, all assumptions of Theorem 4.4 are satisfied.

Finally,

$$\mathscr{T}(\xi) = \xi \iff \frac{\xi}{4} = \xi \iff \xi = 0.$$

Thus,

$$\xi^\star = 0$$

is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

Example 5.5. Let

$$\mathscr{X} = [0,4], \qquad \mathscr{D}(\xi,\eta) = |\xi - \eta|,$$

and define

$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$

Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \begin{cases} \dfrac{\xi}{3}, & 0 \le \xi \le 2, \\ \dfrac{\xi+2}{6}, & 2 < \xi \le 4. \end{cases}$$

The mapping is continuous on $[0,4]$ and satisfies

$$\mathscr{T}(\mathscr{X}) \subseteq [0,1].$$

Moreover,

$$|\mathscr{T}\xi - \mathscr{T}\eta| \le \frac{1}{3}|\xi - \eta|, \qquad \xi,\eta \in \mathscr{X},$$

and hence

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta) \le \frac{1}{3}\,\mathscr{D}(\xi,\eta).$$

Choosing

$$\lambda = \frac{1}{3}, \qquad \mu = \frac{1}{4},$$

it follows immediately that $\mathscr{T}$ satisfies the DC–mixed rational contractive condition (3.5).

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,

$$\delta_n = \mathscr{D}(\xi_n,\xi_{n+1}) \le \frac{1}{3}\,\delta_{n-1},$$

so that

$$q_{\mathrm{x}} = \frac{1}{3}.$$

Furthermore,

$$\alpha^* = \beta^* = 1, \qquad \beta^* q_{\mathrm{x}} = \frac{1}{3} < 1.$$

Hence, all assumptions of Theorem 4.5 are fulfilled.

Finally,

$$\mathscr{T}(\xi) = \xi$$

gives

$$\frac{\xi}{3} = \xi \quad (0 \le \xi \le 2),$$

or

$$\frac{\xi+2}{6} = \xi \quad (2 < \xi \le 4).$$

The first equation yields $\xi = 0$, whereas the second gives $\xi = \frac{2}{5}$, which does not belong to $(2,4]$. Therefore,

$$\xi^\star = 0$$

is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

6 Application to a Nonlinear Integral Equation

Let

$$\mathscr{X} = C([0,1],\mathbb{R})$$

be the Banach space of real-valued continuous functions on $[0,1]$ equipped with the supremum norm

$$\|\varphi\|_\infty = \max_{t\in[0,1]} |\varphi(t)|.$$

Define

$$\mathscr{D}(\varphi,\psi) = \|\varphi-\psi\|_\infty, \qquad \alpha(\varphi,\psi) \equiv 1, \qquad \beta(\varphi,\psi) \equiv 1.$$

Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Consider the nonlinear Fredholm integral equation

$$\varphi(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi(s))\,ds, \qquad t \in [0,1]. \tag{6.1}$$

Define the associated integral operator

$$(\mathscr{T}\varphi)(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi(s))\,ds.$$

Theorem 6.1. *Assume that*

(I) $\mathscr{G} \in C([0,1],\mathbb{R})$*;*
(II) $\mathscr{K} \in C([0,1]\times[0,1],\mathbb{R})$*;*
(III) $\mathscr{F} : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous;*
(IV) *there exist* $\lambda,\mu \in [0,1)$ *with* $\lambda+\mu<1$ *such that*

$$\mathscr{D}(\mathscr{T}\varphi,\mathscr{T}\psi) \le \lambda\mathscr{D}(\varphi,\psi) + \mu\frac{\mathscr{D}(\varphi,\mathscr{T}\varphi)\mathscr{D}(\psi,\mathscr{T}\psi)}{1+\mathscr{D}(\varphi,\psi)}, \quad \forall\,\varphi,\psi \in \mathscr{X}. \tag{6.2}$$

Then the integral equation (6.1) *admits a unique solution in* $C([0,1],\mathbb{R})$*.*

Proof. The continuity of $\mathscr{G}$, $\mathscr{K}$, and $\mathscr{F}$ ensures that $\mathscr{T} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is well defined.

Condition (6.2) is precisely the *DC*–rational contractive condition (3.1). Since $\alpha \equiv \beta \equiv 1$, the control boundedness assumptions of Theorem 4.1 are automatically satisfied. Hence, $\mathscr{T}$ possesses a unique fixed point $\varphi^\star \in \mathscr{X}$.

Finally,

$$\mathscr{T}\varphi^\star = \varphi^\star$$

is equivalent to

$$\varphi^\star(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi^\star(s))\,ds,$$

which is exactly (6.1). Therefore, (6.1) has a unique continuous solution. □

7 Conclusion

In this paper, we introduced several classes of rational-type contractions in double controlled metric type spaces and established fixed point results for rational, quadratic, maximum, interpolative, and mixed rational contractive mappings. Under suitable control conditions, existence and uniqueness were obtained for the rational, quadratic, maximum, and mixed contractions, while existence together with Picard convergence was proved for the interpolative case. Illustrative examples and an application to a nonlinear Fredholm integral equation demonstrated the applicability of the proposed results. Future work may focus on multivalued mappings, graph and ordered structures, coupled and common fixed points, and applications to fractional and nonlinear evolution equations.

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NEW GENERALIZED RESULTS ON GRÜSS-TYPE FRACTIONAL INTEGRAL INEQUALITIES INVOLVING THE CAPUTO–FABRIZIO INTEGRAL OPERATOR

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Abstract

The Grüss inequality is a well-known result in mathematical analysis that gives an upper estimate for the difference between the integral of the product of two functions and the product of their individual integrals. Its generalizations in various mathematical contexts, including fractional calculus. In this paper, we present a comprehensive generalization of certain Grüss-type integral inequalities and other related integral inequalities using the Caputo-Fabrizio fractional integral operator. Specifically, we extend and generalize classical integral inequalities to obtain new results within the framework of fractional calculus. Unlike traditional fractional operators, the Caputo-Fabrizio operator is characterized by a non-singular exponential kernel that provides a more regular and physically meaningful approach to fractional integration. By utilizing this operator into the analysis, we derive new generalized versions of the classical Grüss inequality.

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1 Introduction

Chebyshev [5] introduced the well-known Chebyshev inequality

$$\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\wp(\theta)\,d\theta \geq \left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\,d\theta\right)\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\wp(\theta)\,d\theta\right), \tag{1.1}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ which are synchronous on $[\varepsilon_1,\varepsilon_2]$, that is,

$$(\mathfrak{H}(\theta)-\mathfrak{H}(\varphi))(\wp(\theta)-\wp(\varphi))\geq 0, \quad \theta,\varphi\in[\varepsilon_1,\varepsilon_2].$$

The well-known Chebyshev functional [5] is defined by

$$T(\mathfrak{H},\wp)=\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\wp(\theta)\,d\theta-\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\,d\theta\right)\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\wp(\theta)\,d\theta\right), \tag{1.2}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ which are synchronous on $[\varepsilon_1,\varepsilon_2]$ then

$$T(\mathfrak{H},\wp)\geq 0. \tag{1.3}$$

A weighted extension of the Chebyshev functional was presented by the Chebyshev in [5], generalizing the formulation in (1.1).

$$T(\mathfrak{H},\wp,u)=\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\mathfrak{H})\zeta(\theta)\wp(\theta)\,d\theta-\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\mathfrak{H}(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\wp(\theta)\,d\theta, \tag{1.4}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ and u is positive and integrable functions on a finite interval $[\varepsilon_1,\varepsilon_2]$.

Again Chebyshev [5] considered the extended Chebyshev functional

$$T(\mathfrak{H},\wp,u,v)=\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}v(\theta)\mathfrak{H}(\theta)\wp(\theta)\,d\theta+\int_{\varepsilon_1}^{\varepsilon_2}v(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\mathfrak{H}(\theta)\wp(\theta)\,d\theta$$

$$- \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\mathfrak{H}(\theta)\,d\theta \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\wp(\theta)\,d\theta - \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\wp(\theta)\,d\theta \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\mathfrak{H}(\theta)\,d\theta, \tag{1.5}$$

where $\mathfrak{H}$ and $\wp$ are two real-valued integrable functions which are synchronous on $[\varepsilon_1, \varepsilon_2]$ and u, v are two positive integrable functions on a finite interval $[\varepsilon_1, \varepsilon_2]$.

In 1935, Grüss [15] introduced the following inequality

$$\left| \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\wp(\theta)\,d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\,d\theta \right) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\,d\theta \right) \right| \leq \frac{1}{4}(\mathfrak{p} - \mathfrak{q})(\mathfrak{d} - \mathfrak{e}), \tag{1.6}$$

such that $\mathfrak{H}$ and $\wp$ are integrable functions on $[\varepsilon_1, \varepsilon_2]$ and satisfy the condition

$$\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}, \mathfrak{d} \leq \wp(\theta) \leq \mathfrak{e}; \mathfrak{p}, \mathfrak{q}, \mathfrak{d}, \mathfrak{e} \in \mathbb{R}, \theta \in [\varepsilon_1, \varepsilon_2].$$

In 1999, Matić *et al.* [21] introduced the pre-Grüss inequality

$$\begin{aligned} &\left| \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\wp(\theta)\,d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\,d\theta \right) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\,d\theta \right) \right| \\ &\leq \frac{1}{2}(\mathfrak{p} - \mathfrak{q}) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp^2(\theta)\,d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\,d\theta \right)^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{1.7}$$

such that $\mathfrak{H}$ is integrable functions on $[\varepsilon_1, \varepsilon_2]$ and satisfy the condition

$$\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}; \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, \theta \in [\varepsilon_1, \varepsilon_2].$$

In 2018, Choi [9] proved Grüss inequality for two weighted functions.
Let $\mathfrak{H}, \wp : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be integrable functions such that $\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}$ and $\mathfrak{d} \leq \wp(\theta) \leq \mathfrak{e}$ for all $\theta \in [\varepsilon_1, \varepsilon_2]$, where $\mathfrak{p}, \mathfrak{q}, \mathfrak{d}, \mathfrak{e}$ are constants. Also let $u, v : [\varepsilon_1, \varepsilon_2] \to [0, \infty)$ be integrable functions on $[\varepsilon_1, \varepsilon_2]$ such that

$$\min \left\{ \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\,d\theta, \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\,d\theta \right\} > 0,$$

then

$$\begin{aligned} |T(\mathfrak{H}, \wp, u, v)| \leq &\left[\left(\frac{(\mathfrak{p} - \mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|_\infty^2 \right) \left(\frac{(\mathfrak{d} - \mathfrak{e})^2}{2} + 2\|\wp\|_\infty^2 \right) \right]^{\frac{1}{2}} \\ &\left(\int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\,d\theta \right) \left(\int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\,d\theta \right), \end{aligned} \tag{1.8}$$

$$\|\mathfrak{H}\|_\infty = \sup_{\theta \in [\varepsilon_1, \varepsilon_2]} |\mathfrak{H}(\theta)| \quad \text{and} \quad \|\wp\|_\infty = \sup_{\theta \in [\varepsilon_1, \varepsilon_2]} |\wp(\theta)|.$$

Fractional calculus generalizes classical calculus to derivatives and integrals of non-integer order. Fractional integral inequalities provide bounds for fractional operators and are important tools in the analysis of fractional differential and integral equations.

The introduction of fractional integral operators has played a significant role in extending classical integral inequalities. In particular, Grüss inequalities and their various extensions have been established for the Riemann-Liouville, Hadamard, Erdlyi-Kober, Saigo, Katugampola fractional integral operators, including those involving the Mittag-Leffler kernel and the references therein [1, 8, 12, 16, 20, 25, 26, 27, 28, 29].

In recent years, fractional integral inequalities have received considerable attention in fractional analysis. Numerous extensions of classical inequalities, including Grüss, weighted Grüss, Chebyshev, and pre-Grüss inequalities, have been established for a variety of fractional integral operators and the references therein [2, 11, 13, 14, 18, 19].

In [4] Caputo and Fabrizio introduced a novel class of fractional derivatives characterized by exponential kernels, which has been successfully applied to both time- and space-fractional models. Subsequently,[24] in 2019 Nchama et al. derived new fractional integral inequalities using the Caputo-Fabrizio operator.More recently, several authors have established Chebyshev, Grüss, weighted, Pólya-Szegö, and Minkowski-type inequalities involving the Caputo-Fabrizio fractional integral operator [6, 7, 22, 23].

Motivated by the aforementioned studies and results, the primary objective of this paper is to derive new generalized fractional integral inequalities involving the Caputo-Fabrizio integral operator. In particular, we establish novel inequalities based on Youngs inequality, the arithmetic-geometric mean inequality, as well as Grüss-type, weighted Grüss-type, and pre-Grüss-type integral inequalities.

Definition 1.1 ([3] (Cauchy–Schwarz Inequality))**.** *Let $\mathfrak{H}(\alpha, \beta)$ and $\wp(\alpha, \beta)$ be measurable functions defined on a product measure space $D = [c_1, c_2] \times [b_1, b_2]$. Then the Cauchy–Schwarz inequality states that*

$$\left(\iint_D \mathfrak{H}(x, y) \wp(x, y)\, dx\, dy \right)^2 \leq \left(\iint_D \mathfrak{H}(x, y)^2\, dx\, dy \right) \left(\iint_D \wp(x, y)^2\, dx\, dy \right), \tag{1.9}$$

where c_1, c_2, b_1, b_2 are real numbers and measurable set $D \subset \mathbb{R}^2$.

Definition 1.2 ([10, 17])**.** *Let $\mathfrak{H}$ and $\wp$ be continuous, strictly increasing and mutually inverse for non-negative argument, with $\mathfrak{H}(0) = \wp(0) = 0$. Then*

$$mn \leq \int_0^m \mathfrak{H}(x)\, dx + \int_0^n \wp(x)\, dx, \tag{1.10}$$

equality is satisfied if and only if $n = \mathfrak{H}(m)$. In this form, if m, n are non-negative and $r, s > 1$ such that $\frac{1}{r} + \frac{1}{s} = 1$, then

$$\frac{1}{r} m^r + \frac{1}{s} n^s \geq mn, \qquad m, n > 0, \quad \frac{1}{r} + \frac{1}{s} = 1.$$

Definition 1.3 ([4])**.** *Let $\eta \in \mathbb{R}$ such that $0 < \eta \leq 1$. The Caputo–Fabrizio fractional integral of order η of a function $\mathfrak{f}$ is defined by*

$$\mathfrak{U}^{\eta}_{0,\theta} \mathfrak{f}(\theta) = \frac{1}{\eta} \int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta - \xi)} \mathfrak{f}(\xi)\, d\xi. \tag{1.11}$$

For $\eta = 1$, we get

$$\mathfrak{U}^{1}_{0,\theta} \mathfrak{f}(\theta) = \int_0^\theta \mathfrak{f}(\xi)\, d\xi. \tag{1.12}$$

2 On certain fractional integral inequalities

This section explores the establishment and generalization of new fractional integral inequalities and their extensions through classical integral inequalities such as Young's inequality and the arithmetic–geometric mean inequality.

Theorem 2.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0, \infty)$ and if $r, s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. For $\theta \geq 0$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$ then the following inequalities hold:*

$$\begin{aligned}
(a)\quad & \frac{1}{r} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^r(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \wp^s(\theta) + \frac{1}{s} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^s(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \wp^r(\theta) \\
& \geq \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}(\theta) \wp(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \mathfrak{H}(\theta) \wp(\theta), \\
(b)\quad & \frac{1}{r} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^r(\theta) + \frac{1}{s} \mathfrak{U}^{\vartheta}_{0,\theta} \wp^s(\theta) \geq \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^{r-1}(\theta) \wp^{s-1}(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \mathfrak{H}(\theta) \wp(\theta), \\
(c)\quad & \frac{1}{r} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^r(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \wp^2(\theta) + \frac{1}{s} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^s(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \wp^2(\theta) \\
& \geq \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^{\frac{2}{r}}(\theta) \wp^{\frac{2}{s}}(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \mathfrak{H}(\theta) \wp(\theta), \\
(d)\quad & \frac{1}{r} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^2(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \wp^s(\theta) + \frac{1}{s} \mathfrak{U}^{\eta}_{0,\theta} \wp^2(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \mathfrak{H}^r(\theta) \\
& \geq \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^{r-1}(\theta) \wp^{s-1}(\theta) \mathfrak{U}^{\vartheta}_{0,\theta} \mathfrak{H}^{\frac{2}{r}}(\theta) \wp^{\frac{2}{s}}(\theta).
\end{aligned}$$

Proof. We use the following Young's inequality given by

$$\frac{1}{r} m^r + \frac{1}{s} n^s \geq mn, \quad m, n > 0, \quad \frac{1}{r} + \frac{1}{s} = 1. \tag{2.1}$$

Now for $m = \mathfrak{H}(\alpha) \wp(\beta)$ and $n = \mathfrak{H}(\beta) \wp(\alpha)$, $\alpha, \beta > 0$, we get

$$\frac{1}{r} (\mathfrak{H}(\alpha) \wp(\beta))^r + \frac{1}{s} (\mathfrak{H}(\beta) \wp(\alpha))^s \geq (\mathfrak{H}(\alpha) \wp(\beta))(\mathfrak{H}(\beta) \wp(\alpha)). \tag{2.2}$$

Multiplying (2.2) by $\frac{1}{\eta} e^{-\frac{1-\eta}{\eta}(\theta - \alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{\wp^r(\beta)}{r} \mathfrak{U}^{\eta}_{0,\theta} \mathfrak{H}^r(\theta) + \frac{\mathfrak{H}^s(\beta)}{s} \mathfrak{U}^{\eta}_{0,\theta} \wp^s(\theta) \geq \mathfrak{H}(\beta) \wp(\beta) \mathfrak{U}^{\eta}_{0,\theta} (\mathfrak{H}(\theta) \wp(\theta)). \tag{2.3}$$

Multiplying (2.3) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^s(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta)\wp(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}(\theta)\wp(\theta)). \tag{2.4}$$

Similarly, inequalities (b), (c), (d) can be established by substituting the following identities in (2.1), respectively

$$m=\frac{\mathfrak{H}(\alpha)}{\mathfrak{H}(\beta)},\quad n=\frac{\wp(\alpha)}{\wp(\beta)},\quad \mathfrak{H}(\beta),\wp(\beta)\neq 0, \tag{2.5}$$

$$m=\mathfrak{H}(\alpha)\wp^{2r}(\beta),\quad n=\mathfrak{H}^{2s}(\beta)\wp(\alpha), \tag{2.6}$$

$$m=\mathfrak{H}^{2r}(\alpha)\mathfrak{H}(\beta),\quad n=\wp^{2s}(\alpha)\wp(\beta). \tag{2.7}$$

□

Theorem 2.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable function on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\geq 0$, $0<\eta\leq 1$ then following inequalities satisfied:*

$$(a)\ \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\geq\left[\frac{1-\eta}{1-e^{-\frac{1-\eta}{\eta}\theta}}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta),$$

$$(b)\ \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2,$$

$$(c)\ \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)$$
$$\geq\left(\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp)^{r-1}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp)^{s-1}(\theta)\right),$$

$$(d)\ \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\wp^{s-1}(\theta)\right).$$

Proof. (a) We use the following Youngs inequality given by

$$\frac{1}{r}m^r+\frac{1}{s}n^s\geq mn,\qquad m,n>0,\quad \frac{1}{r}+\frac{1}{s}=1. \tag{2.8}$$

Now for $m=\mathfrak{H}(\alpha)$ and $n=\wp(\beta)$, $\alpha,\beta>0$ in (2.8), we get

$$\frac{1}{r}\mathfrak{H}^r(\alpha)+\frac{1}{s}\wp^s(\beta)\geq\mathfrak{H}(\alpha)\wp(\beta). \tag{2.9}$$

Multiplying (2.9) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{\wp^s(\beta)}{s(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\geq\wp(\beta)\mathfrak{U}^{\eta}_{0,\theta}\zeta(\theta). \tag{2.10}$$

Multiplying (2.10) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\geq\frac{1-\eta}{1-e^{-\frac{1-\eta}{\eta}\theta}}\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta). \tag{2.11}$$

(b) For proof of (b), put $m=\mathfrak{H}(\alpha)\wp(\beta)$ and $n=\mathfrak{H}(\beta)\wp(\alpha)$ in (2.8) we get the desired result.

(c) For proof of (c), put $m=\dfrac{\mathfrak{H}(\alpha)}{\wp(\alpha)}$ and $n=\dfrac{\mathfrak{H}(\beta)}{\wp(\beta)}$, $\wp(\alpha)\neq 0$, $\wp(\beta)\neq 0$ in (2.8) we get the desired result.

(d) For proof of (d), put $m=\dfrac{\mathfrak{H}(\alpha)}{\mathfrak{H}(\beta)}$ and $n=\dfrac{\wp(\alpha)}{\wp(\beta)}$, $\mathfrak{H}(\beta)\neq 0$, $\wp(\beta)\neq 0$ in (2.8) we get the desired result. □

Theorem 2.3. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\geq 0$, $0<\eta\leq 1$ then following inequalities satisfied:*

$$\begin{aligned}
&(a)\quad \frac{1}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right)+\frac{1}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right)\\
&\qquad\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{r}(\theta)\mathfrak{H}^{s}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right),\\
&(b)\quad \frac{1}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)+\frac{1}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)\\
&\qquad\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\delta^{s-1}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\wp^{s}(\theta)\right).
\end{aligned}$$

Proof. **(a)** Substituting $m=\mathfrak{H}(\alpha)\wp^{\frac{2}{r}}(\beta), n=\mathfrak{H}^{\frac{2}{s}}(\beta)\wp(\alpha), \alpha,\beta>0$ in (2.8), we get

$$\frac{1}{r}\mathfrak{H}^{r}(\alpha)\wp^{2}(\beta)+\frac{1}{s}\wp^{s}(\alpha)\mathfrak{H}^{2}(\beta)\geq\mathfrak{H}(\alpha)\wp(\alpha)\mathfrak{H}^{\frac{2}{r}}(\beta)\wp^{\frac{2}{s}}(\beta). \tag{2.12}$$

Multiplying (2.12) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$\frac{\wp^{2}(\beta)}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right)+\frac{\mathfrak{H}^{2}(\beta)}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right)\geq\mathfrak{H}^{\frac{2}{r}}(\beta)\wp^{\frac{2}{s}}(\beta)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right). \tag{2.13}$$

Now multiplying (2.13) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ we get (a).

(b) Substituting $m=\frac{\mathfrak{H}^{\frac{2}{r}}(\alpha)}{\mathfrak{H}(\beta)}, n=\frac{\wp^{\frac{2}{s}}(\alpha)}{\wp(\beta)}, \mathfrak{H}(\beta),\wp(\beta)\neq 0$ in (2.8), we get

$$\frac{\mathfrak{H}^{2}(\alpha)}{r\mathfrak{H}^{r}(\beta)}+\frac{\wp^{2}(\alpha)}{s\wp^{s}(\beta)}\geq\frac{\mathfrak{H}^{\frac{2}{r}}(\alpha)\wp^{\frac{2}{s}}(\alpha)}{\mathfrak{H}(\beta)\wp(\beta)},$$

which can be written as

$$\frac{\mathfrak{H}^{2}(\alpha)\wp^{s}(\beta)}{r}+\frac{\wp^{2}(\alpha)\mathfrak{H}^{r}(\beta)}{s}\geq\mathfrak{H}^{\frac{2}{r}}(\alpha)\wp^{\frac{2}{s}}(\alpha)\mathfrak{H}^{r-1}(\beta)\wp^{s-1}(\beta). \tag{2.14}$$

Again, multiplying (2.14) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{\wp^{s}(\beta)}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)+\frac{\mathfrak{H}^{r}(\beta)}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)\geq\mathfrak{H}^{r-1}(\beta)\wp^{s-1}(\beta)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{\frac{2}{s}}(\theta)\right). \tag{2.15}$$

Multiplying (2.15) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ we get (b). □

Theorem 2.4. *Let $\theta\geq 0$, $0<\eta\leq 1$, $\mathfrak{H}$ and $\wp$ be two positive and integrable functions on $[0,\infty)$ and if $r,s>1$ be such that*

$$\frac{1}{r}+\frac{1}{s}=1.$$

Let

$$\mathbb{S}=\min_{0\leq\alpha\leq\theta}\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}\quad\textit{and}\quad\mathbb{P}=\max_{0\leq\alpha\leq\theta}\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}. \tag{2.16}$$

Then the following inequalities satisfied:

$$\begin{aligned}
&(a)\quad 0\leq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)\leq\frac{(\mathbb{P}+\mathbb{S})^{2}}{4\mathbb{PS}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^{2},\\
&(b)\quad 0\leq\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)-\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)}\\
&\qquad\leq\frac{(\sqrt{\mathbb{P}}-\sqrt{\mathbb{S}})^{2}}{2\sqrt{\mathbb{SP}}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right),\\
&(c)\quad 0\leq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)-\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)\\
&\qquad\leq\frac{(\mathbb{P}-\mathbb{S})^{2}}{4\mathbb{SP}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^{2}.
\end{aligned}$$

Proof. **(a)** From equation (2.16) and inequality

$$\left(\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}-\mathbb{S}\right)\left(\mathbb{P}-\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}\right)\wp^2(\alpha)\geq 0, 0\leq\alpha\leq\theta, \tag{2.17}$$

which can be written as

$$(\mathfrak{H}(\alpha)-\mathbb{S}\wp(\alpha))(\mathbb{P}\wp(\alpha)-\xi(\alpha))\geq 0,$$

it follows that

$$(\mathbb{P}+\mathbb{S})\mathfrak{H}(\alpha)\wp(\alpha)\geq\mathfrak{H}^2(\alpha)+\mathbb{S}\mathbb{P}\,\wp^2(\alpha). \tag{2.18}$$

Multiplying (2.18) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$(\mathbb{P}+\mathbb{S})\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta). \tag{2.19}$$

Now, since $\mathbb{S}\mathbb{P}>0$,

$$\left(\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}-\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}\right)^2\geq 0, \tag{2.20}$$

so that

$$\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\geq 2\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}. \tag{2.21}$$

Using equations (2.19) and (2.21) we get

$$2\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}\leq(\mathbb{P}+\mathbb{S})\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta), \tag{2.22}$$

which follows that

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)\leq\frac{(\mathbb{P}+\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

(b) By using equation (2.22)

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}\leq\frac{\mathbb{P}+\mathbb{S}}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta). \tag{2.23}$$

Subtracting $\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)$ from both sides of (2.23), we get

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\leq\frac{\mathbb{P}+\mathbb{S}}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta).$$

It follows that

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\leq\frac{(\sqrt{\mathbb{P}}-\sqrt{\mathbb{S}})^2}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta).$$

(c) By using equation (2.22)

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)\leq\frac{(\mathbb{P}+\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2. \tag{2.24}$$

Subtracting $\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2$ from both sides of (2.24), we get

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)-\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

$$(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta))(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta))-(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta))^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

It follows that

$$(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta))(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta))-(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta))^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

□

Theorem 2.5. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\geq 0$, $0<\eta\leq 1$, $0<\vartheta\leq 1$ then following inequalities satisfied.*

$$\begin{aligned}
&(a)\quad r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\\
&\qquad\geq \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\,,\\
&(b)\quad r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}(\theta)\wp^{s}(\theta))+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp^{s-1}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}(\wp^{s}(\theta)\mathfrak{H}(\theta))\\
&\qquad\geq \mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^{r}(\theta),\\
&(c)\quad r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp^{\frac{2}{r}}(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\,\mathfrak{U}\,-^{\vartheta}_{0,\theta}\,\mathfrak{H}^{\frac{2}{s}}(\theta)\\
&\qquad\geq \mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}^{r}(\theta)\wp^{2}(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(\wp^{r}(\theta)\mathfrak{H}^{2}(\theta)\big)\,,\\
&(d)\quad r\,\mathfrak{U}^{\eta}_{0,\theta}\Big(\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{s}(\theta)\Big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp^{s-1}(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp^{s-1}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\Big(\wp^{\frac{2}{s}}(\theta)\mathfrak{H}^{r}(\theta)\Big)\\
&\qquad\geq \mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^{2}(\theta).
\end{aligned}$$

Proof. (a) From the arithmetic mean and geometric mean inequality (AM-GM) given by

$$rm+sn\geq m^{r}n^{s},\quad m,n\geq 0,\quad r+s=1. \tag{2.25}$$

Now putting $m=\mathfrak{H}(\alpha)\wp(\beta)$ and $n=\mathfrak{H}(\beta)\wp(\alpha)$, $\alpha,\beta>0$ in (2.25), we get

$$r\mathfrak{H}(\alpha)\wp(\beta)+s\mathfrak{H}(\beta)\wp(\alpha)\geq(\mathfrak{H}(\alpha)\wp(\beta))^{r}(\mathfrak{H}(\beta)\wp(\alpha))^{s}. \tag{2.26}$$

Multiplying (2.26) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$r\wp(\beta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)+s\mathfrak{H}(\beta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\geq\mathfrak{H}^{r}(\beta)\wp^{s}(\beta)\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta)). \tag{2.27}$$

Multiplying (2.27) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ, and using (1.11) we have

$$r\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)+s\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(\wp^{r}(\theta)\mathfrak{H}^{s}(\theta)),$$

which is the required inequality.

Other inequalities can be arrived by substituting identities in (2.25), respectively

$$\text{(b)}\quad m=\frac{\mathfrak{H}(\beta)}{\mathfrak{H}(\alpha)}\quad\text{and}\quad n=\frac{\wp(\alpha)}{\wp(\beta)},\quad \mathfrak{H}(\alpha),\wp(\beta)\neq 0, \tag{2.28}$$

$$\text{(c)}\quad m=\mathfrak{H}(\alpha)\wp^{\frac{2}{5}}(\beta)\quad\text{and}\quad n=\mathfrak{H}^{\frac{2}{5}}(\beta)\wp(\alpha), \tag{2.29}$$

$$\text{(d)}\quad m=\frac{\mathfrak{H}^{\frac{2}{5}}(\alpha)}{\wp(\beta)}\quad\text{and}\quad n=\frac{\mathfrak{H}^{\frac{2}{5}}(\beta)}{\wp(\alpha)},\quad \wp(\alpha),\wp(\beta)\neq 0. \tag{2.30}$$

□

3 Grüss-type fractional integral inequalities

This section presents new generalized forms of Grüss-type fractional integral inequalities formulated via the Caputo-Fabrizio fractional integral operator.

($\mathcal{H}$)Assume that $\mathfrak{H},\wp:[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$ are two integrable functions on $[\varepsilon_1,\varepsilon_2]$ for which there exist constants $\mathfrak{p},\mathfrak{q},\mathfrak{d},\mathfrak{e}\in\mathbb{R}$ such that

$$\mathfrak{p}\leq\mathfrak{H}(\theta)\leq\mathfrak{q},\qquad \mathfrak{d}\leq\wp(\theta)\leq\mathfrak{e},\qquad \theta\in[\varepsilon_1,\varepsilon_2]. \tag{3.1}$$

Lemma 3.1. *Suppose $\mathfrak{H}$ is integrable function on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\leq 1$, we have:*

$$\begin{aligned}
&\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)-\Big(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\Big)^{2}\\
&=\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\xi(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\\
&\quad-\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big).
\end{aligned} \tag{3.2}$$

Proof. Suppose $\mathfrak{H}$ is integrable function on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. For any $\alpha,\beta \in [0,\infty)$, we have

$$\begin{aligned}&(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\alpha)-\mathfrak{q})+(\mathfrak{p}-\mathfrak{H}(\alpha))(\mathfrak{H}(\beta)-\mathfrak{q})-(\mathfrak{p}-\mathfrak{H}(\alpha))(\mathfrak{H}(\alpha)-\mathfrak{q})\\&-(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\beta)-\mathfrak{q})=\mathfrak{H}^2(\alpha)+\mathfrak{H}^2(\beta)-2\mathfrak{H}(\alpha)\mathfrak{H}(\beta).\end{aligned}\tag{3.3}$$

Multiplying (3.3) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$\begin{aligned}&(\mathfrak{p}-\mathfrak{H}(\beta))\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\\&+\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)(\mathfrak{H}(\beta)-\mathfrak{q})\\&-\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)-(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\beta)-\mathfrak{q})\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\\&=\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathfrak{H}^2(\beta)\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{H}(\beta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta).\end{aligned}\tag{3.4}$$

Now, multiplying (3.4) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ which gives (3.2) and proves the lemma. □

Theorem 3.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$ and $0<\eta\le 1$, we have the inequality*

$$\begin{aligned}&\left|\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\right|\\&\le\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)^2(\mathfrak{p}-\mathfrak{q})(\mathfrak{d}-\mathfrak{e}).\end{aligned}\tag{3.5}$$

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Define

$$J(\alpha,\beta)=\big(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta)\big)\big(\wp(\alpha)-\wp(\beta)\big).\tag{3.6}$$

Multiplying (3.6) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to α and β over $(0,\theta)^2$, we get

$$\begin{aligned}&\frac{1}{\eta^2}\int_0^{\theta}\int_0^{\theta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}\big(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta)\big)\big(\wp(\alpha)-\wp(\beta)\big)\,d\alpha\,d\beta\\&=2\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-2\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big).\end{aligned}\tag{3.7}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\right)^2\\&\le\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\left[\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}^2(\theta)\big)-\Big(\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\Big)^2\right]\\&\quad\times\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\left[\mathfrak{U}^{\eta}_{0,\theta}\big(\wp^2(\theta)\big)-\Big(\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\Big)^2\right].\end{aligned}\tag{3.8}$$

Since $(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\ge 0$ and $(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\ge 0$, we have

$$\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\ge 0,\tag{3.9}$$

and

$$\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)\ge 0.\tag{3.10}$$

Using (3.2), (3.9) and (3.10), we have

$$\frac{1}{(1-\eta)}\left[1-e^{-(\frac{1-\eta}{\eta})\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\,\mathfrak{H}^2(\theta)-\Big(\mathfrak{U}^{\eta}_{0,\theta}\,\mathfrak{H}(\theta)\Big)^2$$

$$\leq\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.11}$$

and

$$\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\wp^{2}(\theta)-\left(\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)^{2}$$
$$\leq\left(\wp\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)-e\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.12}$$

By using (3.2) and inequalities (3.8), (3.11) and (3.12), we have

$$\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}(\mathfrak{H}\wp(\theta))-\mathfrak{U}_{0,\theta}^{\eta}(\mathfrak{H}(\theta))\mathfrak{U}_{0,\theta}^{\eta}(\wp(\theta))\right)^{2}$$
$$\leq\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\cdot\left(\wp\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)-e\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.13}$$

Now using the elementary identities $4ab\leq(a+b)^2$, $a,b\in\mathbb{R}$, we can state that

$$4\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\leq\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right](\mathfrak{p}-\mathfrak{q})\right)^{2}, \tag{3.14}$$

$$4\left(\mathfrak{d}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{e}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\leq\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right](\mathfrak{d}-\mathfrak{e})\right)^{2}. \tag{3.15}$$

Again using (3.13), (3.14) and (3.15), we get the desired result. □

Lemma 3.2. *Suppose that* $\mathfrak{H}$ *and* $\wp$ *are two integrable functions on* $[0,\infty)$*. Then for* $0<\theta<\varepsilon_2$, $0<\eta\leq1$, $0<\vartheta\leq1$*, we have the inequality*

$$\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}\wp(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\zeta\wp(\theta)-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)\right.$$
$$\left.-\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)^{2}$$
$$\leq\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}^{2}(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}^{2}(\theta)-2\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\right)$$
$$\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\wp^{2}(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\wp^{2}(\theta)-2\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)\right). \tag{3.16}$$

Proof. Multiplying (3.7) by $\dfrac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}\dfrac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to α and β over $(0,\theta)^2$ and applying Cauchy-Schwarz inequality for double integrals, we get the desired result. □

Lemma 3.3. *Suppose that* $\mathfrak{H}$ *be an integrable function on* $[0,\infty)$ *satisfying the condition* $\mathcal{H}$ *on* $[0,\infty)$*. Then for* $0<\theta<\varepsilon_2$, $0<\eta\leq1$, $0<\vartheta\leq1$*, we have:*

$$\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}^{2}(\theta)+\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}^{2}(\theta)-2\,\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\,\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)$$
$$=\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)$$

$$
\begin{aligned}
&+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\\
&-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\\
&-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big). \qquad (3.17)
\end{aligned}
$$

Proof. Multiplying (3.5) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ which gives (3.17) and proves the lemma. □

Theorem 3.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, $0<\vartheta\le 1$ we have the inequality*

$$
\begin{aligned}
&\Bigg(\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}\wp(\theta)+\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}\wp(\theta)\\
&-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\Bigg)^2\\
&\le\Bigg[\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\Bigg]\\
&\times\Bigg[\left(\mathfrak{d}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)\\
&\quad+\left(\mathfrak{d}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)-e\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\Bigg]. \qquad (3.18)
\end{aligned}
$$

Proof. Since $(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\ge 0$ and $(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\ge 0$,

Therefore we write

$$
\begin{aligned}
&-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\\
&-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\le 0. \qquad (3.19)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)\\
&-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)\le 0. \qquad (3.20)
\end{aligned}
$$

Now applying (3.17) to $\mathfrak{H}$ and $\wp$ we get

$$
\begin{aligned}
&\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^2(\theta)\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\\
&\le\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right). \qquad (3.21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)+\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\\
&\le\left(\mathfrak{d}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\right)
\end{aligned}
$$

$$+\left(\mathfrak{d}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right). \tag{3.22}$$

Using (3.21) and (3.22) in (3.16) we get the desired result. □

4 Grüss-type fractional integral inequalities for two weighted functions

This section is devoted to the study of generalized versions of Grüss-type fractional integral inequalities for two weighted functions involving the Caputo–Fabrizio fractional integral operator.

Lemma 4.1. *Assume Ω is an integrable function on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathfrak{p}\le\Omega(\theta)\le\mathfrak{q}$ on $[\varepsilon_1,\varepsilon_2]$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, we have:*

$$\begin{aligned}
&\left(\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p}-\Omega(\theta))v(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[(\Omega(\alpha)-q)u(\theta)]\right)\\
&+\left(\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p}-\Omega(\theta))u(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[(\Omega(\theta)-q)v(\theta)]\right)\\
&-\left(\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})u(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right)\\
&-\left(\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})v(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\right)\\
&=\left(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[v\Omega^2(\theta)]\right)+\left(\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[u\Omega^2(\theta)]\right)\\
&\quad-2\left(\mathfrak{U}^{\eta}_{0,\theta}[v\Omega(\theta)]\right)\left(\mathfrak{U}^{\eta}_{0,\theta}[u\Omega(\theta)]\right).
\end{aligned} \tag{4.1}$$

Proof. Let Ω be an integrable function on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathfrak{p}\le\Omega(\theta)\le\mathfrak{q}$ on $[\varepsilon_1,\varepsilon_2]$. For any $\alpha,\beta\in[\varepsilon_1,\varepsilon_2]$, we have

$$\begin{aligned}
&(\mathfrak{p}-\Omega(\beta))(\Omega(\alpha)-\mathfrak{q})+(\mathfrak{U}-\Omega(\alpha))(\Omega(\beta)-\mathfrak{q})\\
&\quad-(\mathfrak{p}-\Omega(\alpha))(\Omega(\alpha)-\mathfrak{q})-(\mathfrak{p}-\Omega(\beta))(\Omega(\beta)-\mathfrak{q})\\
&=\Omega^2(\alpha)+\Omega^2(\beta)-2\Omega(\alpha)\Omega(\beta).
\end{aligned} \tag{4.2}$$

Multiplying (4.2) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$, we get the required lemma. □

Theorem 4.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, we have the inequality:*

$$\begin{aligned}
&\Big|\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\zeta\delta(\theta))+\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta))\\
&\quad-\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))-\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\Big|\\
&\le\left[\left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2}+2\|\mathfrak{H}\|^2_\infty\right)\left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2}+2\|\wp\|^2_\infty\right)\right]^{\frac{1}{2}}\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta)),
\end{aligned} \tag{4.3}$$

where $\mathfrak{H},\wp\in L_\infty[\varepsilon_1,\varepsilon_2]$.

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathcal{H}$. Define the functional

$$J(\alpha,\beta)=(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta))(\wp(\alpha)-\wp(\beta)). \tag{4.4}$$

Multiplying (4.4) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$ we get

$$\begin{aligned}
&\frac{1}{\eta^2}\int_0^\theta\int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)(\zeta(\alpha)-\zeta(\beta))(\delta(\alpha)-\delta(\beta))\,d\alpha d\beta\\
&=\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta))+\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta))-\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\
&\quad-\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta)).
\end{aligned} \tag{4.5}$$

Applying Cauchy–Schwarz inequality for double integral and using Lemma 4.1 and using the result $4ab\le(a+b)^2;\ a,b\in\mathbb{R}$ and utilizing the two inequalities

$$\left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))u^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\right]^2\le\|\mathfrak{H}\|^2_\infty\left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2,$$

$$\left[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right]^2 \le \|\mathfrak{H}\|^2_\infty \left[u^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2, \tag{4.6}$$

we get

$$\begin{aligned}&(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)]) + (\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)])\\ &\qquad - 2(\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}(\theta)])\\ &\le \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v(\theta)).\end{aligned} \tag{4.7}$$

Similarly, we obtain

$$\begin{aligned}&(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[v\wp^2(\theta)]) + (\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)])\\ &\qquad - 2(\mathfrak{U}^{\eta}_{0,\theta}[v\wp(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\wp(\theta)])\\ &\le \left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2} + 2\|\wp\|^2_\infty\right)\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v(\theta)).\end{aligned} \tag{4.8}$$

Using (4.5), (4.7) and (4.8) we get the desired result. □

Corollary 4.1. *Suppose that $\mathfrak{H}$ is integrable function on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u is positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \le 1$, we have the inequality:*

$$\left|\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}^2(\theta)) - \left(\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right)^2\right| \le \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{4} + \|\mathfrak{H}\|^2_\infty\right)\left(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\right)^2. \tag{4.9}$$

where $\mathfrak{H} \in L_\infty[\varepsilon_1, \varepsilon_2]$.

Proof. Applying Theorem 4.1 for $\mathfrak{H}(\theta) = \wp(\theta)$ and $u(\theta) = v(\theta)$, we get (4.9). □

Lemma 4.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \le 1$, $0 < \vartheta \le 1$, we have the inequality:*

$$\begin{aligned}&\Big[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}\wp(\theta))\\ &\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\ &\quad - \mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big]^2\\ &\le \Big(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}^2(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}^2(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}^2(\theta))\\ &\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}^2(\theta)) - 2\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - 2\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big)\\ &\quad \times \Big(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,{}^{\vartheta}_{0,\theta}(v\wp^2(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp^2(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\wp^2(\theta))\\ &\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\wp^2(\theta)) - 2\mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - 2\mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\Big).\end{aligned} \tag{4.10}$$

Proof. Let us consider the functional

$$J(\alpha, \beta) = (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta)). \tag{4.11}$$

Multiplying (4.17) by $\frac{1}{\eta\vartheta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$, we get

$$\begin{aligned}&\frac{1}{\eta\vartheta}\int_0^\theta\int_0^\theta (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta))e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)\,d\alpha d\beta\\ &= \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}\wp(\theta))\\ &\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\ &\quad - \mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta)).\end{aligned} \tag{4.12}$$

Applying Cauchy–Schwarz inequality for double integral, we get the lemma. □

Lemma 4.3. *Assume Ω is an integrable function on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathfrak{p} \leq \Omega(\theta) \leq \mathfrak{q}$ on $[\varepsilon_1, \varepsilon_2]$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$, we have:*

$$
\begin{aligned}
&\mathfrak{U}^{\eta}_{0,\theta}\big[(\Omega(\theta) - \mathfrak{q})u(\theta)\big]\, \mathfrak{U}^{\vartheta}_{0,\theta}\big[(\mathfrak{p} - \Omega(\theta))v(\theta)\big] \\
&+ \mathfrak{U}^{\vartheta}_{0,\theta}\big[(\Omega(\theta) - \mathfrak{q})u(\theta)\big]\, \mathfrak{U}^{\eta}_{0,\theta}\big[(\mathfrak{p} - \Omega(\theta))v(\theta)\big] \\
&+ \mathfrak{U}^{\eta}_{0,\theta}\big[(\mathfrak{p} - \Omega(\theta))u(\theta)\big]\, \mathfrak{U}^{\vartheta}_{0,\theta}\big[(\Omega(\theta) - \mathfrak{q})v(\theta)\big] \\
&+ \mathfrak{U}^{\vartheta}_{0,\theta}\big[(\mathfrak{p} - \Omega(\theta))u(\theta)\big]\, \mathfrak{U}^{\eta}_{0,\theta}\big[(\Omega(\theta) - \mathfrak{q})v(\theta)\big] \\
&- \Big(\mathfrak{U}^{\eta}_{0,\theta}\left[(\mathfrak{p} - \Omega(\theta))(\Omega(\theta) - \mathfrak{q})u(\theta)\right]\Big)\left(\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\right) \\
&- \left(\mathfrak{U}^{\vartheta}_{0,\theta}\left[(\mathfrak{p} - \Omega(\theta))(\Omega(\theta) - \mathfrak{q})u(\theta)\right]\right)\Big(\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\Big) \\
&- \Big(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\Big)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\left[(\mathfrak{p} - \Omega(\theta))(\Omega(\theta) - \mathfrak{q})v(\theta)\right]\right) \\
&- \left(\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\right)\Big(\mathfrak{U}^{\eta}_{0,\theta}\left[(\mathfrak{p} - \Omega(\theta))(\Omega(\theta) - \mathfrak{q})v(\theta)\right]\Big) \\
&= \mathfrak{U}^{\eta}_{0,\theta}[u\Omega^2(\theta)]\, \mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\vartheta}_{0,\theta}[u\Omega^2(\theta)]\, \mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] \\
&+ \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\, \mathfrak{U}^{\vartheta}_{0,\theta}[v\Omega^2(\theta)] + \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\, \mathfrak{U}^{\eta}_{0,\theta}[v\Omega^2(\theta)] \\
&- 2\, \mathfrak{U}^{\eta}_{0,\theta}[u\Omega(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\Omega(\theta)] - 2\mathfrak{U}^{\vartheta}_{0,\theta}[u\Omega(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\Omega(\theta)]
\end{aligned} \tag{4.13}
$$

Proof. Multiplying (4.2) by $\dfrac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}\dfrac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$ we get the required lemma. □

Theorem 4.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$, we have the inequality:*

$$
\begin{aligned}
&\Big|\mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{U}\wp(\theta)) \\
&+ \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta)) \\
&- \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big| \\
&\leq \left[\left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2} + 2\|\wp\|^2_\infty\right)\right]^{\frac{1}{2}} \Big[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\Big],
\end{aligned} \tag{4.14}
$$

where $\mathfrak{H}, \wp \in L_\infty[\varepsilon_1, \varepsilon_2]$.

Proof. Since

$$(\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q}) \geq 0 \quad \text{and} \quad (\mathfrak{d} - \wp(\theta))(\wp(\theta) - \mathfrak{e}) \geq 0,$$

therefore we write

$$
\begin{aligned}
&- \mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(p - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q})u(\theta)] \\
&- \mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q})u(\theta)] \\
&- \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q})v(\theta)] \\
&- \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q})v(\theta)] \leq 0,
\end{aligned} \tag{4.15}
$$

and

$$
\begin{aligned}
&- \mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(b - \wp(\theta))(\wp(\theta) - e)u(\theta)] \\
&- \mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{d} - \wp(\theta))(\wp(\theta) - e)u(\theta)] \\
&- \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{d} - \wp(\theta))(\wp(\theta) - \mathfrak{e})v(\theta)] \\
&- \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{d} - \wp(\theta))(\wp(\theta) - \mathfrak{e})v(\theta)] \leq 0,
\end{aligned} \tag{4.16}
$$

Now applying (4.13) to $\mathfrak{H}$ and $\wp$ and using the result $4ab \leq (a+b)^2$, $a, b \in \mathbb{R}$ and utilize the two inequalities

$$\begin{aligned}\left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\right]^2 &\leq \|\mathfrak{H}\|^2_\infty \left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2,\\ \left[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right]^2 &\leq \|\mathfrak{H}\|^2_\infty \left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2,\end{aligned} \tag{4.17}$$

we get

$$\begin{aligned}&\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)]\\ &+ \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)] - 2\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)]\\ &- 2\mathfrak{U}^{\vartheta}_{0,\theta}[u\mathfrak{H}(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)]\\ &\leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\left[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right].\end{aligned} \tag{4.18}$$

Similarly, we obtain

$$\begin{aligned}&\mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\wp^2(\theta)]\\ &+ \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\wp^2(\theta)] - 2\mathfrak{U}^{\eta}_{0,\theta}[u\wp(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\wp(\theta)] - 2\mathfrak{U}^{\vartheta}_{0,\theta}[u\wp(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\wp(\theta)]\\ &\leq \left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2} + 2\|\wp\|^2_\infty\right)\left[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right].\end{aligned} \tag{4.19}$$

Using (4.10), (4.18) and (4.19) we get the desired result. □

Corollary 4.2. *Suppose that $\mathfrak{H}$ is an integrable function on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$, we have the inequality*

$$\begin{aligned}&\Big|\mathfrak{U}^{\eta}_{0,\theta}\big(v(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(u\mathfrak{H}^2(\theta)\big) + \mathfrak{U}^{\eta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v\mathfrak{H}^2(\theta)\big) + \mathfrak{U}^{\vartheta}_{0,\theta}\big(v(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(u\mathfrak{H}^2(\theta)\big)\\ &\quad + \mathfrak{U}^{\vartheta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v\mathfrak{H}^2(\theta)\big) - 2\,\mathfrak{U}^{\eta}_{0,\theta}\big(u\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v\mathfrak{H}(\theta)\big) - 2\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(u\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v\mathfrak{H}(\theta)\big)\Big|\\ &\leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\Big[\mathfrak{U}^{\eta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v(\theta)\big) + \mathfrak{U}^{\vartheta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v(\theta)\big)\Big].\end{aligned} \tag{4.20}$$

where $\mathfrak{H} \in L_\infty[\varepsilon_1, \varepsilon_2]$.

Proof. Applying Theorem 4.2 for $\mathfrak{H}(\theta) = \wp(\theta)$ and $u(\theta) = v(\theta)$, we get (4.20). □

5 η-order pre-Grüss fractional integral inequality

This section investigates generalized forms of the η-order pre-Grüss fractional integral inequality using the Caputo–Fabrizio fractional integral operator.

Theorem 5.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0, \infty)$ satisfying the condition $\mathcal{H}$ on $[0, \infty)$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, we have the inequality*

$$\begin{aligned}&\left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta))\right)^2\\ &\leq \left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)^2 (\mathfrak{p}-\mathfrak{q})^2\\ &\quad\times\left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta) - \left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2\right).\end{aligned} \tag{5.1}$$

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Define

$$J(\alpha,\beta) = (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta)). \tag{5.2}$$

Multiplying (5.2) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to α and β over $(0,\theta)^2$ we get

$$\begin{aligned}&\frac{1}{\eta^2}\int_0^\theta\int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta))(\wp(\alpha)-\wp(\beta))\,d\alpha d\beta\\ &= 2\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}\varrho(\theta)) - 2\mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^\eta_{0,\theta}(\wp(\theta)).\end{aligned} \tag{5.3}$$

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^\eta_{0,\theta}(\wp(\theta))\right)^2\\ &\leq \left(\frac{1}{1-\eta}\right)\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}^2(\theta) - \left(\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\right)^2\\ &\times\left(\frac{1}{1-\eta}\right)\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}\wp^2(\theta) - \left(\mathfrak{U}^\eta_{0,\theta}\wp(\theta)\right)^2.\end{aligned} \tag{5.4}$$

Since $(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q}) \geq 0$, we have

$$\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big) \geq 0. \tag{5.5}$$

Using (3.2) and (5.5), we have

$$\begin{aligned}&\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}^2(\theta) - \left(\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\right)^2\\ &\leq \left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right).\end{aligned} \tag{5.6}$$

By using (3.2) and inequalities (5.4) and (5.6), we have

$$\begin{aligned}&\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^\eta_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^\eta_{0,\theta}(\wp(\theta))\right)^2\\ &\leq \left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{U}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)\\ &\times\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^\eta_{0,\theta}\wp^2(\theta) - \left(u^\eta_{0,\theta}\wp(\theta)\right)^2\right).\end{aligned} \tag{5.7}$$

Now using the elementary identity $4ab \leq (a+b)^2$, $a,b\in\mathbb{R}$, we can state that

$$\begin{aligned}&4\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)\\ &\leq \left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right](\mathfrak{p}-\mathfrak{q})\right)^2.\end{aligned} \tag{5.8}$$

Further using (5.7) and (5.8) we get the desired result. □

6 (η,ϑ)-order pre-Grüss fractional integral inequality

In this section, we present and analyze the generalized version of the (η,ϑ)-order pre-Grüss fractional integral inequality using the Caputo–Fabrizio fractional integral operator.

Theorem 6.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\leq 1$, $0<\vartheta\leq 1$ we have the inequality*

$$\begin{aligned}&\Bigg(\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}\wp(\theta) + \frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^\eta_{0,\theta}\mathfrak{H}\wp(\theta)\\ &\quad - \mathfrak{U}^\eta_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^\vartheta_{0,\theta}\wp(\theta) - \mathfrak{U}^\vartheta_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^\eta_{0,\theta}\wp(\theta)\Bigg)^2\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\\
&\quad+\left(\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)+\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right.\\
&\qquad\left.-2\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\right). \qquad (6.1)
\end{aligned}
$$

Proof. Since

$$(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\geq 0.$$

Therefore we write

$$
\begin{aligned}
&-\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\\
&-\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\leq 0. \qquad (6.2)
\end{aligned}
$$

Now applying (3.17) to $\mathfrak{H}$ we get

$$
\begin{aligned}
&\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^2(\theta)\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-2\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\\
&\leq\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \qquad (6.3)
\end{aligned}
$$

Using (6.3) in (3.16), we get the desired result. □

7 Conclusion

In the present study, we employ the Caputo–Fabrizio fractional integral operator to investigate several fractional integral inequalities, utilizing Youngs inequality and the arithmetic–geometric mean inequality to establish generalized and newly extended results. Specifically, we have derived generalized forms of Grüss-type inequalities, weighted Grüss inequalities, and pre-Grüss inequalities.The nonsingular kernel of the Caputo–Fabrizio operator provides a useful framework for developing new bounds and has potential applications in fractional differential equations, norm estimates, error analysis, and the qualitative analysis of fractional models.

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ON SOFT M^*-OPEN SET IN SOFT TOPOLGICAL SPACES

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Abstract

In this paper we introduce a new class of generalized soft open sets called soft M^*-open ($\tilde{M}^*O$) sets in soft topological space. Also we discuss some of their properties. We hope that the result findings in this paper will help researcher enhance and promote the further study on soft M^*-open set in soft topological spaces to carry out a general framework for their applications in practical life.

2020 Mathematical Sciences Classification: 54A05, 54A10, 54C05, 54C08, 54C10.
Keywords and Phrases: soft M^*-open sets, soft M^*-closed sets, soft M^*-interior, soft M^*-closure, soft M^*-limit point, soft M^*-derived set, soft M^*-neighborhood, soft M^*-bases and soft M^*-subbases.

1 Introduction

In 1999, Molodtsov [7] developed the concept of a soft set to handle difficult problem in economics, engineering and the environment, where no mathematical methods could effectively deal with the many types of uncertainty. The notion of soft topological spaces was introduced by Shabir and Naz [9] on initial universe with a fixed set of parameters and they investigated some properties of soft topological spaces. The concept of generalzed closed set was introduced by N. Levine [6].Many mathematicians extended the results of generalized closed sets and open sets in many directions. Weak forms of open set were first studied by Chen [4]. He investigate soft semi-open sets in soft topological spaces and studied some properties of it. Akdag and Ozkan [1] defined soft α-open (soft α-closed) sets. Alzahran, EL-Maghrabi, AL-Juhani and Badr [2] introduce soft M-open sets. M^*-open sets were introduced into general topology by A. Devika and Thilagavathi [5]. The purpose of this article is to conduct a theoretical investigation of the new set termed $\tilde{M}^*O$ and $\tilde{M}^*C$ sets over soft topological space (U, E, τ) and to analysis some of their properties.

2 Preliminaries

In this segment, we recall some necessary definitions and notations. Also in this research, we express soft operations by ' $\sim$ ', soft M^*-open ($\tilde{M}^*O$), soft M^*-closed ($\tilde{M}^*C$). Throughout this entire article, we will refer to U as an initial universal set, (U, E, τ) is a soft topological space and (F, E) is a soft set over U.

Definition 2.1 ([7]). *Molodtsov Soft Set Let U be an initial universe set and let E be a set of parameters. Let $P(U)$ denote the power set of U and let $A \subseteq E$. A pair (F, E) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2.2 ([7]). *For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if*
(i) $A \subseteq B$ and
(ii) for all $e \in A, F(e)$ and $G(e)$ are identical approximations. We write $(F, A)\tilde{\subset}(G, B)$.$(F, A)$ is said to be a soft super set of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\tilde{\supset}(G, B)$.

Definition 2.3 ([7]). *The complement of a soft set (F, A) is denoted by $(F, A)'$ and is defined by $(F, A)' = (F', /A)$ where, $F' : /A \to P(U)$ is a mapping given by $F'(/\alpha) = U \backslash F(\alpha)$, for all $\alpha \in A$.*

Let us call F' the soft complement function of F. Clearly $(F')'$ is the same as F and $((F, A)')' = (F,\ A)$.

Definition 2.4 ([7])**.** *A soft set (F, A) over U is said to be a NULL soft set denoted by* Φ *if for all e* $\in A, F(e) = \Phi$ *(null set).*

Definition 2.5 ([8])**.** *Let* τ *be the collection of soft sets over* X*, then* τ *is said to be a soft topology on* X *if*
(i) $\Phi, \tilde{X}$ *belong to* τ.
(ii) The union of any number of soft sets in τ *belongs to* τ.
(iii) The intersection of any two soft sets in τ *belongs to* τ.
The triplet (X, τ, E) *is called a soft topological space over* X.

Definition 2.6 ([9])**.** *Let* (X, τ, E) *be a soft space over* X*, then the members of* τ *are said to be soft open sets in* X.

Definition 2.7 ([8])**.** *Let* (X, τ, E) *be a soft space over* X*. A soft set* (F, E) *over* X *is said to be a soft closed set in* X*, if its relative complement* $(F, E)'$ *belongs to* τ.

Definition 2.8. *A soft set* (F, A) *in a soft topological space* X *is called:*
- *(i) Soft regular open (resp. soft regular closed) set* [3] *if* $(F, A) = \mathrm{int}(cl\ (F, A))$ *[resp.* $(F,\ A) = cl(\mathrm{int}(F, A))$*].*
- *(ii) Soft semi-open (resp. soft semi-closed) set* [4] *if* $(F, A)\tilde{\subseteq}\, \mathrm{cl}(int(F, A))$*[resp.* $int\ (cl(F,\ A))\tilde{\subseteq}(F, A)$*].*
- *(iii) Soft pre-open (resp. soft pre-closed)* [3] *if* $(F, A)\tilde{\subseteq}\, \mathrm{int}(\mathrm{cl}(F, A))$ *[resp.* $cl(int\ (F,\ A))\tilde{\subseteq}(F, A)$*].*
- *(iv) Soft* α*-open (resp. soft-*α *closed)* [3] *if* $(F, A)\tilde{\subseteq}\ int\ (cl(int\ (\ F, A)\ ))$ *[resp.* $cl(int\ (cl(F,\ A)))\ \tilde{\subseteq}(F, A)$*].*
- *(v) Soft* β*-open (resp. soft-*β *closed) set* [3] *if* $(F, A)\tilde{\subseteq}\, \mathrm{cl}(\mathrm{int}(\mathrm{cl}(F, A)))$ *[resp.* $int\ (cl(int\ (F, A)))\tilde{\subseteq}(F, A)$*].*
- *(vi) A soft* (F_1, H) *in a soft topological space* (U_1, τ, S) *is called:*
  - *(i) Soft* θ*-semi open set iff* $(F_1, H)\tilde{\subseteq}\ cl(\mathrm{int}_\theta(F_1, H))$.
  - *(ii) Soft* θ*-semi closed set iff* $(F_1, H)\tilde{\supseteq}\ int(\mathrm{cl}_\theta(F_1, H))$.
- *(vii) A soft* (F_1, H) *in a soft topological space* (U_1, τ, S) *is called:*
  - *(i) Soft* δ*-pre open set iff* $(F_1, H)\tilde{\subseteq}\mathrm{int}(\mathrm{cl}_\delta(F_1, H)$..
  - *(ii) Soft* δ*-pre closed set iff* $cl(int_\delta(F_1, H))\tilde{\subseteq}(F_1, H)$.

The union of all Soft δ*-pre open sets contained in a soft* (F_1, H) *in a soft topological space* $(U_1, \tau.\ ,S)$ *is called the Soft* δ*-pre interior of* (F_1, H)*. The intersection of all Soft* δ*-pre closed sets containing a soft set* (F_1, H) *in a soft topological space* (U_1, τ, S) *is called the Soft* δ*-pre closure of* (F_1, H).

Definition 2.9 ([2])**.** *In a soft topological spaces (* U_1, τ, H *) a soft set (* F_1, H *) is called :*
- *(i) Soft* M*-open set if* $(F_1, H)\tilde{\subseteq}\, \mathrm{cl}(\mathrm{int}_\theta(F_1, H))\tilde{\cup} int(\mathrm{cl}_\delta(F_1, H))$
- *(ii)* Soft M*-closed set if* $(F_1, H)\tilde{\supseteq}\, \mathrm{int}(cl_\theta(F_1, H))\tilde{\cap} cl(int_\delta(F_1, H))$

3 Soft M^*-open sets

In this section we introduce soft M^*-open sets in soft topological spaces and study some of their properties.

Definition 3.1. *Let* (U, E, τ) *be a soft topological space. Then a soft subset* (F, E) *of a soft topological spaces* (U, E, τ) *is said to be,*
- *(i) A soft* M^**-open set (briefly* $\tilde{M}^*O$ *) if* $(F, E)\tilde{\subseteq}(\mathrm{int}(\mathrm{cl}(\mathrm{int}_\theta(F, E))))$.
- *(ii) A soft* M^**-closed set (briefly* $\tilde{M}^*C$ *) if* $(F, E)\tilde{\supseteq}(\mathrm{cl}(\mathrm{int}(cl_\theta(F, E))))$.

Lemma 3.1. *Let (* F, E *) be a soft subset of a soft topological space (* U, E, τ *),*
- *(I) Every* $\tilde{M}^*O$ *is a soft* θ*-semi open set.*
- *(II) Every* $\tilde{M}^*O$ *is an soft* M*-open set.*

Proof. (i) Obvious from the definition.
(ii) Let (F, E) be $\tilde{M}^*O$.
Then $(F, E)\tilde{\subseteq}\, \mathrm{int}(cl(\mathrm{int}_\theta(F, E)))$
$\tilde{\subseteq} cl(\mathrm{int}_\theta(F, E))$
$\tilde{\subseteq}\, \mathrm{cl}(\mathrm{int}_\theta(F, E))\tilde{\cup}\, \mathrm{int}(cl_\delta(F, E))$.
Hence (F, E) is a $\tilde{M}^*O$. □

Lemma 3.2. *In a soft topological space* (U, E, τ)*, we have*
- *(i) The arbitrary union of* $\tilde{M}^*O$ *is* $\tilde{M}^*O$.

(ii) The arbitrary intersection of $\tilde{M}^*C$ sets is $\tilde{M}^*C$.

Proof. (i) Let $\{(F,E)_\alpha : \alpha \in \wedge.$ an index set $\}$ is a collection of $\tilde{M}^*O$.
Hence for each $\alpha, (F,E)_\alpha \tilde{\subseteq} \text{int}(\text{cl}(\text{int}_\theta(F,E)_\alpha))$. Taking the union of all such relations we get,
$\tilde{\cup}\{(F,E)_\alpha\} \tilde{\subseteq} \tilde{\cup}(\text{int}(\text{cl}(\text{int}_\theta(F,E)_\alpha)))$

$\tilde{\subseteq} \text{int}(\text{cl}(\text{int}_\theta(\tilde{\cup}(F,E)_\alpha))$.

Hence $\tilde{\cup}(F,E)_\alpha$ is $\tilde{M}^*O$.
(ii) As (i) by taking the complements.

□

Theorem 3.1. *For a soft set (F,E) in the soft topological space (U,E,τ)*
(i) (F,E) is a soft M^*-open set iff $(F,E)^c$ is a soft M^*-closed set.
(ii) (F,E) is a soft M^*-closed set iff $(F,E)^c$ is a soft M^*-open set.

Proof. Obvious. □

4 Soft M^*-Closure and Soft M^*-Interior

Definition 4.1. *Let (F,E) be a soft subset of a soft topological space (U,E,τ). Then,*
(i) The intersection of all soft M^*-closed sets containing (F,E) is called the soft M^*-closure of (F,E) and is denoted by $\tilde{M}^*$-cl(F,E).
(ii) The union of all soft M^*-open sets contained in (F,E) is called the soft M^*-interior of (F,E) and is denoted by $\tilde{M}^*$-int(F,E).

Theorem 4.1. *The following hold for a soft subset of a soft topological space (U,E,τ).*
(i) (F,E) is $\tilde{M}^*O$ if and only if $(F,E) = (F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$.
(ii) (F,E) is $\tilde{M}^*C$ if and only if $(F,E) = (F,E)\tilde{\cup}\,\text{cl}(\text{int}(cl_\theta(F,E)))$.

Proof. (i) Let (F,E) be an $\tilde{M}^*O$.
Then $(F,E)\tilde{\subseteq}\text{int}(\text{cl}(\text{int}_\theta(F,E)))$.
Hence $(F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E))) = (F,E)$.
Conversely let $(F,E) = (F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$. Then, $(F,E)\tilde{\subseteq}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$. Hence (F,E) is $\tilde{M}^*O$.
□

Theorem 4.2. *The following hold for a soft subset of a soft topological space (U,E,τ).*
(i) $\tilde{M}^*$-int$(F,E) = (F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$.
(ii) $\tilde{M}^*$-cl$(F,E) = (F,E)\tilde{\cup}\,\text{cl}(\text{int}(\text{cl}_\theta(F,E)))$.

Proof. (i) since $\tilde{M}^*$-int(F,E) is $\tilde{M}^*O$,
$\tilde{M}^*\text{-int}(F,E)\tilde{\subseteq}\,\text{int}(\text{cl}(\text{int}_\theta(\tilde{M}^*\text{-int}(F,E))))$
$\tilde{\subseteq}\text{int}(\text{cl}(\text{int}_\theta(F,E)))$. Also,
$(F,E)\tilde{\cap}\tilde{M}^*\text{-int}(F,E)\tilde{\subseteq}(F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$.
Hence $\tilde{M}^*\text{-int}(F,E)\tilde{\subseteq}(F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E)))$.
Conversely since, $\text{int}(\text{cl}(\text{int}_\theta((F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E))))))$.
$\tilde{\supseteq}\,\text{int}(\text{cl}(\text{int}_\theta((F,E)\tilde{\cap}\,\text{int}_\theta(\text{cl}(\text{int}_\theta(F,E))))))$
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$= \text{int}(\text{cl}(\text{int}_\theta((F,E)\tilde{\cap}\,\text{int}_\theta(F,E))))$
$= \text{int}(\text{cl}(\text{int}_\theta(F,E)))$
$= (F,E)\tilde{\cap}\,\text{int}(\text{cl}\,(\text{int}_\theta(F,E)))$.
Hence, $\text{int}(\text{cl}(\text{int}_\theta((F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E))))))\tilde{\supseteq}(F,E)\tilde{\cap}\,\text{int}(\text{cl}\,(int_\theta(F,E)))$.
This implies that,
$(F,E)\tilde{\cap}\,\text{int}(\text{clc}\,\text{int}_\theta(F,E))$ is an $\tilde{M}^*O$ contained in (F,E). Hence, $(F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E))\tilde{\subseteq}\tilde{M}^*\text{-int}(F,E)$.
Therefore $\tilde{M}^*\text{-int}(F,E) = (F,E)\tilde{\cap}\,\text{int}(\text{cl}(\text{int}_\theta(F,E))$. □

5 Soft M^*-Limit Point, Soft M^*-Derived Set and Soft M^*-Neighborhood

Definition 5.1. *If (F,E) is a soft subset of soft topological space(U,E,τ). Then a point $x \in U$ is called soft M^*-limit point of a soft set $(F,E)\tilde{\subseteq}U$ if every soft M^*-open set $(T,E)\tilde{\subseteq}U$ containing x contains a point other than x.*

Definition 5.2. *The set of all soft M^*-limit points of (F,E) is called soft M^*-derived set of (F, E) and is denoted by $\tilde{M}^*$-$d(F,E)$.*

Definition 5.3. *A soft subset (F,E) of a soft topological spaces (U,E,τ) is said to be a soft M^*-neighborhood (Briefly $\tilde{M}^*$-nbd) of a point $x \in U$ if there exists an soft M^*-open set (G,E) such that $x \in (G,E)\tilde{\subseteq}(N,E)$. Then class of soft M^*-neighborhoods of $x \in U$ is called the soft M^*-neighborhood system of x and denoted by $\tilde{M}^*$-N_x.*

Theorem 5.1. *If (F,E) is a soft subset of (G,E). Then every soft M^*-limit point of (F,E) is also a soft M^*-limit point of (G,E) i.e.*
$(F,E)\tilde{\subseteq}(G,E)$ then $\tilde{M}^*$-$d(F,E)\tilde{\subseteq}\tilde{M}^*$-$d(G,E)$.

Proof. Let $p \in \tilde{M}^*$-$d(F,E)$. Then
(5.1) $((M,E)$-$\{p\})\tilde{\cap}(F,E) \neq \varnothing, \forall p \in \tau$ s.t. $p \in (M,E)$
$(F,E)\tilde{\subseteq}(G,E) \Rightarrow (F,E)\tilde{\cap}((M,E)$-$p)\tilde{\subseteq}(G,E)\tilde{\cap}((M,E)$-$\{p\}) \Rightarrow (G,E)\tilde{\cap}((M,E)$-$\{p\}) \neq \varnothing$ according to eq. (5.1) $\Rightarrow p \in \tilde{M}^*$-$d(G,E)$
Therefore $p \in \tilde{M}^*$-$d(F,E) \Rightarrow p \in \tilde{M}^*$-$d(G,E) \Rightarrow \tilde{M}^*$-$d(F,E)\tilde{\subseteq}\tilde{M}^*$-$d(G,E)$. □

Theorem 5.2. *A soft subset (G,E) of a soft topological space (U,E,τ) is soft M^*-open if and only if it is soft M^*-nbd for every point $p \in (G,E)$.*

Proof. Let (G,E) be a soft M^*-open set. Then (G,E) is a $\tilde{M}^*$-*nbd* for each $p \in (G,E)$.
Conversely let (G,E) be an $\tilde{M}^*$-*nbd* for each $p \in (G,E)$. Then there exists an soft M^*-open set $(W,E)_p$ Containing p such that $p \in (W,E)_p\tilde{\subseteq}(G,E)$. So
$(G,E) = \tilde{\cup}_{p\in(G,E)}(W,E)_p$. Therefore (G,E) is a soft M^*-open set. □

6 Soft M^*-Bases and Soft M^*-Subbases

Definition 6.1. *Let (U,E,τ), be a soft topological spaces. A class $\mathfrak{B}^*$ of soft M^*-open subsets of U, i.e. $\mathfrak{B}^*\tilde{\subseteq}\tau$, is a soft M^*-base for the soft topology τ iff*

(i) Every soft M^*-open set $(G,E) \in \tau$ is the union of members of $\mathfrak{B}^*$
(ii) For any point p belonging to an soft M^*-open set (G,E), there exists $(B,E) \in \mathfrak{B}^*$ with $p \in (B,E)\tilde{\subseteq}(G,E)$.

Definition 6.2. *Let (U,E,τ), be a soft topological spaces. A class ψ of soft M^*-open subsets of U, i.e. $\psi\tilde{\subseteq}\tau$, is a soft M^*-subbase for the soft topology τ on U iff finite intersection of members of ψ form a soft M^* base for τ.*

Theorem 6.1. *Let $\mathfrak{B}^*$ and $\mathfrak{B}^{**}$ be soft M^*-bases, respectively, for soft topologies τ^* and τ^{**} on a set U. If $(B,E) \in \mathfrak{B}^*$ is the union of members of $\mathfrak{B}^{**}$. Then $\tau^*\tilde{\subseteq}\tau^{**}$.*

Proof. Let (G,E) be a soft τ^*-open set. Then (G,E) is the soft union of members of $\mathfrak{B}^*$, i.e. $(G,E) = \cup_i(B_i,E)$ where $(B_i,E) \in \mathfrak{B}^*$. But by hypothesis, each $(B_i,E) \in \mathfrak{B}^*$ is the soft union of members of $\mathfrak{B}^{**}$ and so $(G,E) = \cup_i(B_i,E)$ is also the soft union of members of $\mathfrak{B}^*$ which are τ^{**}-open sets. Hence (G,E) is also a τ^{**}-open set and so $\tau^*\tilde{\subseteq}\tau^{**}$. □

Theorem 6.2. *If ψ is a soft M^*-subbases for soft topologies τ^* and τ^{**} on U, then $\tau^* = \tau^{**}$.*

Proof. Suppose $(G,E) \in \tau^*$. Since ψ is a soft M^*-subbases for $\tau^*, (G,E) = \cup_i(s_{i1} \cap \ldots \cap s_{ini})$ where $s_{ik} \in \psi$. But ψ is also a soft M^*-subbases for τ^{**} and so $\psi\tilde{\subseteq}\tau^{**}$; hence each $sik \in \tau^{**}$.since τ^{**} is a soft topology, $(s_{i1} \cap \ldots \cap s_{ini}) \in \tau^{**}$ and hence $(G,E) \in \tau^{**}$. Thus $\tau^*\tilde{\subseteq}\tau^{**}$. Similarly $\tau^{**}\tilde{\subseteq}\tau^*$ and so $\tau^* = \tau^{**}$. □

7 Conclusion

In the present work, we have introduced the concept of soft M^*-open set in soft topological space such as soft M^*-interior, soft M^*-closure, soft M^*-limit point, soft M^*-derived set and soft M^*-neighborhood. We also define and discuss the properties of soft M^*-bases and soft M^*-subbases.

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HALL CURRENT AND VISCOUS DISSIPATION ON *MHD* NATURAL CONVECTION FLOW WITH THERMAL STRATIFICATION

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Abstract

The study's unsteady *MHD* free convective flow traverses a porous material that is influenced by thermal radiation, heat absorption, the hall effect and thermal stratification as it passes over a vertically inclined temperature-varying flat plate. The governing equations are transformed into a dimensionless form by applying the appropriate transformation and solved by using Crank-Nicolson method, numerically. The graphical results have been achieved using Mathematica. The results demonstrate that radiation and heat absorption have a significant effect on the velocity and temperature profiles on thermally stratified fluid.

2020 Mathematical Sciences Classification: 76W05, 76D10, 76S05, 80A20, 65M06.
Keywords and Phrases: MHD, Thermally stratification, Viscous dissipation, Hall current, Crank-Nicolson method.

Nomenclature

B_0 Strength of magnetic field
u' Dimensional velocity of plate in x-direction
v' Dimensional velocity of plate in y-direction
w' Dimensional velocity of plate in z-direction
T' Dimensional Fluid temperature
u Dimensionless velocity
g Gravitational acceleration
β_T Coefficient of thermal expansion
α angle of inclination
σ Electrical conductivity
σ_s Stefan-Boltzmann Constant
ρ Fluid density
ν Kinematic viscosity
K' Dimensional permeability of porous medium
κ Thermal conductivity
k_e mean absorption coefficient
C_p Specific heat at constant pressure
μ viscosity of the fluid
q_r Radiative heat flux
Q_0 heat absorption coefficient
T_∞ Fluid temperature far away from the plate
T_w wall temperature
A Constant buoyancy frequency
Gr Thermal Grashof number
θ Non-dimensional temperature
M Magnetic parameter
Pr Prandtl number
R Radiation parameter
S stratification parameter
Ec Eckert number
Q Heat absorption parameter
P Pressure
τ Skin friction
m hall parameter

1 Introduction

The importance of magnetohydrodynamic free convection stems from its presence in both natural and fluid engineering situations. One example of how free convection is used in a natural phenomenon is the ocean currents, which are caused by variations in salt and temperature difference in the layers of the water. This temperature difference between the layers of the water is known as thermal stratification, Which is applicable in energy extraction procedures from sea waves. Hall current, heat source or sinks, thermal radiation and thermal stratification all have a significant impact on the pace of heat transmission, which determines the quality of the finished product.

Many scholars have examined the dynamics of flow and heat transmission. Chakrabarti and Gupta [2] who measured the heat transfer in hydro-magnetic flow. Grubka and Bobba [7] investigated the properties of heat transfer and fluid flow on a stretching sheet at different temperatures. In a boundary layer flow through a porous media, Chaudhary *et al*[3] investigated the complexities on *MHD* and Khan *et al.*[10] worked on dissipation energy and stress work on unsteady *MHD* free convection flow of heat transfer. A porous stretching sheet in porous media with a range of effects were discussed by Vijayalakshmi *et al.*[17] and Raju *et al.*[13]. In their research on heat transport in a porous media, temperature-dependent heat sources have been taken into consideration by Moalem [11] and Chamkha [4]. Soid and Ishak [15] estimates thermal boundary layer flow over an increasingly stretched surface with radiation effect.

Recently, The effects of varied fuid characteristics and viscous dissipation on the transport phenomenon were discussed by Reddy *et al.*[14], Das *et al.*[5], Ferdows *et al.*[6] and Megahed *et al.*[12]. Researchers Barik *et al.*[1] studied the effects of thermal radiation on an unstable *MHD* flow past an inclined porous heated plate in the context of viscous dissipation, and Iranian D [9] studied the effects of varying viscosity, thermal conductivity, and stratification parameter with velocity and temperature profiles. Senge *et al.*[16] investigated the *MHD* in a thermally stratified porous medium in the presence of heat source. Goud *et al.*[8] used an accelerating perpendicular plate with *MHD* to examine the viscous dissipation-affected time-dependent thermally stratified flow through a porous medium.

In an effort to close the gap with the previously mentioned literature, the study investigated the effects of thermal stratification, radiation, viscous dissipation, and the hall effect using an unsteady *MHD* flow across a temperaturevarying vertically angled plate.

2 Mathematical Formulation

This paper considers radiation, thermal stratification, and viscous dissipation with hall effect in the context of an incompressible, unsteady, free convective viscous *MHD* flow via a semi-infinite nonconducting vertically angled plates inserted in a porous medium. In contrast to the direction of gravity, the x-axis rises vertically while the y-axis is perpendicular to the plate. Figure 2.1 depicts the problem's schematic arrangement. In the x' and y' directions, the plate is positioned after an α, tilt from the vertical x-axis. The homogeneous magnetic field, B_0, is speculated transversely to the flow direction. Hall currents are generated as a result of the strong magnetic field which ultimately result in the magnetic forces exerted on the fluid in motion. The Hall parameter $m(=\omega_e\tau_e)$ is introduced in order to account for the Hall effect, where ω_e and τ_e represent the electron frequency and electron collision time respectively. At $t' \leq 0$, the fluid and plate are initially at rest with a temperature T_∞. The plate experiences a sudden jerk after time t', which causes the fluid to flow with a velocity $A\sqrt{A\nu}t'$ and raise its temperature to T_w.

With the use of the generalized Ohm law, the conservation of electric charge equation yields the electric current density component in x and z direction as follows:

$$J'_x = \frac{\sigma B_0}{1+m^2}(mu' - w') \, , \tag{2.1}$$

$$J'_z = \frac{\sigma B_0}{1+m^2}(u' + mw') \, . \tag{2.2}$$

Here, u' and w' are the velocity and secondary velocity for the fluid flow in x' and z' direction respectively. The flow variables are only functions of y' and t' because of the infinite length in the x' direction.

Under the prior assumptions, the governing boundary layer equations using Boussinesq's approximation are

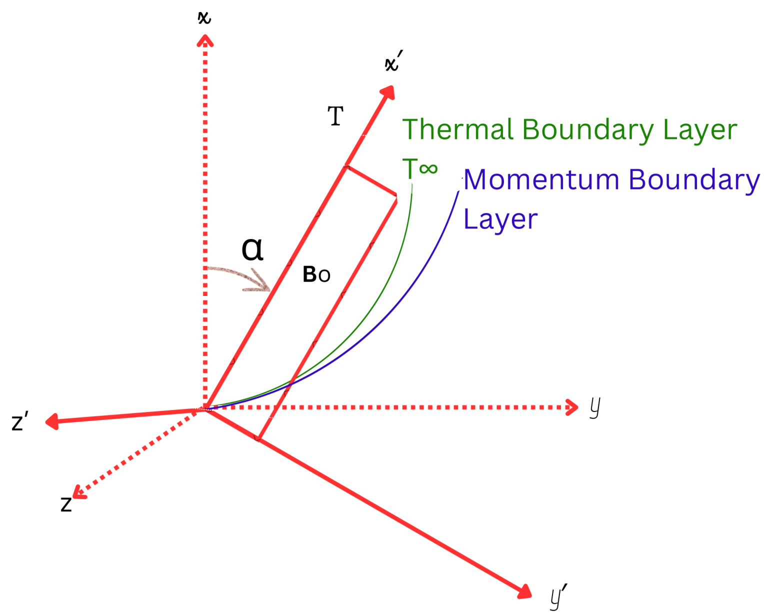

Figure 2.1: *Configuration of the physical model of the problem*

$$\frac{\partial v'}{\partial y'} = 0 \; , \tag{2.3}$$

$$\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'} = \nu\frac{\partial^2 u'}{\partial y'^2} + g\beta_T(T' - T_\infty)cos\alpha - \frac{\sigma B_0^2}{\rho(1+m^2)}(u' + mw') - \frac{\nu}{K'}u' \tag{2.4}$$

$$\frac{\partial w'}{\partial t'} + v'\frac{\partial w'}{\partial y'} = \nu\frac{\partial^2 w'}{\partial y'^2} + \frac{\sigma B_0^2}{\rho(1+m^2)}(mu' - w') - \frac{\nu}{K'}w', \tag{2.5}$$

$$\frac{\partial T'}{\partial t'} + v'\frac{\partial T'}{\partial y'} = \frac{\kappa}{\rho C_p}\frac{\partial^2 T'}{\partial y'^2} - \gamma(u' + w') + \frac{\mu}{\rho C_p}((\frac{\partial u'}{\partial y'})^2 + (\frac{\partial w'}{\partial y'})^2) - \frac{1}{\rho Cp}\frac{\partial q_r}{\partial y'} - \frac{Q_0}{\rho C_p}(T' - T_\infty). \tag{2.6}$$

where

$$\gamma = \frac{dT_w}{dy'} + \frac{g}{C_p}.$$

In this case, pressure work is denoted by $\frac{g}{C_P}$ and thermal stratification by $\frac{dT_\infty}{dz}$. It is assumed that the fluid flow has the following initial and boundary conditions

$$u'(y', 0) = 0, \;\; w'(y', 0) = 0, \;\; T'(y', 0) = T_\infty \;\; \forall \; y' > 0, \tag{2.7}$$

$$u'(0, t') = A\sqrt{A\nu}\, t', \;\; w'(0, t') = A\sqrt{A\nu}\, t', \;\; T'(0, t') = T_\infty + (T_w - T_\infty)e^{At'} \;\; \forall \; t' > 0, \tag{2.8}$$

$$u'(y', t') \to 0, \;\; w'(y', t') \to 0, \;\; T'(y', t') \to T_\infty \;\; as \;\; y' \to \infty. \tag{2.9}$$

Where $v' = -\sqrt{\nu A}$ be the velocity in z-drection and A $= \frac{\gamma u'}{\Delta T'}$ is given as buoyancy frequency which measures a fluid's resistance to vertical displacements like those carried on by convection. It is presumed that thermal radiation exists. The radiative heat flow, denoted by q_r, can be calculated using Rosseland's approximation.

$$q_r = -\frac{4\sigma_s}{3k_e}\frac{\partial T'^4}{\partial y'}.$$

If the flow's temperature gradients are low enough, we can get a linear version by substituting T'^4 in a Taylor series by leaving out higher order components around T_∞ as-

$$T'^4 = 4{T_\infty}^3 T' - 3{T_\infty}^4.$$

We introduce the subsequent non-dimensional quantities:

$$y=\sqrt{\frac{A}{\nu}}y',\ \ u=\frac{u'}{\sqrt{\nu A}},\ \ w=\frac{w'}{\sqrt{\nu A}},\ \ t=At',\ \ K=\frac{K'A}{\nu},\ \ Gr=\frac{g\beta_T(T_w-T_\infty)}{A\sqrt{\nu A}},$$

$$\theta=\frac{T'-T_\infty}{T_w-T_\infty},\ \ M=\frac{\sigma {B_0}^2}{\rho A},\ \ Pr=\frac{\mu C_P}{\kappa},\ \ \mu=\rho\nu,\ \ S=\frac{\sqrt{\nu}\,\gamma}{\sqrt{A}(T_w-T_\infty)},$$

$$Q=\frac{Q_0}{\rho C_p A},\ \ R=\frac{k_e\kappa}{4\sigma_s {T_\infty}^3},\ \ Ec=\frac{\nu A}{C_p(T_w-T_\infty)}.$$

Dimensionless values can be used to simplify equations (2.4) to (2.9). Therefore, equations in their non-dimensionalized form (2.4) to (2.9) convert as follows:

$$\frac{\partial u}{\partial t}-\frac{\partial u}{\partial y}=\frac{\partial^2 u}{\partial y^2}+Gr\theta cos\alpha-(\frac{M}{1+m^2}+\frac{1}{K})u-\frac{mM}{1+m^2}w, \tag{2.10}$$

$$\frac{\partial w}{\partial t}-\frac{\partial w}{\partial y}=\frac{\partial^2 w}{\partial y^2}-(\frac{M}{1+m^2}+\frac{1}{K})w+\frac{mM}{1+m^2}u, \tag{2.11}$$

$$\frac{\partial \theta}{\partial t}-\frac{\partial \theta}{\partial y}=\frac{1}{Pr}(1+\frac{4}{3R})\frac{\partial^2 \theta}{\partial y^2}-S(u+w)+Ec((\frac{\partial u}{\partial y})^2+(\frac{\partial w}{\partial y})^2)-Q\theta. \tag{2.12}$$

Following are the relevant non-dimensional boundary conditions:

$$u(y,0)=0,\ \ w(y,0)=0,\ \ \theta(y,0)=0,\ \ \forall\ y>0, \tag{2.13}$$

$$u(0,t)=t,\ \ w(0,t)=t,\ \ \theta(0,t)=e^t\ \ \forall\ t>0, \tag{2.14}$$

$$u(y,t)\to 0,\ \ w(y,t)\to 0,\ \ \theta(y,t)\to 0\ \ as\ \ y\to\infty\ \ \forall\ t>0. \tag{2.15}$$

Equations (2.16) to (2.18) can be used to estimate the values of Skin Friction as well as Nusselt Number.

Skin Friction. Skin friction is determined by:

$$\tau_1=(\frac{\partial u}{\partial y})_{y=0}, \tag{2.16}$$

$$\tau_2=(\frac{\partial w}{\partial y})_{y=0}. \tag{2.17}$$

Nusselt Number Nusselt number is determined by:

$$Nu=-(\frac{\partial \theta}{\partial y})_{y=0}. \tag{2.18}$$

Solution. Set up the proper beginning and boundary conditions, then use the implicit finite difference approach of the Crank and Nicolson model to solve the non-linear momentum and energy equations (2.10) to (2.15). As such, we use the Nicolson approach to obtain the relevant finite difference equations

$$\begin{aligned}\frac{u_{i,j+1}-u_{i,j}}{\Delta t}-\frac{u_{i+1,j}-u_{i,j}}{\Delta y}&=\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}+u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{2(\Delta y)^2}\right)\\&+Gr\cos(\alpha)\left(\frac{\theta_{i,j+1}+\theta_{i,j}}{2}\right)-\left(\frac{M}{1+m^2}+\frac{1}{K}\right)\left(\frac{u_{i,j+1}+u_{i,j}}{2}\right)\\&-\left(\frac{mM}{1+m^2}\right)\left(\frac{w_{i,j+1}+w_{i,j}}{2}\right),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\frac{w_{i,j+1}-w_{i,j}}{\Delta t}-\frac{w_{i+1,j}-w_{i,j}}{\Delta y}&=\left(\frac{w_{i+1,j+1}-2w_{i,j+1}+w_{i-1,j+1}+w_{i+1,j}-2w_{i,j}+w_{i-1,j}}{2(\Delta y)^2}\right)\\&-\left(\frac{M}{1+m^2}+\frac{1}{K}\right)\left(\frac{w_{i,j+1}+w_{i,j}}{2}\right)+\left(\frac{mM}{1+m^2}\right)\left(\frac{u_{i,j+1}+u_{i,j}}{2}\right),\end{aligned} \tag{2.20}$$

$$\begin{aligned}\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta t}-\frac{\theta_{i,j+1}-\theta_{i,j}}{\Delta y}&=\frac{1}{Pr}\left(1+\frac{4}{3R}\right)\left(\frac{\theta_{i+1,j+1}-2\theta_{i,j+1}+\theta_{i-1,j+1}+\theta_{i+1,j}-2\theta_{i,j}+\theta_{i-1,j}}{2(\Delta y)^2}\right)\\&-S\left(\frac{u_{i,j+1}+u_{i,j}}{2}\right)+\left(\frac{w_{i,j+1}+w_{i,j}}{2}\right)\\&+Ec\left(\left(\frac{u_{i+1,j}-u_{i,j}}{\Delta y}\right)^2+\left(\frac{w_{i+1,j}-w_{i,j}}{\Delta y}\right)^2\right)-Q\left(\frac{\theta_{i,j+1}+\theta_{i,j}}{2}\right).\end{aligned} \tag{2.21}$$

The related boundary conditions are as follows:

$$u_{i,0} = 0, \quad w_{i,0} = 0, \quad \theta_{i,0} = 0 \ \ \forall \ i, \tag{2.22}$$

$$u_{0,j} = j\Delta t, \quad w_{0,j} = j\Delta t, \quad \theta_{0,j} = e^{j\Delta t} \ \ \forall \ j > 0, \tag{2.23}$$

$$u_{L,j} \to 0, \quad w_{L,j} \to 0, \quad \theta_{L,j} \to 0 \ \ as \ \ L \to \infty \ \ \forall \ j > 0. \tag{2.24}$$

Here, time t is indicated by index j, while space y is shown by index i. Furthermore, accordingly, the mesh sizes along the time t-direction and the y-direction are represented by Δt and Δy. The tri-diagonal system of equations consisting of the finite difference equations (2.19) to (2.21) at each internal nodal point on a given n-level is solved using the Thomas algorithm.

3 Results and Discussion

The temperature(θ) and velocity (u and w) profiles are shown using graphs based on various flow parameter values. The graphic analysis of the effects of different parameters, such as the heat absorption parameter Q, radiation parameter (R), thermal stratification parameter (S), thermal Grashof number (Gr), Hartmann number (M), hall parameter (m), angle of inclination (α), Eckert number (Ec) and Prandtl number (Pr) on the temperature field and velocity distribution is shown in Figures 3.1 to 3.21.

When seen in Figure 3.1 and Figure 3.2, the angle of inclination (α) decreases, both u and w increases. The link of both fluid velocity u as well as w between K, Gr and m is depicted in Figures 3.2 to 3.8 as, these parameters values increase, consequently increases the fluid's velocity. Figures 3.9 and 3.10 illustrates how velocity u falls and w rise with increasing M. In Figures 3.11 and 3.12 velocity decreases in nature for increasing values of Prandtl number. In addition, velocity u increases with increasing value of Ec in Figure 3.13. Decreasing effect of Q, R and S have been seen in Figures 3.14 to 3.16 for u. Figure 3.17 illustrates how increasing the Ec causes the fluid's temperature to rise. Figures 3.18 to 3.21 shows the decreasing influence of Pr, Q, R and S on θ The temperature falls when Pr, Q, R and S rises

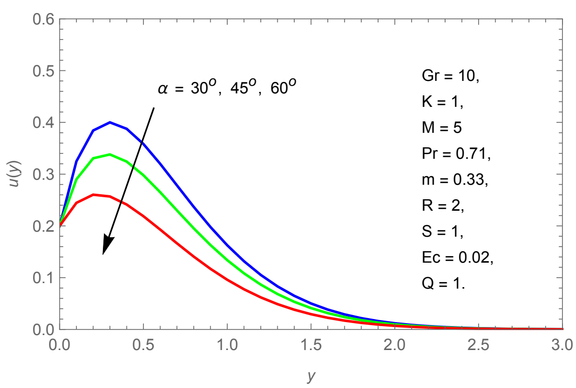

Figure 3.1: *Velocity* (u) *v/s* α

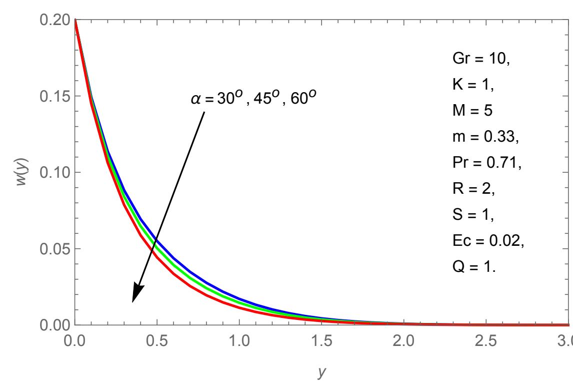

Figure 3.2: *Velocity* (w) *v/s* α

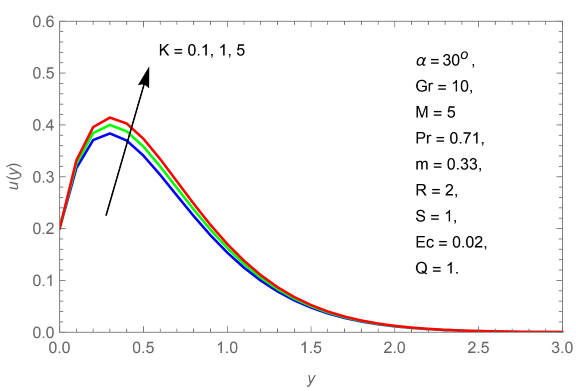

Figure 3.3: *Velocity* (u) *v/s* K

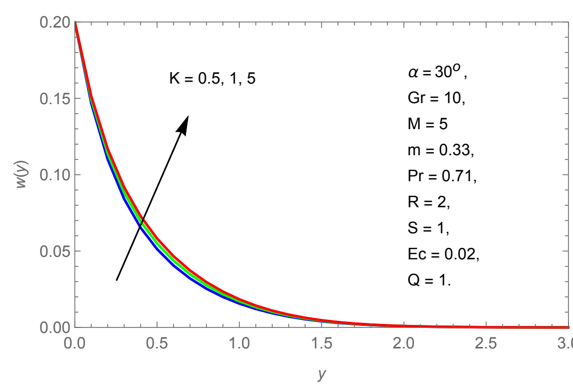

Figure 3.4: *Velocity* (u) *v/s* K

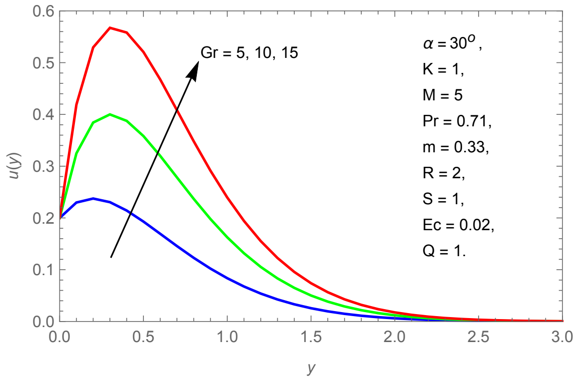

Figure 3.5: *Velocity (u) v/s Gr*

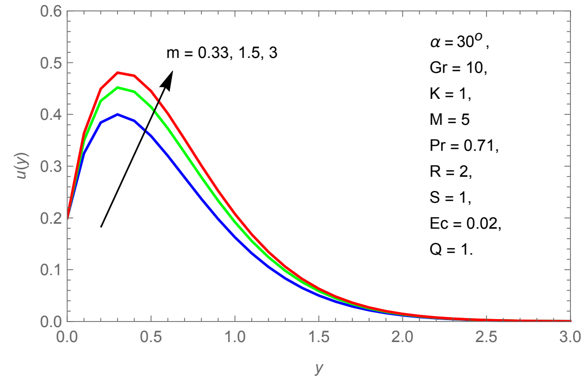

Figure 3.6: *Velocity (u) v/s m*

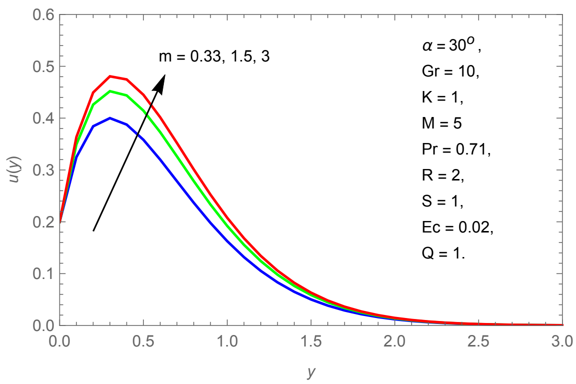

Figure 3.7: *Velocity (u) v/s m*

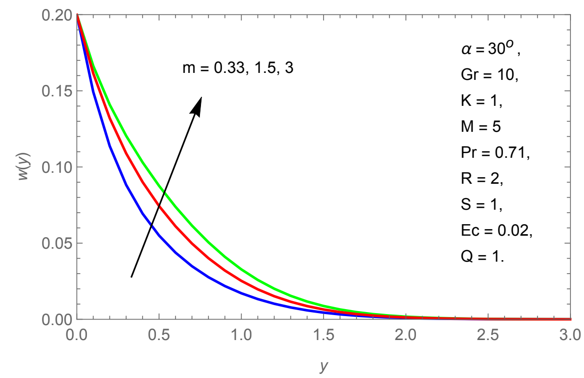

Figure 3.8: *Velocity (w) v/s m*

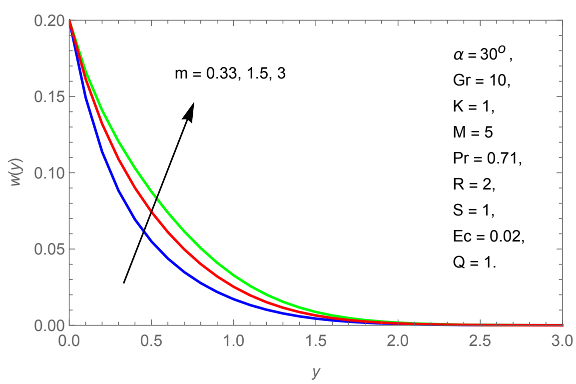

Figure 3.9: *Velocity (w) v/s m*

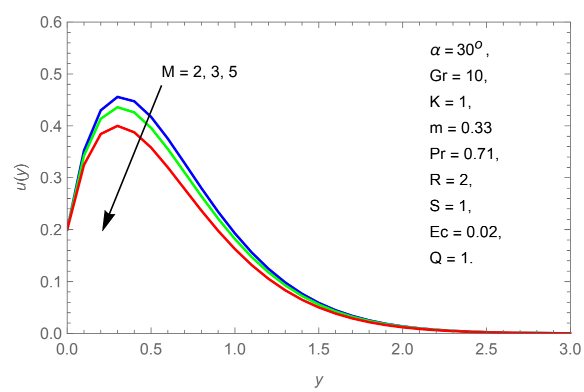

Figure 3.10: *Velocity (u) v/s M*

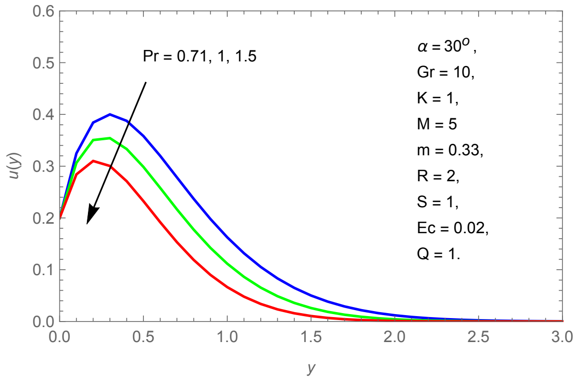

***Figure* 3.11:** *Velocity* (u) *v/s* Pr

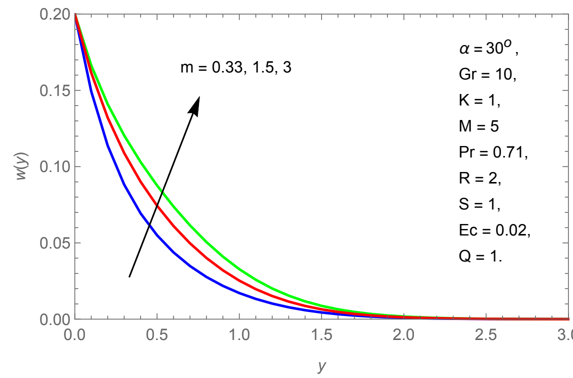

***Figure* 3.12:** *Velocity* (w) *v/s* m

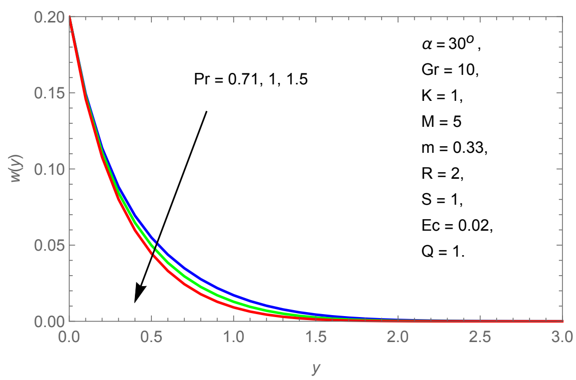

***Figure* 3.13:** *Velocity* (w) *v/s* Pr

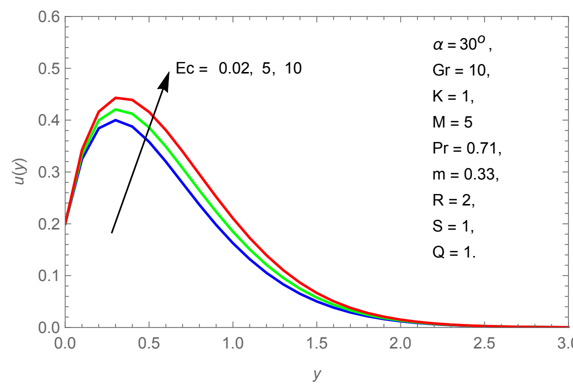

***Figure* 3.14:** *Velocity* (u) *v/s* Ec

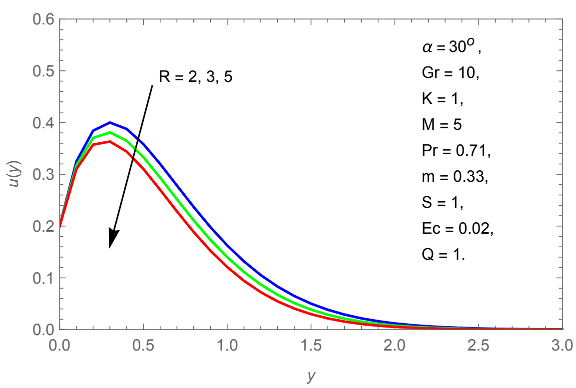

***Figure* 3.15:** *Velocity* (u) *v/s* R

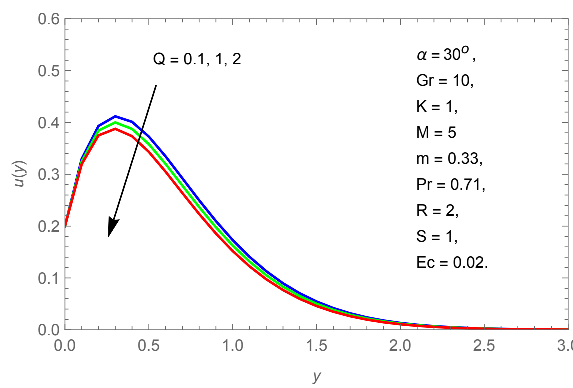

***Figure* 3.16:** *Velocity* (u) *v/s* Q

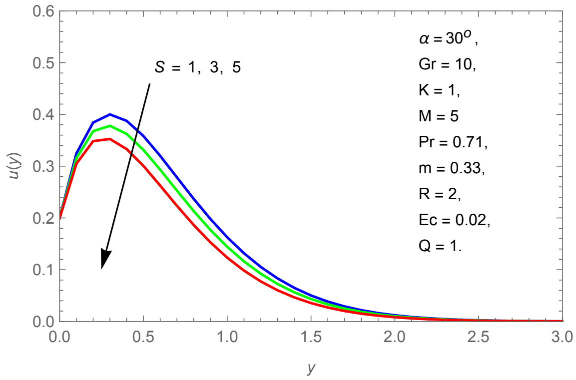

***Figure* 3.17:** *Velocity* (u) *v/s* S

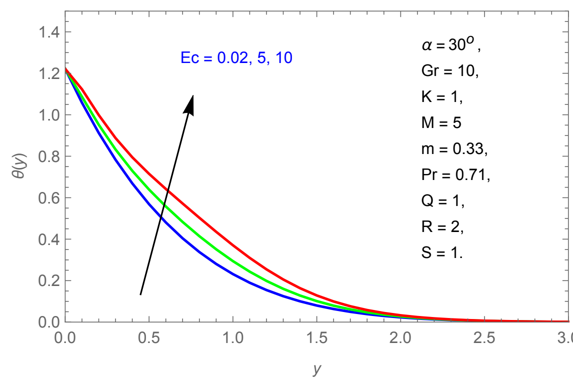

***Figure* 3.18:** *Temperature* θ *v/s* Ec

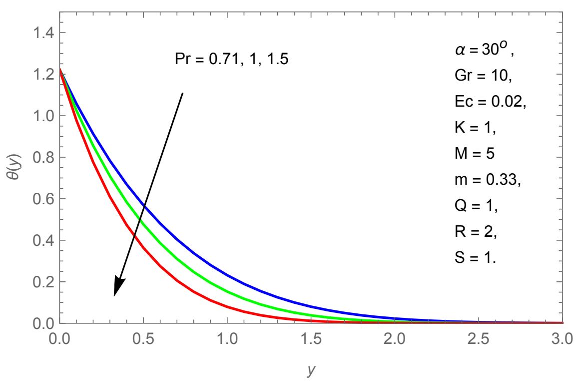

***Figure* 3.19:** *Temperature* θ *v/s* Pr

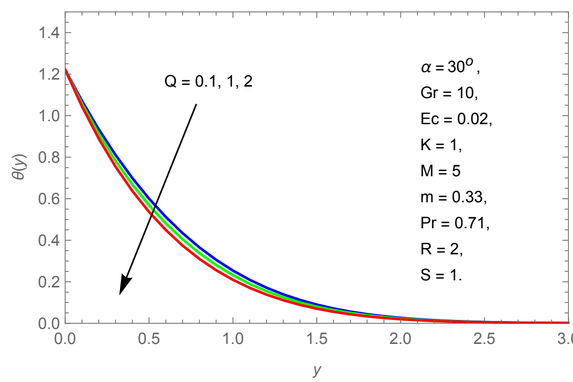

***Figure* 3.20:** *Temperature* θ *v/s* Q

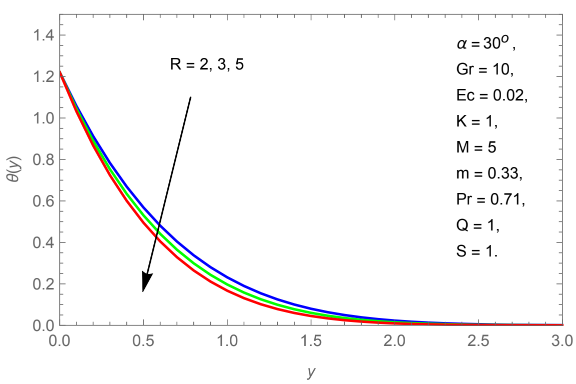

***Figure* 3.21:** *Temperature* θ *v/s* R

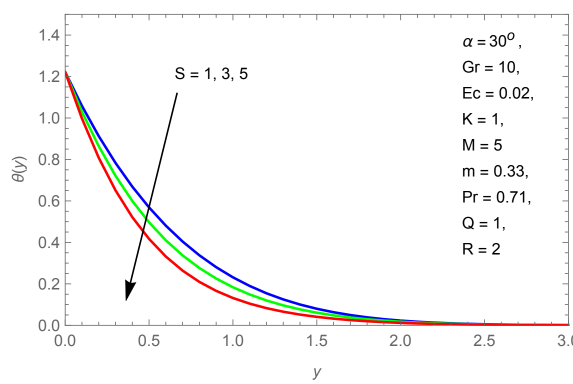

***Figure* 3.22:** *Temperature* θ *v/s* S

Tables 3.1 and 3.2 give the values of skin friction and nusselt number respectively and show how different parameters affect these quantities such as- Based on Table 3.1, skin friction values τ_1 and τ_2 grow in proportion to increases in α, Pr, R, Q, and S. The contrary effect is seen when the figures for K, Gr, and Ec rise. In contrast, the skin frictions are surprisingly affected by m and M; that is, τ_1 decreases and τ_2 increases as m increases but τ_1 increases and τ_2 lowers as M increases. The Nusselt figure for Ec dropped, according to Table 3.2 , but when Pr, R, Q and S were increased, the opposite outcome was seen.

Table 3.1: *Coefficients of skin friction τ_1 and τ_2 for varying parameters value*

α	**K**	**Gr**	**M**	**m**	**Pr**	**R**	**Ec**	**Q**	**S**	τ_1	τ_2
30^0	**1**	**10**	**5**	**0.33**	**0.71**	**2**	**0.02**	**1**	**1**	1.24768	0.507061
45^0	1	10	5	0.33	0.71	2	0.02	1	1	0.900179	0.525578
60^0	1	10	5	0.33	0.71	2	0.02	1	1	0.445472	0.549808
30^0	**0.5**	10	5	0.33	0.71	2	0.02	1	1	1.16679	0.532558
30^0	**5**	10	5	0.33	0.71	2	0.02	1	1	1.31643	0.485516
30^0	1	**5**	5	0.33	0.71	2	0.02	1	1	0.297948	0.557669
30^0	1	**15**	5	0.33	0.71	2	0.02	1	1	2.18847	0.456932
30^0	1	10	**2**	0.33	0.71	2	0.02	1	1	1.52217	0.521526
30^0	1	10	**3**	0.33	0.71	2	0.02	1	1	1.42499	0.514181
30^0	1	10	5	**1.5**	0.71	2	0.02	1	1	1.49436	0.336609
30^0	1	10	5	**3**	0.71	2	0.02	1	1	1.63466	0.385574
30^0	1	10	5	0.33	**1**	2	0.02	1	1	1.06429	0.522542
30^0	1	10	5	0.33	**1.5**	2	0.02	1	1	0.841479	0.53961
30^0	1	10	5	0.33	0.71	**3**	0.02	1	1	1.17182	0.513623
30^0	1	10	5	0.33	0.71	**5**	0.02	1	1	1.1012	0.519531
30^0	1	10	5	0.33	0.71	2	**5**	1	1	1.33266	0.502503
30^0	1	10	5	0.33	0.71	2	**10**	1	1	1.42726	0.497618
30^0	1	10	5	0.33	0.71	2	0.02	**0.1**	1	1.29702	0.50406
30^0	1	10	5	0.33	0.71	2	0.02	**2**	1	1.19637	0.5102
30^0	1	10	5	0.33	0.71	2	0.02	1	**5**	1.15601	0.511781
30^0	1	10	5	0.33	0.71	2	0.02	1	**10**	1.04956	0.517345

Table 3.2: *Nusselt Number coefficients for various parameter values*

Pr	**R**	**Ec**	**Q**	**S**	Nu
0.71	**2**	**0.02**	**1**	**1**	1.63305
1	2	0.02	1	1	1.96001
1.5	2	0.02	1	1	2.43302
0.71	**3**	0.02	1	1	1.76233
0.71	**5**	0.02	1	1	1.89014
0.71	2	**5**	1	1	1.34621
0.71	2	**10**	1	1	0.983352
0.71	2	0.02	**0.1**	1	1.50071
0.71	2	0.02	**2**	1	1.77101
0.71	2	0.02	1	**5**	1.91603
0.71	2	0.02	1	**10**	2.24096

4 Conclusion

This paper investigates radiation effects on thermally stratified fluid with Hall effects, viscous dissipation with heat source/sink and unsteady *MHD* free convectionan incompressible electrically conducting fluid flowing past an inclined plate while a uniform magnetic field is applied. The leading equations are numerically solved by use of the Crank-Nicolson method. The findings show the temperature and velocity as well as other flow properties. Based on the flow analysis, the following findings are made:

- As the angle of inclination α or magnetic parameter M increases, the magnitude of velocity u decreases. The results for the heat absorption parameter Q, radiation parameter R, stratification parameter S and Prandtl number Pr are all the same. With regard to the remaining parameters as the porosity parameter K, hall parameter m, Grashof number Gr and Ekert numbers Ec the opposite conclusions are valid.
  Similar findings apply to velocity w, with the exception of the M, velocity w grows along with M.
- The temperature rises when the Ec rises, whereas the remaining parameters such as Q, R, S and Pr do the reverse.
- As the K, Gr and Ec increase, so does the local skin friction τ_1 and τ_2. As the angle of inclination α, Pr, R, S and Q increase, it also rises. When the M increases, the value of τ_1 rises and τ_2 falls. As the value of m increases, τ_1 declines and τ_2 rises.
- The Nusselt number Nu rises as the Pr, R, Q and S all rise. As the value of the Ec increases, it lowers.

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ON $\Psi_{\alpha,\beta}$-CONTRACTIVE MAPPINGS AND FIXED POINTS

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Abstract

In this paper we introduce a new displacement–distance control function $\Psi_{\alpha,\beta}$ that nonlinearly couples the interpoint distance $d(\xi,\eta)$ with the self–displacements $d(\xi,\mathscr{T}\xi)$ and $d(\eta,\mathscr{T}\eta)$. This rational gauge generates a novel contractive mechanism that is simultaneously sensitive to the relative geometry of points and to their deviation from fixedness.Under a natural domination condition linking self–displacements to interpoint distances, we establish a new fixed point theorem for $\Psi_{\alpha,\beta}$–contractive mappings in complete metric spaces.The results guarantee existence, uniqueness, and global convergence of Picard iterates.Several refined corollaries show that the classical Banach, Reich, and Kannan contraction principles are recovered as special cases of the present framework.An orbit–dominated version of the main theorem is also derived, which extends the applicability of the method to nonuniform and state–dependent contractions that are not covered by standard Lipschitz conditions.As an application, the theory is applied to a nonlinear integral equationnon a space of continuous functions, where the associated integral operator is shown to satisfy the new $\Psi_{\alpha,\beta}$–contractive condition. The proposed displacement–distance paradigm provides a flexible and unifying approach to fixed point theory and opens new directions for generalizations in nonlinear analysis and operator equations.

2020 Mathematical Sciences Classification: 47H10; 54H25; 54E50.
Keywords and Phrases: metric space; fixed point; control function; displacement; rational contraction; Picard iteration; nonlinear integral equation.

1 Introduction

Fixed point theory in metric spaces is a cornerstone of nonlinear analysis and its applications to integral and differential equations, optimization, and numerical schemes; comprehensive background and applications can be found in standard monographs and handbooks such as [4, 11, 18, 25, 29]. The starting point is the Banach contraction principle [1], which guarantees existence, uniqueness, and convergence of Picard iterates for strict Lipschitz contractions. Many influential generalizations weaken the Lipschitz requirement by involving self-displacements, cross distances, multivaluedness, or implicit controls; representative classical directions include Kannan-type contractions [15], Chatterjea-type contractions [5], Reich-type contractions and their refinements [9, 23], as well as comparison frameworks [24] and metric-completeness characterizations [26]. Beyond these, nonlinear and rational-type mechanisms have been developed to cover settings not captured by uniform Lipschitz constants [3, 6, 7, 13, 22]. Modern control-function approaches include altering distances [16], α–ψ-contractive mappings [27], simulation functions [19], and $\mathscr{F}$-contractions [12, 28]. For convenient reference and positioning, we record five classical theorems that motivate the present work.

Theorem 1.1 (Banach contraction principle [1]). *Let $(\mathscr{X},d)$ be complete and let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ satisfy $d(\mathscr{T}x,\mathscr{T}y)\le k\,d(x,y)$ for all $x,y\in\mathscr{X}$ with some $k\in[0,1)$. Then $\mathscr{T}$ has a unique fixed point $p\in\mathscr{X}$, and for every $x_0\in\mathscr{X}$ the Picard sequence $x_{n+1}=\mathscr{T}x_n$ converges to p.*

Theorem 1.2 (Kannan fixed point theorem [15]). *Let $(\mathscr{X},d)$ be complete and let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ satisfy*

$$d(\mathscr{T}x,\mathscr{T}y)\le\kappa\big(d(x,\mathscr{T}x)+d(y,\mathscr{T}y)\big)\qquad\forall x,y\in\mathscr{X},$$

for some $\kappa\in\big[0,\tfrac{1}{2}\big)$. Then $\mathscr{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 1.3 (Chatterjea fixed point theorem [5]). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}x, \mathscr{T}y) \leq \chi\big(d(x, \mathscr{T}y) + d(y, \mathscr{T}x)\big) \qquad \forall x, y \in \mathscr{X},$$

for some $\chi \in \left[0, \frac{1}{2}\right)$. Then $\mathscr{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 1.4 (Reich type theorem [23]). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}x, \mathscr{T}y) \leq a\, d(x, y) + b\, d(x, \mathscr{T}x) + c\, d(y, \mathscr{T}y) \qquad \forall x, y \in \mathscr{X},$$

with constants $a, b, c \geq 0$ satisfying $a + b + c < 1$. Then $\mathscr{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 1.5 ($\mathscr{F}$-contraction principle [28]). *Let $(\mathscr{X}, d)$ be complete. If $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is an $\mathscr{F}$-contraction in the sense of Wardowski, then $\mathscr{T}$ admits a unique fixed point and Picard iteration converges to it.*

The development of rational and implicit contractive conditions is another strong theme, notably the rational-type extension due to Dass and Gupta [7] and subsequent comparative frameworks [9, 24]. Motivated by these directions, we propose a new *displacement–distance* control function that depends simultaneously on $d(x, y)$ and on the two self-displacements $d(x, \mathscr{T}x)$ and $d(y, \mathscr{T}y)$. This produces a rational, nonlinear contractive reduction that is sensitive to how far points are from their images and complements existing control-function paradigms [16, 19, 27, 28].

2 Preliminaries

We recall standard notions and notations used throughout; see [4, 18, 25] for background.

Definition 2.1 (Metric space [18, 25]). *Let $\mathscr{X} \neq \varnothing$. A mapping*

$$d : \mathscr{X} \times \mathscr{X} \to [0, \infty)$$

is called a metric *on $\mathscr{X}$ if, for all $\xi, \eta, \zeta \in \mathscr{X}$, the following conditions hold:*
(m1) $d(\xi, \eta) = 0$ if and only if $\xi = \eta$,
(m2) $d(\xi, \eta) = d(\eta, \xi)$,
(m3) $d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$.
In this case, the pair $(\mathscr{X}, d)$ is called a metric space.

Definition 2.2 (Open ball, convergence [18, 25]). *Let $(\mathscr{X}, d)$ be a metric space. For $\xi \in \mathscr{X}$ and $r > 0$, the* open ball *centered at ξ with radius r is*

$$B(\xi, r) := \{\eta \in \mathscr{X} : d(\xi, \eta) < r\}.$$

A sequence $\{\xi_n\} \subset \mathscr{X}$ converges *to $\xi \in \mathscr{X}$ if $d(\xi_n, \xi) \to 0$ as $n \to \infty$.*

Definition 2.3 (Cauchy sequence, completeness [18, 25]). *Let $(\mathscr{X}, d)$ be a metric space. A sequence $\{\xi_n\} \subset \mathscr{X}$ is called a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(\xi_n, \xi_m) < \varepsilon \qquad \text{for all } m, n \geq N.$$

The metric space $(\mathscr{X}, d)$ is called complete *if every Cauchy sequence in $\mathscr{X}$ converges to a point of $\mathscr{X}$.*

Definition 2.4 (Self-map, fixed point [4, 25]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* self-map. *A point $\pi \in \mathscr{X}$ is called a* fixed point *of $\mathscr{T}$ if*

$$\mathscr{T}\pi = \pi.$$

The set of fixed points of $\mathscr{T}$ is denoted by $\mathrm{Fix}(\mathscr{T}) = \{\pi \in \mathscr{X} : \mathscr{T}\pi = \pi\}$.

Definition 2.5 (Picard iteration and Picard operator [4]). *Let $(\mathscr{X}, d)$ be a metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. For an initial point $\xi_0 \in \mathscr{X}$, the* Picard iteration *is the sequence $\{\xi_n\}$ defined by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

The mapping $\mathscr{T}$ is called a Picard operator *if it has a unique fixed point $\pi \in \mathscr{X}$ and, for every $\xi_0 \in \mathscr{X}$, the Picard iteration converges to π.*

Definition 2.6 (Banach contraction [1]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Banach contraction *if there exists a constant $k \in [0, 1)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.7 (Kannan contraction [15]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Kannan contraction *if there exists $\kappa \in \left[0, \frac{1}{2}\right)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \kappa\big(d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta)\big) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.8 (Chatterjea contraction [5]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Chatterjea contraction *if there exists $\chi \in \left[0, \frac{1}{2}\right)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \chi\big(d(\xi, \mathscr{T}\eta) + d(\eta, \mathscr{T}\xi)\big) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.9 (Reich-type contraction [23]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Reich-type contraction *if there exist constants $a, b, c \geq 0$ with $a + b + c < 1$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq a\, d(\xi, \eta) + b\, d(\xi, \mathscr{T}\xi) + c\, d(\eta, \mathscr{T}\eta) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.10 ($\mathscr{F}$-contraction (Wardowski) [28]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called an $\mathscr{F}$-*contraction *if there exist a function $\mathscr{F} : (0, \infty) \to \mathbb{R}$ satisfying Wardowskis axioms and a constant $\tau > 0$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) > 0 \;\Rightarrow\; \tau + \mathscr{F}\big(d(\mathscr{T}\xi, \mathscr{T}\eta)\big) \leq \mathscr{F}\big(d(\xi, \eta)\big), \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.11 (Fixed point and Picard iteration [4, 25]). *Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. A point $\pi \in \mathscr{X}$ is called a* fixed point *of $\mathscr{T}$ if*

$$\mathscr{T}\pi = \pi.$$

Given $\xi_0 \in \mathscr{X}$, the Picard iteration *is the sequence $\{\xi_n\}$ defined recursively by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

Definition 2.12 (Classical contraction types [1, 5, 15, 23]). *A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a Banach contraction if it satisfies Definition* 2.6*. It is called a Kannan contraction (resp. Chatterjea contraction, Reich-type contraction) if it satisfies Definition* 2.7 *(resp. Definition* 2.8*, Definition* 2.9*).*

Definition 2.13 (Rational-type contraction (prototype) [7]). *A* rational-type contraction *is a self-map whose contractive control for $d(\mathscr{T}\xi, \mathscr{T}\eta)$ is expressed through a rational combination of distances, typically involving $d(\xi, \eta)$ together with displacement or cross terms. The first systematic formulation of this type was introduced by Dass and Gupta* [7].

3 New Control Function

In this section, we introduce a new nonlinear rational control function, called the *displacement–distance gauge*, which couples the interpoint distance with the self-displacements of the mapping. This control will be the main tool in establishing our fixed point results.

Definition 3.1. *Let $\alpha, \beta > 0$. The mapping*

$$\Psi_{\alpha,\beta} : [0, \infty)^3 \to [0, 1), \qquad \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) := \frac{\mathfrak{c}^\alpha}{1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha},$$

is called the displacement–distance gauge*.*

In the special case $\alpha = \beta = 1$, we write $\Psi := \Psi_{1,1}$, and hence

$$\Psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}}{1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c}}.$$

Proposition 3.1. *Fix $\alpha, \beta > 0$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$. Then*

$$0 \leq \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) < 1, \qquad \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = 0 \iff \mathfrak{c} = 0.$$

Moreover, the function $\Psi_{\alpha,\beta}$ is strictly increasing in the variable $\mathfrak{c}$ and strictly decreasing in each of the variables $\mathfrak{a}$ and $\mathfrak{b}$.

Proof. Let $\alpha, \beta > 0$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$ be arbitrary. By definition,

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha}.$$

Since $\alpha, \beta > 0$, the functions $t \mapsto t^\alpha$ and $t \mapsto t^\beta$ are continuous and strictly increasing on $[0, \infty)$. Consequently, the denominator $1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha$ is strictly positive for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$. Moreover,

$$1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha > \mathfrak{c}^\alpha \quad \text{for all } \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0,$$

which immediately implies

$$0 \le \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} < 1.$$

Next, observe that

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 0 \iff \mathfrak{c}^\alpha = 0 \iff \mathfrak{c} = 0,$$

because $\alpha > 0$ and the only nonnegative real number whose α-th power is zero is zero itself.

To verify the monotonicity with respect to $\mathfrak{c}$, fix $\mathfrak{a},\mathfrak{b} \ge 0$ and consider the real-valued function

$$\phi(\mathfrak{c}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}, \qquad \mathfrak{c} \ge 0.$$

Since $\alpha > 0$, the function $\mathfrak{c} \mapsto \mathfrak{c}^\alpha$ is strictly increasing, and both numerator and denominator are differentiable for $\mathfrak{c} > 0$. A direct computation gives

$$\phi'(\mathfrak{c}) = \frac{\alpha\mathfrak{c}^{\alpha-1}\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta\big)}{\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha\big)^2}, \qquad \mathfrak{c} > 0.$$

All factors in the numerator and denominator are nonnegative, and the numerator is strictly positive for every $\mathfrak{c} > 0$. Hence $\phi'(\mathfrak{c}) > 0$ for all $\mathfrak{c} > 0$, which proves that $\Psi_{\alpha,\beta}$ is strictly increasing in $\mathfrak{c}$.

Finally, fix $\mathfrak{b},\mathfrak{c} \ge 0$ and consider

$$\psi(\mathfrak{a}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}, \qquad \mathfrak{a} \ge 0.$$

Since $\beta > 0$, the map $\mathfrak{a} \mapsto \mathfrak{a}^\beta$ is strictly increasing. Differentiating with respect to $\mathfrak{a}$ yields

$$\psi'(\mathfrak{a}) = -\frac{\beta\mathfrak{a}^{\beta-1}\mathfrak{c}^\alpha}{\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha\big)^2} \le 0, \qquad \mathfrak{a} > 0,$$

and the inequality is strict whenever $\mathfrak{c} > 0$. Thus $\Psi_{\alpha,\beta}$ is strictly decreasing in $\mathfrak{a}$. An identical argument applies to $\mathfrak{b}$.

This completes the proof. □

Proposition 3.2. *Fix $\alpha,\beta > 0$ and let $\mathfrak{a},\mathfrak{b},\mathfrak{c} \ge 0$. Then the following hold:*

(P1) (Upper bound by a distance-only gauge)

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \le \frac{\mathfrak{c}^\alpha}{1+\mathfrak{c}^\alpha}.$$

(P2) (Domination lower bound) If $\mathfrak{a}^\beta+\mathfrak{b}^\beta \le \theta\,\mathfrak{c}^\alpha$ for some $\theta \ge 0$, then

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \ge \frac{\mathfrak{c}^\alpha}{1+(1+\theta)\mathfrak{c}^\alpha}.$$

(P3) (Lipschitz-type sensitivity in the distance variable) For fixed $\mathfrak{a},\mathfrak{b}$, the function $\mathfrak{c} \mapsto \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ is continuous on $[0,\infty)$ and satisfies

$$\lim_{\mathfrak{c}\downarrow 0} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 0, \qquad \lim_{\mathfrak{c}\to\infty} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 1.$$

Proof. Fix $\alpha,\beta > 0$ and $\mathfrak{a},\mathfrak{b},\mathfrak{c} \ge 0$. Recall that

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}.$$

(P1). Since $\mathfrak{a}^\beta \ge 0$ and $\mathfrak{b}^\beta \ge 0$, we have

$$1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha \ge 1+\mathfrak{c}^\alpha.$$

Dividing the nonnegative numerator $\mathfrak{c}^\alpha$ by both sides yields

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} \le \frac{\mathfrak{c}^\alpha}{1+\mathfrak{c}^\alpha},$$

which proves (P1).

(P2). Assume $\mathfrak{a}^\beta+\mathfrak{b}^\beta \le \theta\,\mathfrak{c}^\alpha$ for some $\theta \ge 0$. Then

$$1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha \le 1+\theta\,\mathfrak{c}^\alpha+\mathfrak{c}^\alpha = 1+(1+\theta)\mathfrak{c}^\alpha.$$

Dividing $\mathfrak{c}^\alpha$ by both sides gives

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} \geq \frac{\mathfrak{c}^\alpha}{1+(1+\theta)\mathfrak{c}^\alpha},$$

which proves (P2).

(P3). Fix $\mathfrak{a},\mathfrak{b} \geq 0$ and consider the function

$$\varphi(\mathfrak{c}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}.$$

Since $t \mapsto t^\alpha$ is continuous on $[0,\infty)$ and all algebraic operations preserving continuity are involved, φ is continuous on $[0,\infty)$.

For the one-sided limits, note that

$$\lim_{\mathfrak{c}\downarrow 0} \mathfrak{c}^\alpha = 0, \qquad \lim_{\mathfrak{c}\to\infty} \mathfrak{c}^\alpha = +\infty.$$

Hence,

$$\lim_{\mathfrak{c}\downarrow 0} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{0}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+0} = 0,$$

and

$$\lim_{\mathfrak{c}\to\infty} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \lim_{\mathfrak{c}\to\infty} \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} = 1,$$

which completes the proof. □

Example 3.1. Each of the following mappings sends $[0,\infty)^3$ into $[0,1)$ and is a particular instance of the displacement–distance gauge in Definition 3.1 (possibly after a harmless regularization by a parameter $\varepsilon > 0$). For $\mathfrak{a},\mathfrak{b},\mathfrak{c} \geq 0$, define

$$\Psi_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}, \qquad \Psi_{2,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^2}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^2},$$

$$\Psi_{1,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}}, \qquad \Psi_{3,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^3}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}^3},$$

$$\Psi_{\frac{1}{2},1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^{1/2}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^{1/2}}, \qquad \Psi_{2,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^2}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}^2},$$

$$\Psi^{\varepsilon}_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{\varepsilon+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}} \quad (\varepsilon > 0), \qquad \Psi_{1,3}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}^3+\mathfrak{b}^3+\mathfrak{c}},$$

$$\Psi_{4,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^4}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^4}.$$

In particular, each displayed function is obtained by fixing (α,β) in Definition 3.1, except $\Psi^{\varepsilon}_{1,1}$ which replaces the constant 1 in the denominator by $\varepsilon > 0$ to allow additional scaling flexibility.

Example 3.2. Let $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (1,3,2)$. Then

$$\Psi_{1,1}(1,3,2) = \frac{2}{1+1+3+2} = \frac{2}{7}, \qquad \Psi_{1,2}(1,3,2) = \frac{2}{1+1^2+3^2+2} = \frac{2}{13},$$

which illustrates that increasing the displacement exponent β reduces the value of the gauge and hence weakens the contractive decrement for fixed $\mathfrak{c}$.

If $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (0,0,5)$, then

$$\Psi_{2,1}(0,0,5) = \frac{25}{1+0+0+25} = \frac{25}{26},$$

which is close to 1 and reflects a strong distance-driven contractive effect when the self-displacements vanish.

4 Main Results

We now establish a purely contractive fixed point theorem governed by the displacement–distance gauge $\Psi_{\alpha,\beta}$. The essential feature is that the nonlinear inequality (4.1) becomes a *uniform Banach contraction* once the natural domination of selfdisplacements by the interpoint distance is imposed through (4.2).

Theorem 4.1 ($\Psi_{\alpha,\beta}$-contractive fixed point theorem)**.** *Let $(\mathscr{X}, d)$ be a complete metric space and $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ a self-mapping. Fix $\alpha, \beta > 0$ and assume that there exists $\lambda \in (0, 1]$ such that for all $\xi, \eta \in \mathscr{X}$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \Big(1 - \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi), d(\eta, \mathscr{T}\eta), d(\xi, \eta)\big)\Big)\, d(\xi, \eta). \tag{4.1}$$

Assume moreover that there exists $\theta \geq 0$ such that

$$d(\xi, \mathscr{T}\xi)^\beta + d(\eta, \mathscr{T}\eta)^\beta \leq \theta\, d(\xi, \eta)^\alpha \qquad \forall\, \xi, \eta \in \mathscr{X}. \tag{4.2}$$

Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and, for every initial point $\xi_0 \in \mathscr{X}$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π.

Proof. Let $\xi, \eta \in \mathscr{X}$ with $\xi \neq \eta$. Set

$$\mathfrak{a} = d(\xi, \mathscr{T}\xi), \qquad \mathfrak{b} = d(\eta, \mathscr{T}\eta), \qquad \mathfrak{c} = d(\xi, \eta) > 0.$$

The domination hypothesis (4.2) yields

$$\mathfrak{a}^\beta + \mathfrak{b}^\beta \leq \theta\, \mathfrak{c}^\alpha.$$

By Definition 3.1,

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha}.$$

Since $\mathfrak{a}^\beta + \mathfrak{b}^\beta \leq \theta\mathfrak{c}^\alpha$, we obtain

$$1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha \leq 1 + \theta\mathfrak{c}^\alpha + \mathfrak{c}^\alpha = 1 + (1 + \theta)\mathfrak{c}^\alpha.$$

Dividing the positive quantity $\mathfrak{c}^\alpha$ by both sides gives

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \geq \frac{\mathfrak{c}^\alpha}{1 + (1 + \theta)\mathfrak{c}^\alpha},$$

which is precisely the estimate asserted in Proposition 3.2 (P2).

Substituting this bound into the contractive condition (4.1), we infer that

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \left(1 - \lambda \frac{\mathfrak{c}^\alpha}{1 + (1 + \theta)\mathfrak{c}^\alpha}\right) \mathfrak{c}.$$

After algebraic simplification, this can be rewritten as

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{1 + (\theta + 1 - \lambda)\mathfrak{c}^\alpha}{1 + (1 + \theta)\mathfrak{c}^\alpha}\, \mathfrak{c}.$$

Define the real-valued function

$$\varphi(t) = \frac{1 + (\theta + 1 - \lambda)t}{1 + (1 + \theta)t}, \qquad t \geq 0.$$

Then the above inequality takes the form

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \varphi(\mathfrak{c}^\alpha)\, d(\xi, \eta).$$

Observe that φ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. A direct computation shows that

$$\varphi'(t) = -\frac{\lambda}{(1 + (1 + \theta)t)^2} < 0 \qquad \forall\, t > 0,$$

hence φ is strictly decreasing. Moreover,

$$\varphi(0) = 1, \qquad \lim_{t \to \infty} \varphi(t) = \frac{\theta + 1 - \lambda}{\theta + 1} < 1,$$

because $\lambda > 0$ and $\theta \geq 0$.

Therefore, $\varphi(t) < 1$ for every $t > 0$. Since φ is continuous and decreasing on $(0, \infty)$, there exists a constant $q \in (0, 1)$, depending only on λ and θ, such that

$$\varphi(t) \leq q < 1 \qquad \forall\, t > 0.$$

Consequently,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq q\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X},\ \xi \neq \eta.$$

Thus $\mathscr{T}$ is a Banach contraction on the complete metric space $(\mathscr{X}, d)$. By Theorem 1.1, $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$, and for every initial point $\xi_0 \in \mathscr{X}$ the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π. $\square$

We show that the classical Banach, Reich, and Kannan fixed point theorems are contained as special cases of Theorem 4.1. We also deduce an orbitdominated version which is strictly more general than uniform contractions.

Corollary 4.1 (Banach contraction as a consequence). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le k\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X},$$

for some $k \in [0, 1)$. Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and, for every $\xi_0 \in \mathscr{X}$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π.

Proof. Fix $\alpha = \beta = 1$. Since $\Psi_{1,1} \in [0, 1)$, choose $\lambda \in (0, 1)$ such that $1 - \lambda \ge k$. Then for all $\xi, \eta \in \mathscr{X}$,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le k\, d(\xi, \eta) \le (1 - \lambda\Psi_{1,1}(\cdot))\, d(\xi, \eta),$$

so (4.1) holds.

Moreover,

$$d(\xi, \mathscr{T}\xi) \le d(\xi, \eta) + d(\eta, \mathscr{T}\eta) \le 2\, d(\xi, \eta),$$

and similarly for η; hence (4.2) holds with $\theta = 4$. Therefore Theorem 4.1 applies and yields the result. □

Corollary 4.2 (Reich-type consequence). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le a\, d(\xi, \eta) + b\, d(\xi, \mathscr{T}\xi) + c\, d(\eta, \mathscr{T}\eta),$$

with $a, b, c \ge 0$ and $a + b + c < 1$. Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and $\xi_{n+1} = \mathscr{T}\xi_n \to \pi$.

Proof. Since $a + b + c < 1$, one has

$$d(\xi, \mathscr{T}\xi) \le \frac{1}{1-c}\, d(\xi, \eta), \qquad d(\eta, \mathscr{T}\eta) \le \frac{1}{1-c}\, d(\xi, \eta).$$

Hence (4.2) holds with $\alpha = \beta = 1$ and $\theta = \frac{2}{1-c}$. Choosing $\lambda := 1 - (a + b + c)$ gives (4.1), so Theorem 4.1 applies. □

Corollary 4.3 (Kannan-type consequence). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le \kappa\big(d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta)\big),$$

with $\kappa \in \big[0, \frac{1}{2}\big)$. Then $\mathscr{T}$ has a unique fixed point π and $\xi_{n+1} = \mathscr{T}\xi_n \to \pi$.

Proof. From the inequality,

$$d(\xi, \mathscr{T}\xi) \le \frac{1}{1-\kappa}\, d(\xi, \eta), \qquad d(\eta, \mathscr{T}\eta) \le \frac{1}{1-\kappa}\, d(\xi, \eta),$$

so (4.2) holds with $\alpha = \beta = 1$ and $\theta = \frac{2}{1-\kappa}$. Taking $\lambda = 1 - 2\kappa$ yields (4.1), and the conclusion follows from Theorem 4.1. □

Theorem 4.2 (Orbitdominated $\Psi_{\alpha,\beta}$ contraction). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Fix $\alpha, \beta > 0$ and $\lambda \in (0, 1]$. Assume that for all $\xi, \eta \in \mathscr{X}$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \le \Big(1 - \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi), d(\eta, \mathscr{T}\eta), d(\xi, \eta)\big)\Big)\, d(\xi, \eta). \tag{4.3}$$

Let $\xi_0 \in \mathscr{X}$ and define the Picard orbit $\xi_{n+1} = \mathscr{T}\xi_n$. Suppose there exists $\theta \ge 0$ such that

$$d(\xi_n, \mathscr{T}\xi_n)^\beta + d(\xi_{n+1}, \mathscr{T}\xi_{n+1})^\beta \le \theta\, d(\xi_n, \xi_{n+1})^\alpha \qquad \forall\, n \ge 0. \tag{4.4}$$

Then $\{\xi_n\}$ converges to the unique fixed point π of $\mathscr{T}$.

Proof. Let $\xi_{n+1} = \mathscr{T}\xi_n$ and set $\tau_n = d(\xi_n, \xi_{n+1})$. Applying (4.3) to (ξ_n, ξ_{n+1}) gives

$$\tau_{n+1} \le (1 - \lambda\Psi_{\alpha,\beta}(\tau_n, \tau_{n+1}, \tau_n))\, \tau_n.$$

By (4.4) and Proposition 3.2(P2),

$$\Psi_{\alpha,\beta}(\tau_n, \tau_{n+1}, \tau_n) \ge \frac{\tau_n^\alpha}{1 + (1+\theta)\tau_n^\alpha}.$$

Hence

$$\tau_{n+1} \le \left(1 - \lambda \frac{\tau_n^\alpha}{1 + (1+\theta)\tau_n^\alpha}\right) \tau_n.$$

As in the proof of Theorem 4.1, this yields $\tau_{n+1} \le q\, \tau_n$ for some $q \in (0, 1)$. Thus $\tau_n \le q^n \tau_0$, and $\{\xi_n\}$ is Cauchy. Let $\pi = \lim \xi_n$.

Passing to the limit in $\xi_{n+1} = \mathscr{T}\xi_n$ gives $\mathscr{T}\pi = \pi$. Uniqueness follows as in Theorem 4.1. □

Theorem 4.3 (Orbitdominated $\Psi_{\alpha,\beta}$ contraction)**.** *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Fix $\alpha, \beta > 0$ and $\lambda \in (0, 1]$. Assume that* (4.3) *holds for all $\xi, \eta \in \mathscr{X}$.*

Let $\xi_{n+1} = \mathscr{T}\xi_n$. If there exists $\theta \geq 0$ such that (4.4) *holds for all n, then $\{\xi_n\}$ converges to the unique fixed point π of $\mathscr{T}$.*

Proof. Set $\tau_n = d(\xi_n, \xi_{n+1})$. From (4.3) and Proposition 3.2(P2),

$$\tau_{n+1} \leq \left(1 - \lambda \frac{\tau_n^\alpha}{1 + (1+\theta)\tau_n^\alpha}\right) \tau_n.$$

As in Theorem 4.1, this implies $\tau_{n+1} \leq q\tau_n$ for some $q \in (0, 1)$. Hence $\tau_n \to 0$, $\{\xi_n\}$ is Cauchy, and $\xi_n \to \pi$. Passing to the limit gives $\mathscr{T}\pi = \pi$, and uniqueness follows. $\square$

Remark 4.1. *The class of mappings covered by Theorem* 4.1 *is a* strict *superset of the classical Banach contraction class.*

Indeed, if $\mathscr{T}$ is a Banach contraction, that is,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \qquad (0 \leq k < 1),$$

then, for any choice of $\alpha, \beta > 0$ and for $\lambda := 1 - k$, one has

$$1 - \lambda\, \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \geq 1 - \lambda = k \qquad \forall\, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0,$$

because $\Psi_{\alpha,\beta} \in [0, 1)$. Consequently,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \leq \big(1 - \lambda\, \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})\big)\, d(\xi, \eta),$$

so every Banach contraction satisfies the $\Psi_{\alpha,\beta}$contractive inequality (4.1)*. Moreover, in this case the domination condition* (4.2) *is automatically satisfied on bounded subsets, in particular along the Picard orbit. Hence Banachs theorem appears as a special case of Theorem* 4.1*.*

The converse, however, is false: there exist mappings that satisfy the hypotheses of Theorem 4.1 *but are not Banach contractions.*

Example 4.1. Let

$$\mathscr{X} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}, \qquad d(\xi, \eta) = |\xi - \eta|.$$

Then $(\mathscr{X}, d)$ is complete since it is a closed subset of $\mathbb{R}$. Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 0, \qquad \mathscr{T}\left(\frac{1}{n}\right) = \frac{1}{n+1} \quad (n \in \mathbb{N}).$$

For $\xi_n = \frac{1}{n}$ and $\eta = 0$,

$$\frac{d(\mathscr{T}\xi_n, \mathscr{T}\eta)}{d(\xi_n, \eta)} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1,$$

and therefore

$$\sup_{\xi \neq \eta} \frac{d(\mathscr{T}\xi, \mathscr{T}\eta)}{d(\xi, \eta)} = 1.$$

Thus $\mathscr{T}$ is not a Banach contraction.

For $\xi = \frac{1}{n}$ and $\eta = \frac{1}{m}$ with $n < m$,

$$d(\xi, \mathscr{T}\xi) = \frac{1}{n(n+1)}, \qquad d(\eta, \mathscr{T}\eta) = \frac{1}{m(m+1)}, \qquad d(\xi, \eta) = \frac{m-n}{nm}.$$

Using $\frac{1}{n(n+1)} \leq \frac{1}{n^2}$ and $\frac{1}{m(m+1)} \leq \frac{1}{m^2} \leq \frac{1}{nm}$, one gets

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq \frac{1}{n^2} + \frac{1}{nm} = \frac{m+n}{n^2 m}.$$

Since $m - n \geq 1$, $d(\xi, \eta) \geq \frac{1}{nm}$ and hence

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq \frac{m+n}{n}\, d(\xi, \eta).$$

As $m + n \leq 2m$ and $m \leq 2n(m-n)$ for $m - n \geq 1$, it follows that

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq 4\, d(\xi, \eta).$$

Thus the domination condition (4.2) holds globally with $(\alpha, \beta, \theta) = (1, 1, 4)$.

Moreover,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{1}{n+1} - \frac{1}{m+1}\right| = \frac{m-n}{(n+1)(m+1)},$$

so

$$\frac{d(\mathscr{T}\xi, \mathscr{T}\eta)}{d(\xi, \eta)} = \frac{nm}{(n+1)(m+1)} \leq \frac{1}{2} \quad \text{whenever } m \geq 2n.$$

For the finitely many pairs with $m < 2n$, this ratio is still strictly less than 1, hence there exists a uniform constant $\rho \in (0, 1)$ such that

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \rho\, d(\xi, \eta) \qquad \forall\, \xi \neq \eta.$$

Fix $\lambda := 1 - \rho \in (0, 1)$. Since $\Psi_{1,1} \in [0, 1)$,

$$1 - \lambda\, \Psi_{1,1}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \geq 1 - \lambda = \rho,$$

and therefore

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \big(1 - \lambda\, \Psi_{1,1}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})\big)\, d(\xi, \eta),$$

so (4.1) is satisfied.

Consequently, all assumptions of Theorem 4.1 hold on $(\mathscr{X}, d)$ and $\mathscr{T}$ has the unique fixed point $\pi = 0$. Moreover, for every $\xi_0 \in \mathscr{X}$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to 0.

Therefore, Theorem 4.1 *constitutes a genuine and strict extension of the Banach contraction principle, providing fixed point existence and convergence for a substantially larger and more flexible class of mappings.*

Example 4.2. Let $\mathscr{X} = [0, 1]$ with the usual metric $d(\xi, \eta) = |\xi - \eta|$ and define

$$\mathscr{T}\xi = \frac{\xi}{1+\xi} \qquad (\xi \in [0, 1]).$$

Then $\mathscr{T}(\mathscr{X}) \subseteq \mathscr{X}$ and $(\mathscr{X}, d)$ is complete.

For $\xi \neq \eta$,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \left|\frac{\xi}{1+\xi} - \frac{\eta}{1+\eta}\right| = \left|\frac{\xi(1+\eta) - \eta(1+\xi)}{(1+\xi)(1+\eta)}\right| = \frac{|\xi - \eta|}{(1+\xi)(1+\eta)},$$

hence

$$\frac{d(\mathscr{T}\xi, \mathscr{T}\eta)}{d(\xi, \eta)} = \frac{1}{(1+\xi)(1+\eta)}.$$

Since $\frac{1}{(1+\xi)(1+\eta)} \to 1$ as $\xi, \eta \downarrow 0$, no constant $k < 1$ satisfies $d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta)$ for all $\xi, \eta \in [0, 1]$. Therefore $\mathscr{T}$ is not a Banach contraction.

Fix $\xi_0 \in [0, 1]$ and define the Picard orbit $\xi_{n+1} = \mathscr{T}\xi_n$. Then

$$\xi_{n+1} = \frac{\xi_n}{1+\xi_n} \leq \xi_n,$$

so $\{\xi_n\}$ is decreasing and bounded below by 0. Hence it converges to some $\pi \in [0, 1]$.

Moreover, for every n,

$$d(\xi_n, \mathscr{T}\xi_n) = |\xi_n - \xi_{n+1}| = \xi_n - \frac{\xi_n}{1+\xi_n} = \frac{\xi_n^2}{1+\xi_n}.$$

In particular,

$$d(\xi_n, \mathscr{T}\xi_n) = d(\xi_n, \xi_{n+1}) \qquad \forall\, n.$$

Let $\alpha, \beta > 0$ and take $(\alpha, \beta) = (1, 1)$. Then for all n,

$$d(\xi_n, \mathscr{T}\xi_n)^\beta + d(\xi_{n+1}, \mathscr{T}\xi_{n+1})^\beta = d(\xi_n, \xi_{n+1}) + d(\xi_{n+1}, \xi_{n+2}) \leq 2\, d(\xi_n, \xi_{n+1}),$$

so the orbit-domination condition (4.4) holds with $\theta = 2$.

Now we verify the Ψ-contractive inequality (4.3). Fix $\xi \neq \eta$ in $[0, 1]$ and set

$$\mathfrak{a} = d(\xi, \mathscr{T}\xi) = \frac{\xi^2}{1+\xi}, \qquad \mathfrak{b} = d(\eta, \mathscr{T}\eta) = \frac{\eta^2}{1+\eta}, \qquad \mathfrak{c} = d(\xi, \eta) = |\xi - \eta|.$$

Then
$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{\mathfrak{c}}{(1+\xi)(1+\eta)}.$$

Choose $\lambda = 1$. Since
$$\Psi(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}} \in [0,1),$$
we obtain
$$(1-\lambda\Psi(\mathfrak{a},\mathfrak{b},\mathfrak{c}))\,\mathfrak{c} = \frac{1+\mathfrak{a}+\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\,\mathfrak{c}.$$

Thus (4.3) holds once
$$\frac{1}{(1+\xi)(1+\eta)} \le \frac{1+\mathfrak{a}+\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}.$$
This is equivalent to
$$1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c} \le (1+\xi)(1+\eta)\,(1+\mathfrak{a}+\mathfrak{b}).$$
Since $(1+\xi)(1+\eta) \ge 1$ and $\mathfrak{c} = |\xi-\eta| \le \xi+\eta \le (1+\xi)(1+\eta)-1$, we obtain
$$1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c} \le (1+\xi)(1+\eta)+\mathfrak{a}+\mathfrak{b} \le (1+\xi)(1+\eta)(1+\mathfrak{a}+\mathfrak{b}),$$
which proves (4.3).

Therefore all hypotheses of Theorem 4.3 are satisfied. Hence $\mathscr{T}$ has a unique fixed point $\pi \in [0,1]$ and $\xi_n \to \pi$ for every $\xi_0 \in [0,1]$. Finally,
$$\mathscr{T}\pi = \pi \quad \Longrightarrow \quad \pi = \frac{\pi}{1+\pi},$$
so $\pi = 0$.

Example 4.3. Let $\mathscr{X} = C([0,1],\mathbb{R})$ endowed with the supremum metric $d(u,v) = \|u-v\|_\infty$. Then $(\mathscr{X}, d)$ is complete.

Fix $\varphi \in C([0,1],\mathbb{R})$ and define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by
$$(\mathscr{T}u)(t) = \frac{1}{2}u(t) + \varphi(t), \qquad t \in [0,1].$$
For any $u, v \in \mathscr{X}$,
$$d(\mathscr{T}u, \mathscr{T}v) = \left\| \frac{1}{2}(u-v) \right\|_\infty = \frac{1}{2}\,d(u,v).$$
Thus $\mathscr{T}$ is a Banach contraction with $k = \frac{1}{2}$ and therefore has a unique fixed point, which is the unique solution of $u = \frac{1}{2}u + \varphi$, namely $u^* = 2\varphi$.

We now verify compatibility with (4.1). Fix any $\alpha, \beta > 0$. Since $\Psi_{\alpha,\beta}(\cdot) \in [0,1)$, we have $1 - \lambda\Psi_{\alpha,\beta}(\cdot) \ge 1-\lambda$ for every $\lambda \in (0,1]$. Choose $\lambda \in (0, \frac{1}{2}]$. Then $1-\lambda \ge \frac{1}{2}$ and hence
$$d(\mathscr{T}u, \mathscr{T}v) = \frac{1}{2}d(u,v) \le (1-\lambda)\,d(u,v) \le \big(1 - \lambda\Psi_{\alpha,\beta}(d(u,\mathscr{T}u), d(v,\mathscr{T}v), d(u,v))\big)d(u,v),$$
so (4.1) holds. The domination condition (4.2) can be enforced by restricting to bounded subsets where $\|u - \mathscr{T}u\|_\infty$ is controlled by $\|u-v\|_\infty$, or by simply invoking Theorem 4.1 with any θ that dominates the displacements on the considered range. In any case, Theorem 4.1 confirms the fixed point $u^* = 2\varphi$ and Picard convergence.

Example 4.4. Let $\mathscr{X} = C([0,1],\mathbb{R})$ with $d(u,v) = \|u-v\|_\infty$. Fix a continuous kernel $K : [0,1]\times[0,1] \to \mathbb{R}$ and assume
$$M := \sup_{t\in[0,1]} \int_0^1 |K(t,s)|\,ds < 1.$$
Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by
$$(\mathscr{T}u)(t) = \int_0^1 K(t,s)\,u(s)\,ds + g(t),$$
where $g \in C([0,1],\mathbb{R})$ is fixed. Then for any u, v,
$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \le \int_0^1 |K(t,s)|\,|u(s)-v(s)|\,ds \le \|u-v\|_\infty \int_0^1 |K(t,s)|\,ds,$$

so taking the supremum over t yields

$$d(\mathscr{T}u, \mathscr{T}v) \le M\, d(u,v).$$

Thus $\mathscr{T}$ is a Banach contraction. The fixed point is the unique solution of the integral equation $u = \mathscr{T}u$.

Moreover, for any $\alpha, \beta > 0$, choose $\lambda \in (0,1]$ with $1-\lambda \ge M$. Since $\Psi_{\alpha,\beta}(\cdot) \in [0,1)$, we have

$$1 - \lambda\Psi_{\alpha,\beta}(\cdot) \ge 1-\lambda \ge M,$$

and therefore

$$d(\mathscr{T}u, \mathscr{T}v) \le M\, d(u,v) \le \big(1 - \lambda\Psi_{\alpha,\beta}(d(u,\mathscr{T}u), d(v,\mathscr{T}v), d(u,v))\big)\, d(u,v),$$

so (4.1) holds. Under standard boundedness constraints on the orbit, one can also ensure (4.2). Hence Theorem 4.1 applies and produces the same unique solution and convergence of successive approximations.

Example 4.5. Let $\mathscr{X} = \{p,q,r\}$ and define a metric d by

$$d(p,q) = 1, \quad d(p,r) = 2, \quad d(q,r) = 2, \qquad d(\xi,\xi) = 0,$$

extended symmetrically. Then $(\mathscr{X}, d)$ is complete.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(p) = p, \qquad \mathscr{T}(q) = p, \qquad \mathscr{T}(r) = q.$$

Compute the displacements:

$$d(p,\mathscr{T}p) = 0, \qquad d(q,\mathscr{T}q) = d(q,p) = 1, \qquad d(r,\mathscr{T}r) = d(r,q) = 2.$$

We verify (4.2) for $(\alpha,\beta) = (1,1)$. For each unordered pair $\{\xi,\eta\}$:

$$\begin{aligned}
\{\xi,\eta\} = \{p,q\}: &\quad d(p,\mathscr{T}p) + d(q,\mathscr{T}q) = 0 + 1 \le 1 \cdot d(p,q),\\
\{\xi,\eta\} = \{p,r\}: &\quad d(p,\mathscr{T}p) + d(r,\mathscr{T}r) = 0 + 2 \le 1 \cdot d(p,r),\\
\{\xi,\eta\} = \{q,r\}: &\quad d(q,\mathscr{T}q) + d(r,\mathscr{T}r) = 1 + 2 \le 2 \cdot d(q,r).
\end{aligned}$$

Thus (4.2) holds with $\theta = 2$.

Now verify (4.1) for $(\alpha,\beta,\lambda) = (1,1,1)$. For each pair (ξ,η):

$$\begin{aligned}
(\xi,\eta) = (p,q): &\quad d(\mathscr{T}p,\mathscr{T}q) = d(p,p) = 0 \le (1 - \Psi(0,1,1)) \cdot 1,\\
(\xi,\eta) = (p,r): &\quad d(\mathscr{T}p,\mathscr{T}r) = d(p,q) = 1 \le (1 - \Psi(0,2,2)) \cdot 2,\\
(\xi,\eta) = (q,r): &\quad d(\mathscr{T}q,\mathscr{T}r) = d(p,q) = 1 \le (1 - \Psi(1,2,2)) \cdot 2.
\end{aligned}$$

Each inequality is readily checked by direct substitution of $\Psi(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}$. Hence all hypotheses of Theorem 4.1 are satisfied, so $\mathscr{T}$ has a unique fixed point p and every Picard iteration reaches p in finitely many steps.

5 Application to a Nonlinear Integral Equation

Let $\mathscr{I} = [0,1]$ and $\mathscr{X} = C(\mathscr{I},\mathbb{R})$ equipped with the metric $d(u,v) = \|u-v\|_\infty$. Then $(\mathscr{X}, d)$ is a complete metric space.

Consider the nonlinear integral equation

$$v(t) = g(t) + \int_0^1 K(t,s)\, F(s, v(s))\, ds, \qquad t \in \mathscr{I}, \tag{5.1}$$

where $g \in C(\mathscr{I},\mathbb{R})$, the kernel $K : \mathscr{I} \times \mathscr{I} \to \mathbb{R}$ is continuous, and $F : \mathscr{I} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Define the operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}u)(t) = g(t) + \int_0^1 K(t,s)\, F(s,u(s))\, ds.$$

Assume that there exists $\mathscr{L} > 0$ such that

$$|F(s,r) - F(s,q)| \le \mathscr{L}|r-q| \qquad \forall\, s \in \mathscr{I},\ \forall\, r,q \in \mathbb{R}, \tag{5.2}$$

and assume further that

$$\sup_{t\in\mathscr{I}} \int_0^1 |K(t,s)|\, ds \le \mathscr{K} \tag{5.3}$$

for some $\mathscr{K} > 0$ satisfying $\mathscr{K}\mathscr{L} < 1$.

Theorem 5.1. *Under* (5.2)–(5.3) *with* $\mathscr{K}\mathscr{L} < 1$, *the integral equation* (5.1) *has a unique solution* $v^* \in \mathscr{X}$. *Moreover, for every initial function* $u_0 \in \mathscr{X}$, *the Picard iteration* $u_{n+1} = \mathscr{T}u_n$ *converges uniformly to* v^*.

Proof. Fix $u, v \in \mathscr{X}$ and $t \in \mathscr{I}$. By (5.2),

$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \le \int_0^1 |K(t,s)|\,|F(s,u(s)) - F(s,v(s))|\,ds \le \mathscr{L}\int_0^1 |K(t,s)|\,|u(s) - v(s)|\,ds.$$

Using (5.3) and taking the supremum over t yields

$$d(\mathscr{T}u, \mathscr{T}v) \le \mathscr{K}\mathscr{L}\,d(u,v).$$

Since $\mathscr{K}\mathscr{L} < 1$, $\mathscr{T}$ is a Banach contraction, and hence by Theorem 1.1 it has a unique fixed point v^*, which is precisely the unique solution of (5.1), and the Picard iteration converges uniformly.

We now reinterpret this conclusion within the framework of Theorem 4.1. Fix arbitrary $\alpha, \beta > 0$ and set

$$\lambda := 1 - \mathscr{K}\mathscr{L} \in (0,1).$$

Since $\Psi_{\alpha,\beta}(\cdot) \in [0,1)$, we have

$$1 - \lambda\,\Psi_{\alpha,\beta}(\cdot) \ge 1 - \lambda = \mathscr{K}\mathscr{L}.$$

Consequently,

$$d(\mathscr{T}u, \mathscr{T}v) \le \mathscr{K}\mathscr{L}\,d(u,v) \le \big(1 - \lambda\,\Psi_{\alpha,\beta}(d(u,\mathscr{T}u), d(v,\mathscr{T}v), d(u,v))\big)\,d(u,v),$$

which is exactly of the form (4.1).

Finally, on bounded subsets of $\mathscr{X}$ (in particular, along the Picard orbit), the domination condition (4.2) is satisfied with some $\theta \ge 0$, because the displacements $d(u, \mathscr{T}u)$ are uniformly bounded by the contraction estimate. Hence all assumptions of Theorem 4.1 hold, and the unique solution v^* follows as a direct consequence. □

6 Conclusion

In this paper we introduced a new displacement–distance control function $\Psi_{\alpha,\beta}$ that nonlinearly couples the interpoint distance $d(\xi,\eta)$ with the self–displacements $d(\xi,\mathscr{T}\xi)$ and $d(\eta,\mathscr{T}\eta)$. Unlike classical linear or implicit contractive coefficients, this rational gauge adapts dynamically to the geometric position of points and to their deviation from fixedness.

Under a natural domination hypothesis that links self–displacements to interpoint distances, the $\Psi_{\alpha,\beta}$–contractive condition is shown to induce a uniform Banach contraction. As a consequence, existence, uniqueness, and global Picard convergence of fixed points are obtained in complete metric spaces. Several detailed corollaries demonstrate that the celebrated principles of Banach, Reich, and Kannan are recovered as special or limiting cases within the present framework.

The applicability of the new control mechanism is further illustrated by a nonlinear integral equation on a space of continuous functions, where the associated integral operator satisfies the $\Psi_{\alpha,\beta}$–contractive condition. This confirms that the displacement–distance gauge is effective in treating nonlinear operator equations that lie beyond standard Lipschitz regimes.

The proposed control function opens several directions for future research, including extensions to multivalued and cyclic mappings, ordered and generalized metric structures, and dynamical systems with state–dependent perturbations. We expect that the displacement–distance paradigm will provide a flexible and unifying tool for developing further generalizations in fixed point theory and its applications.

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COEFFICIENT INEQUALITIES FOR A CERTAIN CLASS OF CLOSE-TO-STAR FUNCTIONS DEFINED BY SUBORDINATION

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Abstract

A class of close-to-star functions $CS_\phi = \left\{ f \in \mathscr{A} : (1-z)^2 \frac{f(z)}{z} \prec \phi(z) \right\}$ defined by subordination is presented in this study. For the class CS_ϕ, coefficient estimates, bounds for the second and third order Hermitian Toeplitz determinants, and Fekete-Szegö inequality is established. Further, Hermitian Toeplitz determinants and Fekete-Szegö inequality are discussed for specific cases of ϕ.
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1 Introduction

Let $\mathscr{A}$ represent the class of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are regular on the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, where S is a subclass of $\mathscr{A}$ which are univalent functions.

Let the class P of Caratheodory functions which are analytic with $p(0) = 1$ and $\text{Re}(p(z)) > 0$ for $z \in \mathbb{D}$ be of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots . \tag{1.2}$$

For two functions $f, g \in \mathbb{D}$, f is said to be subordinate to g, expressed by $f \prec g$, if there exist a function $\omega(z)$ with $|\omega(z)| < 1,\ z \in \mathbb{D}$ and $\omega(0) = 0$ such that $f(z) = g(\omega(z)), \forall\ z \in \mathbb{D}$. If g is univalent in $\mathbb{D}$, the following equivalency hold:

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{D}$ with $\phi(0) = 1,\ \phi'(0) > 0$ which maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Then, for $z \in \mathbb{D},\ \phi(z)$ can be written as

$$\phi(z) = 1 + D_1 z + D_2 z^2 + \cdots ,\ D_1 > 0. \tag{1.3}$$

We note that all D_i's are real, as $\phi(\mathbb{D})$ is symmetric about the real axis and $\phi(0) = 1$. Furthermore, as ϕ is a Caratheodory function, it follows that $|D_n| \leq 2,\ n \in \mathbb{N}$ [5].

In 1992, Ma and Minda [19] defined the classes

$$S^*_\phi = \left\{ f \in A\ :\ \frac{zf'(z)}{f(z)} \prec \phi(z)\ (z \in \mathbb{D}) \right\}.$$

If $\phi(z) = \frac{1+z}{1-z}$, then the class S^*_ϕ reduces to the class of starlike functions S^*, consists of functions $f \in \mathscr{A}$ which meet the criteria

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}.$$

A function $f \in \mathscr{A}$ is close-to-star if there exists $g \in S^*$ and $\beta \in \mathbb{R}$ such that

$$Re\left(\frac{e^{i\beta} f(z)}{g(z)}\right) > 0,\ z \in \mathbb{D}.$$

Reade [23] introduced the class CST of all close-to-star functions and deduced that the class of close-to-star functions has the same relationship to it as the class of starlike functions has to the class of convex functions. It is observed that [12] $f \in CST$ is close to convex if and only if a function $F(z) = \int_0^z \frac{f(t)}{t} dt, z \in \mathbb{D}$. The geometric characteristics and coefficient inequalities for the class of strongly close-to-star functions were computed by Park and Lee [21]. Dziok [4] examined the generalised problem of starlikeness for the product of close-to-star functions, whereas Arif *et al.* [2] provided the product of multivalent close-to-star functions. A growth and distortion theorem, together with coefficient constraints, was recently derived for certain subclasses of close-to-star functions by Selvaraj and Logu [24]. Further, the radius of starlikeness for specific close-to-star functions was computed by Kanaga and Ravichandran [13].

Jastrzebski *et al.* [11] examined a subclass of CST in 2020 for the choice of $\beta = 0$ and $g(z) = \frac{z}{(1-z)2}$ as $ST_1 = \left\{ f \in \mathscr{A} / \frac{(1-z)^2 f(z)}{z} \geq 0 \right\}$ and established lower and upper bounds for Hermitian-Toeplitz determinants of second and third order.

Motivated by the above works, we define the class CS_ϕ as below.

Definition 1.1. *Let CS_ϕ denote the subclass of functions $f \in \mathscr{A}$ which satisfies*

$$(1-z)^2 \frac{f(z)}{z} \prec \phi(z) \quad , z \in \mathbb{D},$$

where $\phi(z)$ is as given by (1.3).

If $\phi(z) = \frac{1+z}{1-z}$, then the class CS_ϕ reduces to the class ST_1 which was studied by the authors [11]. For different choices of $\phi(z)$, the class CS_ϕ reduces to the classes CS_{ϕ_1}, CS_{ϕ_2} and CS_{ϕ_3} respectively, which are defined as below.

Definition 1.2. *Let CS_{ϕ_1} denote the subclass of functions $f \in \mathscr{A}$ which satisfies*

$$(1-z)^2 \frac{f(z)}{z} \prec 1 + \ sin\ z = \phi_1(z) \quad , z \in \mathbb{D}.$$

Definition 1.3. *Let CS_{ϕ_2} denote the subclass of functions $f \in \mathscr{A}$ which satisfies*

$$(1-z)^2 \frac{f(z)}{z} \prec e^z \ = \phi_2(z)\ , z \in \mathbb{D}.$$

Definition 1.4. *Let CS_{ϕ_3} denote the subclass of functions $f \in \mathscr{A}$ which satisfies*

$$(1-z^2) \frac{f(z)}{z} \prec (1 + \ s\ z)^2 \ = \phi_3(z)\ , z \in \mathbb{D}, 0 < s \leq \frac{1}{\sqrt{2}}.$$

To prove that the classes CS_{ϕ_1}, CS_{ϕ_2} and CS_{ϕ_3} are non-empty, we have the following example.

Example 1.1. Let $f : \mathbb{D} \to \mathbb{C}$ be defined by

$$f(z) = \frac{z + iaz^2}{(1-z)^2(1+ibz)},\ a \in \mathbb{C},$$

where,

$0 \leq |a| \leq 0.85,\ 0 \leq |b| \leq 0.5,\ if\ f \in CS_{\phi_1}$,
$0 \leq |a| \leq 0.65,\ 0 \leq |b| \leq 0.45,\ if\ f \in CS_{\phi_2}$ and
$0 \leq |a| \leq 0.9s,\ 0 \leq |b| \leq 0.1s,\ 0 < s \leq \frac{1}{\sqrt{2}},\ if\ f \in CS_{\phi_3}$.

Therefore, for $z \in \mathbb{D}$,

$$(1-z^2)\frac{f(z)}{z} = \frac{1+iaz}{1+ibz} := g_{(a,b)}(z).$$

Using *MATLAB* R2021a, we obtain the images of $\phi_i(\partial\mathbb{D}),\ i = 1,2,3$ and $g_{(a,b)}(\partial\mathbb{D})$ for different choices of $'a'$ and $'b'$ in the range specified above correspondingly for CS_{ϕ_1}, CS_{ϕ_2} and CS_{ϕ_3} are as given in Figure 1.1, 1.2 and 1.3 respectively.

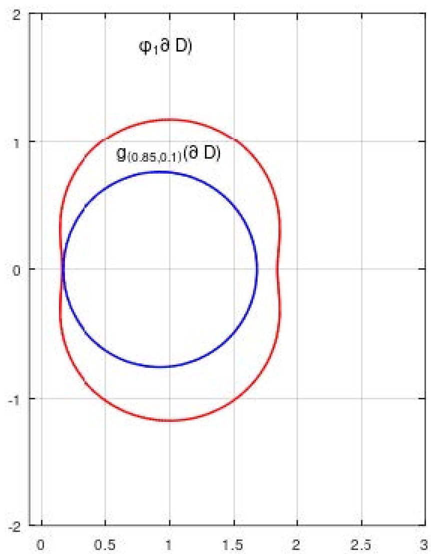

***Figure* 1.1**:
$a = 0.85$, $b = 0.1$

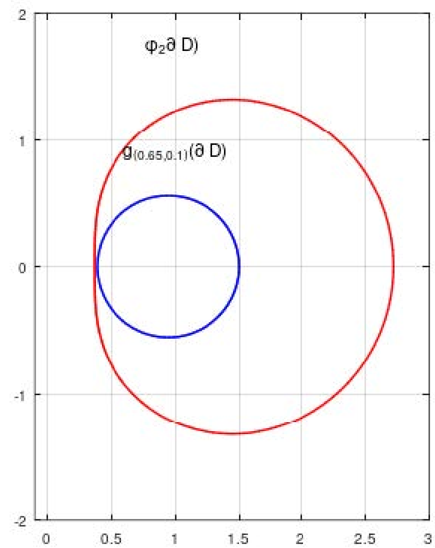

***Figure* 1.2**:
$a = 0.65$, $b = 0.1$

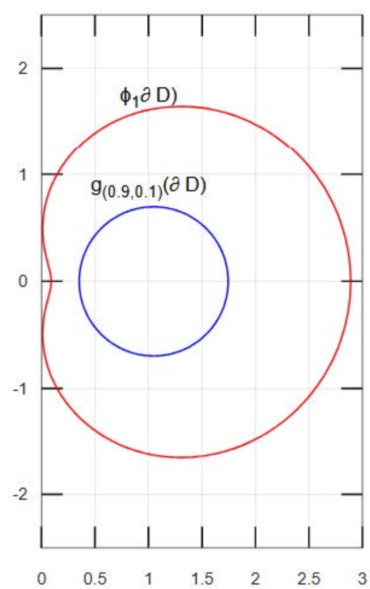

***Figure* 1.3**:
$a = 0.63$, $b = 0.07$

Hartman and Winter [10] started the study of Hermitian-Toeplitz determinant in 1954. It is defined as the coefficients $\{a_k\}_{k\geq 2}$ of functions $f \in \mathscr{A}$ of the form (1.1).

$$\det(T_{m,n}(f)) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+m-1} \\ \overline{a_{n+1}} & a_n & \dots & a_{n+m-2} \\ \vdots & \vdots & \dots & \vdots \\ \overline{a_{n+m-1}} & \overline{a_{n+m-2}} & \dots & a_n \end{vmatrix}. \tag{1.4}$$

In particular,

$$\det(T_{2,1}(f)) = \begin{vmatrix} 1 & a_2 \\ \overline{a_2} & 1 \end{vmatrix} = 1 - |a_2|^2$$

and

$$\det(T_{3,1}(f)) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a_2} & 1 & a_3 \\ \overline{a_3} & \overline{a_2} & 1 \end{vmatrix} = 2Re(a_2^2\overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1.$$

Several uses in the field of signal processing include Hermitian-Toeplitz matrices and their determinants [22, 25]. The sharp lower and upper bounds for second and third-order Hermitian-Toeplitz determinants for the classes of convex and starlike functions were calculated by Cudna *et al.* [3]. The authors in [20] established the sharp bounds for the third-order Hermitian-Toeplitz determinant for the classes of univalent functions. The sharp lower and upper bounds for Hermitian-Toeplitz determinants of order two and three for the classes of Janowski starlike and convex functions were estimated by Kumar *et al.* [14]. Second and third-order Hermitian-Toeplitz determinants for the classes of starlike functions subordinate to $\phi(z)$ were recently determined by Surya Giri and Sivaprasad Kumar [7]. Moreover, other scholars, including [9, 15, 16, 17, 30] have established the bounds for Hermitian-Toeplitz determinants of order two and three for various subclasses of univalent functions. Recently, the authors in [29] have studied the Hankel and Hermitian-Toeplitz determinants for a certain Subclass of analytic functions involving sine domain.

In 1933, Fekete and Szegö [6] obtained the upper bound for the functional $|a_3 - \mu a_2^2|$, where μ is real is often referred as Fekete-Szegö inequality. Numerous authors including [1, 8, 18, 26, 27, 28, 31] have also examined the Fekete-Szegö inequality for various subclasses of $\mathscr{A}$.

The aim of this study is to find the bounds for Fekete-Szegö inequality and Hermitian-Toeplitz determinants of second and third order for $f \in \mathscr{A}$ that belong to the class CS_ϕ.

In this sequel to prove main results, we need following lemmas.

Lemma 1.1 ([11])**.** *If p of the form* (1.2) *is in P, then*

$$|p_n| \leq 2, \quad n \in \mathbb{N}. \tag{1.5}$$

Moreover

$$p_1 = 2c_1, \tag{1.6}$$

$$p_2 = 2c_1^2 + 2(1 - |c_1|^2)c_2 \tag{1.7}$$

for some $c_1, c_2 \in \bar{\mathbb{D}}$.

Lemma 1.2 ([19])**.** *If p of the form* (1.2) *is in P, then*

$$|p_2 - \mu p_1^2| \leq \begin{cases} -4\mu + 2 & \mu \leq 0, \\ 2 & 0 \leq \mu \leq 1, \\ 4\mu - 2 & \mu \geq 1. \end{cases} \tag{1.8}$$

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $p_1(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then the equality holds if and only if $p_1(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\mu = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\left(\frac{1-z}{1+z}\right), \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $\mu = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\mu = 0$.

2 Coefficient Estimates

In this section, the bounds for the second and third coefficients of the function f in CS_ϕ are estimated.

Theorem 2.1. *If f of the form (1.1) is in CS_ϕ, then*
(i) $|a_2| \le 2 + D_1$ and
(ii) $|a_3| \le \sigma$,
where $\sigma = 3 + 2D_1 + |D_2|$.
The result is sharp.

Proof. Let the function $f \in CS_\phi$. Then for some $p \in P$, we have

$$(1-z)^2\frac{f(z)}{z} = \phi\left(\frac{p(z)-1}{p(z)+1}\right).$$

That is,

$$1 + (a_2 - 2)z + (a_3 + 1 - 2a_2)z^2 + \cdots = 1 + \frac{D_1 p_1}{2}z + \left(\frac{D_1(2p_2 - p_1^2)}{4} + \frac{D_2 p_1^2}{4}\right)z^2 + \cdots$$

By comparing the coefficients

$$a_2 = 2 + \frac{D_1 p_1}{2} \tag{2.1}$$

and

$$a_3 = 3 + D_1 p_1 + \left(\frac{D_1(2p_2 - p_1^2)}{4} + \frac{D_2 p_1^2}{4}\right). \tag{2.2}$$

Using Lemma 1.1 in (2.1) we have

$$|a_2| \le 2 + D_1. \tag{2.3}$$

Applying Lemma 1.1 and triangle inequality in (2.2), we have

$|a_3| \le 3 + 2D_1|c_1| + D_1(1 - |c_1|^2)|c_2| + |D_2||c_1|^2 = F(|c_1|, |c_2|)$.

Let $|c_1| = r$ and $|c_2| = s$, where $r, s \in [0, 1]$. Then

$F(r, s) = 3 + D_1(2r + (1 - r^2)s) + |D_2|r^2$.

By the second derivative test, we see that there are no critical points in the domain $(0, 1) \times (0, 1)$.
Thus,

$F(0, s) = 3 + sD_1 \le 3 + D_1$,

$F(1, s) = \sigma, \qquad$ where $\sigma = 3 + 2D_1 + |D_2|$,

$F(r, 0) = 3 + 2D_1 r + |D_2|r^2 \le \sigma$,

$F(r, 1) = 3 + D_1(2r + 1 - r^2) + |D_2|r^2 \le \sigma$.

We observe that

$$|a_3| \le max_{(r,s)\in[0,1]\times[0,1]}F(r, s) = \sigma, \tag{2.4}$$

which completes the proof.
The upper bound is sharp for $f_1(z)$, which is given by

$$f_1(z) = \frac{z\phi(z)}{(1-z)^2} = z + (2 + D_1)z + (3 + 2D_1 + D_2)z^2 + ... \tag{2.5}$$

□

Corollary 2.1. *(i) If f of the form (1.1) is in CS_{ϕ_1}, then $|a_2| \le 3$ and $|a_3| \le 5$.*
(ii) If f of the form (1.1) is in CS_{ϕ_2}, then $|a_2| \le 3$ and $|a_3| \le \frac{11}{2}$.
(iii)If f of the form (1.1) is in CS_{ϕ_3}, then, for $0 < s \le \frac{1}{\sqrt{2}}$, $|a_2| \le 2 + 2s$ and $|a_3| \le 3 + 2s + s^2$.

3 Hermitian Toeplitz Determinants

In this section, bounds for second and third order Hermitian Toeplitz determinants for the function f in CS_ϕ are estimated.

Theorem 3.1. *If f of the form (1.1) is in CS_ϕ, then $1-|2+D_1|^2 \le det(T_{(2,1)})(f) \le 1$. The result is sharp.*

Proof. Consider $det(T_{(2,1)}(f)) = 1-|a_2|^2$.
From (2.1),

$$1-|a_2|^2 = 1-\left|2+\frac{D_1p_1}{2}\right|^2 \le 1.$$

Now from (2.3), we have

$$1-|a_2|^2 \ge 1-|2+D_1|^2.$$

Thus

$$1-|2+D_1|^2 \le det(T_{(2,1)})(f) \le 1.$$

Hence the Theorem.
The upper bound is sharp if $f_0(z)=z$ and the lower bound is sharp for $f_1(z)$ given by (2.5). □

Corollary 3.1. *(i) If f of the form (1.1) is in CS_{ϕ_1}, then $-8 \le det(T_{(2,1)})(f) \le 1$.*
(ii) If f of the form (1.1) is in CS_{ϕ_2}, then $-8 \le det(T_{(2,1)})(f) \le 1$.
(iii)If f of the form (1.1) is in CS_{ϕ_3}, then, for $0<s\le \frac{1}{\sqrt{2}}$, $1-(2+2s)^2 \le det(T_{(2,1)})(f) \le 1$.

Theorem 3.2. *If f of the form (1.1) is in CS_ϕ, then*

$$det\ (T_{(3,1)})(f) \le max\{1, 2(2+D_1)^2(\sigma-1)-\sigma^2+1\},$$

where $\sigma = 3+2D_1+|D_2|$.
The result is sharp.

Proof. Consider

$$\begin{aligned} det\ (T_{(3,1)})(f) &= 2Re(a_2^2\overline{a_3}) - 2|a_2|^2 - |a_3|^2 + 1, \\ &\le 2|a_2|^2|a_3| - 2|a_2|^2 - |a_3|^2 + 1 = G(|a_2|, |a_3|). \end{aligned}$$

Letting $|a_2| = x$ and $|a_3| = y$, $x \in [0, 2+D_1]$, $y \in [0,\sigma]$, we have

$$G(|a_2|, |a_3|) = 2xy - 2x^2 - y^2 + 1 = G(x, y).$$

By the second derivative test, we see that there are no critical points in $(0, 2+D_1)\times(0,\sigma)$.
Now on $[0, 2+D_1]\times[0,\sigma]$, we have

$G(0, y) = 1-y^2 \le 1$,

$G(2+D_1, y) = 2(2+D_1)^2y - 2(2+D_1)^2 - y^2 + 1 \le 2(2+D_1)^2(\sigma-1) - \sigma^2 + 1$,

where $\sigma = 3+2D_1+|D_2|$,

$G(x, 0) = 1-2x^2 \le 1$,

$G(x,\sigma) = 2x^2\sigma - 2x^2 - \sigma^2 + 1 \le 2(2+D_1)^2(\sigma-1) - \sigma^2 + 1$.

We observe that

$$det\ (T_{(3,1)})(f) \le max_{(x,y)\in[0,2+D_1]\times[0,\sigma]} G(x,y) \le max\{1, 2(2+D_1)^2(\sigma-1)-\sigma^2+1\},$$

which completes the theorem.
The upper bound is sharp for $f_1(z)$ given by (2.5). □

Corollary 3.2. *(i) If f of the form (1.1) is in CS_{ϕ_1}, then $det\ (T_{(3,1)})(f) \le 48$.*
(ii) If f of the form (1.1) is in CS_{ϕ_2}, then $det\ (T_{(3,1)})(f) \le \frac{207}{4}$.
(iii) If f of the form (1.1) is in CS_{ϕ_3}, then, for $0<s\le\frac{1}{\sqrt{2}}$,
$det(T_{(3,1)})(f) \le 7s^4 + 28s^3 + 46s^2 + 36s + 8$.

4 Fekete-Szegö Inequality

In this section, Fekete-Szegö inequality for the function f in CS_ϕ is obtained.

Theorem 4.1. *If f of the form (1.1) is in CS_ϕ, then, for $\lambda \in \mathbb{R}$,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} K(\lambda) + 2D_1\left(\frac{D_2}{D_1} - \lambda D_1\right) & \lambda D_1^2 - D_2 \leq -D_1, \\ K(\lambda) + 2D_1 & -D_1 \leq \lambda D_1^2 - D_2 \leq D_1, \\ K(\lambda) + 2D_1\left(\lambda D_1 - \frac{D_2}{D_1}\right) & D_1 \leq \lambda D_1^2 - D_2, \end{cases}$$

where $K(\lambda) = |3 - 4\lambda| + 2D_1|1 - 2\lambda|$.
The result is sharp.

Proof. From (2.1) and (2.2), we have

$$|a_3 - \lambda a_2^2| = \left|(3 - 4\lambda) + D_1 p_1(1 - 2\lambda) + \frac{D_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + D_2\frac{p_1^2}{4} - \lambda\frac{D_1^2 p_1^2}{4}\right|.$$

By applying the triangle inequality and after taking rearrangements, we get

$$|a_3 - \lambda a_2^2| \leq |(3 - 4\lambda)| + D_1|p_1||1 - 2\lambda| + \frac{D_1}{2}\left|p_2 - \left(\frac{1}{2} - \frac{D_2}{2D_1} + \frac{\lambda D_1}{2}\right)p_1^2\right|.$$

In virtue of Lemma 1.1 and Lemma 1.2,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} K(\lambda) + 2D_1\left(\frac{D_2}{D_1} - \lambda D_1\right) & \lambda D_1^2 - D_2 \leq -D_1, \\ K(\lambda) + 2D_1 & -D_1 \leq \lambda D_1^2 - D_2 \leq D_1, \\ K(\lambda) + 2D_1\left(\lambda D_1 - \frac{D_2}{D_1}\right) & D_1 \leq \lambda D_1^2 - D_2, \end{cases}$$

where $K(\lambda) = |3 - 4\lambda| + 2D_1|1 - 2\lambda|$.
The upper bound is sharp for $f_1(z)$ given by (2.5).
Hence the proof. $\square$

Corollary 4.1. *(i) If f of the form (1.1) is in CS_{ϕ_1}, then,*

$$|a_3 - \lambda a_2^2| \leq \begin{cases} |3 - 4\lambda| + 2|1 - 2\lambda| - 2\lambda & \lambda \leq -1, \\ |3 - 4\lambda| + 2|1 - 2\lambda| + 2 & -1 \leq \lambda \leq 1, \\ |3 - 4\lambda| + 2|1 - 2\lambda| + 2\lambda & 1 \leq \lambda. \end{cases}$$

(ii) If f of the form (1.1) is in CS_{ϕ_2}, then,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} |3 - 4\lambda| + 2|1 - 2\lambda| - 2(\lambda - \frac{1}{2}) & \lambda \leq \frac{-1}{2}, \\ |3 - 4\lambda| + 2|1 - 2\lambda| + 2 & \frac{-1}{2} \leq \lambda \leq \frac{3}{2}, \\ |3 - 4\lambda| + 2|1 - 2\lambda| + 2(\frac{1}{2} + \lambda) & \frac{3}{2} \leq \lambda. \end{cases}$$

(iii) If f of the form (1.1) is in CS_{ϕ_3}, then, for $0 < s \leq \frac{1}{\sqrt{2}}$,

$$|a_3 - \lambda a_2^2| \leq \begin{cases} |3 - 4\lambda| + 2s|1 - 2\lambda| - 4s^2(\frac{1}{2} - 2\lambda) & s(\lambda - 1) \leq -2, \\ |3 - 4\lambda| + 2s|1 - 2\lambda| + 4s^2 & -2 \leq s(\lambda - 1) \leq 2, \\ |3 - 4\lambda| + 2|1 - 2\lambda| + 4s^2(2\lambda - \frac{1}{2}) & 2 \leq s(\lambda - 1). \end{cases}$$

Remark: *For $\phi(z) = \frac{(1+z)}{(1-z)}$, Theorem 3.1 and Theorem 3.2 reduce to the results of Jastrzebski et al.* [11].

5 Conclusion

This paper presents a modest attempt to determine the Fekete-Szegö inequality, coefficient estimates and bounds of the second and third order Hermitian Toeplitz Determinants for a class of close-to-star functions CS_ϕ. Also special cases of ϕ are discussed. We hope that these results will motivate the researchers in this field to investigate the higher order Hermitian Toeplitz determinants exploring various choices of $\phi(z)$.

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SOME INFERENCES ON CHARACTERIZATIONS OF PROBABILITY DISTRIBUTIONS, WITH SPECIAL REFERENCE TO THE CONTRIBUTIONS OF PROFESSOR C. R. RAO ON THE CHARACTERIZATIONS OF DISTRIBUTIONS

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Abstract

The characterization of probability distributions plays a fundamental role in statistical theory and mathematical statistics. Before applying any probabilistic model to observed data, it is essential to ensure that the assumed distribution satisfies the relevant defining properties. Characterization theorems provide powerful tools for identifying probability distributions through functional, moment-based, or distributional properties of random variables.

In this paper, we present several important characterization results for classical probability distributions, with special reference to the contributions of Professor C. R. Rao on the characterization of distributions. We also propose and develop some new characterization results for logistic distribution. Logistic distribution occupies an important place in statistical theory and is often regarded as a close analogue of normal distribution. It arises naturally in several theoretical and applied contexts, including logistic growth models and logistic regression. The present work discusses basic properties and selected characterizations of the logistic distribution, thereby contributing to the existing literature on characterization theory.
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1 Introduction

The identification and validation of probability distributions constitute a central theme in probability theory and mathematical statistics. Prior to applying any distributional model to real data, it is necessary to verify that the underlying assumptions of the model are satisfied. The study of characterizations of probability distributions addresses this issue by determining conditions under which a distribution is uniquely specified by certain properties of random variables. As pointed out by Nagaraja [27, 2006], "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". Furthermore, according to Koudou and Ley [19, 2014], "In probability and statistics, a characterization theorem occurs when a given distribution is the only one which satisfies a certain property. Besides their evident mathematical interest per se, characterization theorems also deepen our understanding of the distributions under investigation and sometimes open unexpected paths to innovations which might have been uncovered otherwise". Thus, it is very important to characterize a particular probability distribution subject to certain conditions before applying it to some real-world data.

Professor C. R. Rao occupies a preeminent place in the development of characterization theory for continuous statistical distributions. From the mid-twentieth century onward, as probability and statistics underwent rapid axiomatization and unification, Rao's work helped shape characterization theory into a rigorous and coherent discipline. His contributions clarified why certain distributions, most notably the normal distribution-emerge inevitably from natural probabilistic, geometric, and inferential principles rather than from ad hoc modeling assumptions. One of the early significant contributions in this area was made by Khatri and Rao [18, 1972] in their paper "Functional Equations and Characterization of Probability Laws through Linear Functions of Random Variables." They obtained important characterization results, particularly for the normal distribution, based on properties of linear functions of random variables. Their work stimulated further research and laid the groundwork for subsequent developments in distribution theory.

A major milestone in literature appeared in 1973 with the publication of the monograph Characterization Problems in Mathematical Statistics by Kagan, Linnik, and Rao [16, 1973].

This authoritative work presents a comprehensive collection of characterization results for the normal and several other distributions, including those based on the equality in distribution of linear statistics derived from random samples and on the constancy of regression of one random variable on another. The book continues to serve as a standard reference in the field.

Rao and Shanbhag [31, 1986] achieved further progress, by reviewing and unifying several important results on characterization problems and introduced refined versions of earlier theorems as corollaries of the Lau-Rao-Shanbhag *(LRS)* theorems. This connection led to some useful variants of the Lau-Rao theorem, which have been successfully applied to a variety of characterization problems in statistics (Lau and Rao [22, 1982]).

In recent years, several researchers and authors have investigated the characterizations based on truncated moments by employing simple relationships between two moments truncated from the left at the same point. For details, see Glänzel *et al.* [13, 1984], and Kotz and Shanbhag [21, 1980]. For another approach, which involves both left- and right-truncated moments, often expressed through products involving the reverse hazard rate and suitable functions of the truncation point, the interested readers are referred to Ahsanullah [1, 2017], Ahsanullah *et al.* [5, 2021], Shakil and Ahsanullah [33, 2015], and Shakil *et al.* [32, 2018]. Furthermore, Ahsanullah and Shakil [6, 2024] characterized the continuous uniform distribution using its characteristic function (CF), by analyzing its functional relations and properties. Moreover, for characterizations of logistic distribution through order statistics with independent exponential shifts, see Ahsanullah *et al.* [4, 2012]. These results define the distribution via specific functional forms, aiding in parameter estimation and goodness-of-fit tests.

For further study of characterization problems, the interested reader may consult the following monographs: Kagan *et al.* [16, 1973]; Mathai and Pederzoli [25, 1977]; Patil *et al.* [28, 1974]; Galambos and Kotz [12, 1978]; Kakosyan *et al.* [17, 1984]; and Ahsanullah [1,2, 2017, 2019].

Several characterizations of normal distribution appear in literature investigated by various authors. As pointed out by Kotz [20, 1974] "while the normal distribution with a more complex distribution function is well characterized, the characteristic property of the logistic distribution has not been thoroughly developed". This comment remains valid nowadays, although some characterization results are recently available in the literature (Lin and Hu [23, 2008]). The present paper reviews some well-known characterization results for the normal distribution and proposes new characterization results for the logistic distribution, thereby extending existing results in the theory of probability distributions.

The organization of paper is as follows: In Section 2 we provide some well-known characterization results for the normal distribution. Section 3 contains some basic properties of logistic distribution. In Section 4, we present main results on some new characterization results and inferences of logistic distribution. Some conclusions are drawn in Section 5.

2 Review of Some Well-known Characterization Results for the Normal Distribution

In what follows, we first present some well-known characterization results for the normal distribution, as provided below:

2.1 Theorem of Cramer or Cramér's Decomposition Theorem

Theorem 2.1. *One of the most important results on the characterizations of normal distribution was proved by Cramér* [10, 1936] *as stated below:*

If X_1 and X_2 are two independent random variables, $Z = X_1 + X_2$, and if Z is normally distributed, the X_1 and X_2 are normally distributed.

Proof. See Cramér [10, 1936]. □

2.2 Theorem of Darmois, Skitovich and Basu

Theorem 2.2. *Darmois* [11, 1953]*, Skitovitch* [34, 1953]*, and Basu* [9,1953] *independently obtained the following general result related to the characterization of the normal distribution using the property of* *stochastic independence of linear functions of independent random variables:*

Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose

$$L_1 = a_1X_1 + a_2X_2 + \ldots\ldots + a_nX_n,$$

and

$$L_2 = b_1X_1 + b_2X_2 + \ldots\ldots + b_nX_n.$$

If L_1 and L_2 are independent, then, for each index $i(i = 1, 2, \ldots, n), a_i$ and $b_i \neq 0, X_i$ is normal.

Proof. See Darmois [11], Skitovitch [34], and Basu [9]. □

2.3 Theorem of Heyde

Theorem 2.3. *The Heyde theorem (Heyde [14, 1970]) characterizes the normal distribution by the symmetry of the conditional distribution of one linear form of independent random variables given another. Specifically, let $X_1, X_2, \ldots, X_n$ be independent random variables and $a_i's$ and $b_i's, i = 1, 2, \ldots, n$, be non-zero constants such that $a_ib_i^{-1} + a_jb_j^{-1} \neq 0$ for all $i \neq j$. Suppose*

$$L_1 = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

and

$$L_2 = b_1X_1 + b_2X_2 + \ldots + b_nX_n.$$

Then, if the conditional distribution of L_1 given L_2 is symmetric, the distributions of the $X_i's, (i = 1, 2, \ldots, n)$, are normal.

Proof. See Heyde [14]. □

2.4 Theorem of Kagan-Linnik-Rao

Theorem 2.4 (Kagan *et al.* [16, 1973])**.** *This theorem establishes that the admissibility of sample means as an estimator (specifically, the linearity of regression of the sample mean on residuals) characterizes the normal distribution. It states that if $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $E(X_i) = 0, i = 1, 2, \ldots, n$, and $n \geq 3$, and if*

$$E\left(\bar{X} \mid X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X}\right) = 0,$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the distributions of the $X_i's, i = 1, 2, \ldots, n$, are normal.

Proof. See Kagan *et al.* [16]. □

2.5 Theorem of Polya (or Pólya's characterization of the normal distribution)

Theorem 2.5 (Pólya [29, 1923])**.** *This theorem identifies the normal distribution as the only distribution (with finite variance) that is "stable" under the specific averaging transformation described. Specifically, suppose that X_1 and X_2 are two independent and identically distributed random variables with finite variance σ^2, then $(X_1 + X_2)/\sqrt{2}$ and X_1 are identically distributed if and only if X_1 is normally distributed with mean zero (or a shifted normal distribution if the mean is non-zero).*

Proof. See Pólya [29]. □

2.6 Theorem of Rao

Theorem 2.6 (Rao [30, 1967])**.** *If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $E(X_i) = 0$ and $E\left(X_i^2\right) < \infty, i = 1, 2, \ldots, n$, for $n \geq 3$. Then, if $E\left(\bar{X} \mid X_i - \bar{X}\right) = 0$ for any $i, i = 1, 2, \ldots, n$, the distributions of the $X_i's, i = 1, 2, \ldots, n$, are normal.*

Proof. See Rao [30]. □

Remark 2.1. *It is also noted that, for $n = 2$, the above result (in Theorem of Rao) does not hold, as seen in the following example.*

Example 2.1. Suppose that the random variables X_1 and X_2 are independent and identically distributed Uniform [-1, 1], and have the joint pdf $f(x_1, x_2)$ as

$$f(x_1, x_2) = \frac{1}{4}, -1 \leq x_1, x_2 \leq 1,$$
$$= 0, \text{ otherwise .}$$

Let $2Y_1 = X_1 + X_2$ and $2Y_2 = X_1 - X_2$, where $Y_1 = \bar{X} = \frac{X_1+X_2}{2}$ (the sample mean). Then, the conditional pdf of $Y_1 \mid Y_2 = y_2$ is

$$f(y_1 \mid Y_2 = y_2) = \frac{1}{2(1-|y_2|)}, -(1-|y_2|) \leq y_1 \leq (1-|y_2|).$$

Thus

$$E(Y_1 \mid Y_2 = y_2) = \int_{-(1-|y_2|)}^{(1-|y_2|)} (y_1) \frac{1}{2(1-|y_2|)} dy_1 = 0$$

Since $Y_1 = \bar{X}$ and Y_2 is a linear function of $X_1 - X_2$, the condition $Y_2 = y_2$ is equivalent to specifying a value for $X_1 - X_2$.

Hence

$$E\left(\bar{X} \mid X_1 - X_2\right) = 0.$$

This completes the proof of Example 2.6.

2.7 Theorem of Zinger and Linnik

Theorem 2.7 (Zinger and Linnik [36, 1964]). *Suppose that $a_1, a_2, \ldots, a_n$ are positive numbers and $\varphi_i(t), i = 1, 2, .., n$, are characteristic functions of independent random variables X_i 's repectively. Further, if*

$$(\varphi_1(t))^{a_1} \left(\varphi_2(t)^{a_2} \ldots \left(\varphi_n(t)^{a_n} = e^{-\frac{t^2}{2}} \text{ for } |t| < t_0, t_0 > 0,\right.\right.$$

then $\varphi_i(t), i = 1, 2, .., n$, are characteristic functions of normal distribution. Thus $Xi's, i = 1, 2, \ldots, n$, are normally distributed.

Proof. See Zinger and Linnik [36]. □

2.8 Theorem of Ahsanullah and Hamedani

Theorem 2.8 (Ahsanullah and Hamedani [7, 1988]). *Suppose X_1 and X_2 are independent and identically symmetric (about zero) random variables and $Z = \min(X_1, X_2)$. If Z^2 is distributed as chi-square one degree of freedom, then X_1 and X_2 are normally distributed.*

Proof. See Ahsanullah and Hamedani [7]. □

Remark 2.2. *The above theorem is also true if minimum is replaced by maximum for Z.*

3 Some Basic Properties of Logistic Distribution

This paper aims to contribute to the understanding of logistic distribution by presenting its basic properties and characterizations. We will discuss some of its characterization techniques that enhance its applicability in statistical analysis, thus enhancing its use in model building and its connection to other distributions. In what follows, we present some basic properties of logistic distribution.

3.1 Logistic Distribution

The logistic distribution is a symmetric distribution, unimodal and similar in shape to the normal distribution. The cumulative distribution function of the logistic distribution is also a scaled version of the of the hyperbolic tangent, as given below:

$$F(x, \mu, \sigma) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}} = \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{x-\mu}{2\sigma}\right), \tag{3.1}$$

where $-\infty < x < \infty, -\infty < \mu < x < \infty, \sigma > 0, \mu$ and σ being location and scale parameters respectively. The corresponding probability density function (*pdf*) is given by

$$f(x, \mu, \sigma) = \frac{1}{4\sigma} \operatorname{sech}^2\left(\frac{x-\mu}{2\sigma}\right). \tag{3.2}$$

To illustrate the behavior of the logistic distribution, the graphs of its *pdf* (2.2) for some selected values of the location, μ, and scale, σ, parameters, are given in Figure 3.1. It is observed that, though the shapes of the logistic distribution and the normal distribution resemble each other, the logistic distribution has heavier tails in view of the fact its kurtosis is high which is approximately 1.2 .

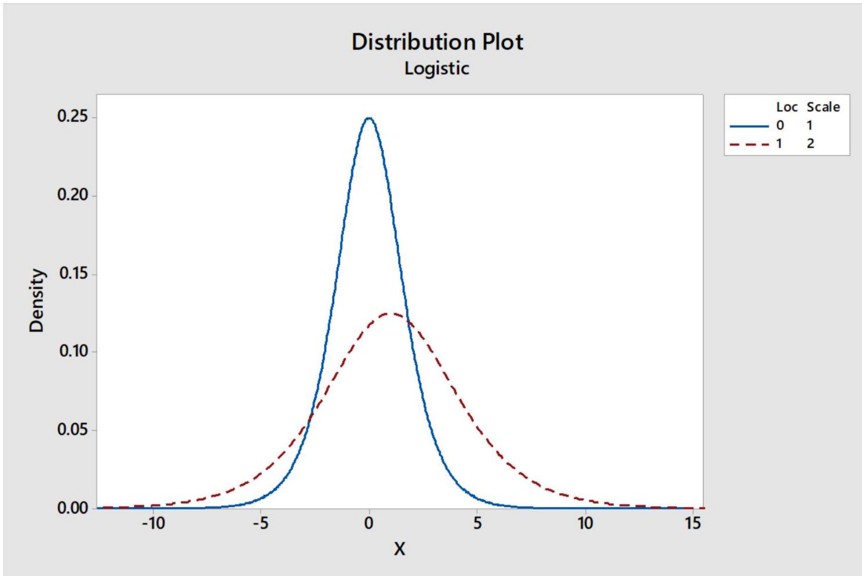

***Figure* 3.1:** *pdfs of the logistic distribution for $\mu = 0, \sigma = 1$; and $\mu = 1, \sigma = 2$, respectively.*

Logistic distribution is widely used in statistical modeling, often preferred over the normal distribution due to its simpler cumulative distribution function (CDF). This simplicity, alongside its symmetric and unimodal shape, makes the logistic distribution particularly useful in applications such as logistic regression and modeling of logistic growth processes. The CDF of the logistic distribution is a scaled version of the hyperbolic tangent, which further enhances its applicability.

Notably, both the United States Chess Federation *(USCF)* and the World Chess Federation (commonly referred to by its French acronym *FIDE*) employ the logistic distribution to assess the skill levels of chess players, replacing the traditional use of the normal distribution for this purpose. This shift underscores the utility of the logistic distribution in various practical settings where normal distribution models might fall short.

3.2 Standard Logistic Distribution

We say a random variable X has a standard logistic distribution if the probability density function (pdf) $f(x)$ is as follows:

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, -\infty < x < \infty. \tag{3.3}$$

The corresponding cumulative distribution function (*cdf*)$F(x)$ is given by

$$F(x) = \frac{1}{1+e^{-x}}. \tag{3.4}$$

The *pdf* and *cdf* of the standard logistic distribution satisfy the following relations:

1. $\frac{f'(x)}{f(x)} = \frac{1-e^{x}}{1+e^{-x}}$,
2. $f(x) = F(x)(1 - F(x))$,
3. The odd moments are zero, that is,

$$E\left(X^{2n+1}\right) = 0, \quad n = 1, 2, \ldots.$$

4. Even moments $E\left(X^{2n}\right), n=1,2,\ldots$, are given as follows:

$$
\begin{aligned}
E\left(X^{2n}\right) &= \int_{-\infty}^{\infty} x^{2n} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx, n=1,2,\ldots \\
&= \int_0^1 \left(\ln \frac{p}{1-p}\right)^n dp \\
&= \pi^n \left(2^n-2\right)\left|B_n\right|,
\end{aligned}
$$

where B_n is the nth Bernoulli number.

4 Some New Characterization Results and Inferences of Logistic Distribution

In what follows, we present some new characterization results and inferences of logistic distribution.

4.1 Characterizations Based on Characteristic Function

Let $\varphi(t)$ be the characteristic function of the standard logistic distribution, given by

$$
\begin{aligned}
\varphi(t) &= \int_{-\infty}^{\infty} e^{itx} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx \\
&= \int_0^1 u^{it}(1-u)^{-it} du \\
&= \Gamma(1-it)E(1+it) \\
&= \frac{\pi t}{\sinh(\pi t)}.
\end{aligned}
$$

The following theorems give characterizations of the standard logistic distribution based on characteristic function.

Theorem 4.1. *If $\varphi(t)$ is the characteristic function of an absolutely continuous random variable X with finite expectation. Then X has the standard logistic random variable if and only if*

$$
\varphi'(t) = \varphi(t)\left(\frac{1}{t} - \pi \frac{\cosh(\pi t)}{\sinh(\pi t)}\right).
$$

Proof. Suppose the random variable has the pdf $f(x)$ as

$$
f(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}.
$$

We have

$$
\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx.
$$

Let $u = \left(1+e^{-x}\right)^{-1}$, then

$$
\begin{aligned}
\varphi(t) &= \int_0^1 e^{it\ln\left(\frac{u}{1-u}\right)} du \\
&= \int_0^1 u^{it}(1-u)^{-it} du \\
&= B(1+it, 1-it) \\
&= \Gamma(1-it)\Gamma(1+it) \\
&= -it\Gamma(-it)\Gamma(1+it) \\
&= -it\pi \csc(\pi it) \\
&= \pi t \operatorname{csech}(\pi t),
\end{aligned}
$$

from which by taking natural log of both sides, we obtain

$$
\ln \varphi(t) = \ln(i\pi t) + \ln \operatorname{csech}(\pi t).
$$

Now differentiating the above expression with respect to t, we obtain

$$
\varphi'(t) = \varphi(t)\left(\frac{1}{t} - \pi \frac{\cosh(\pi t)}{\sinh^2(\pi t)}\right).
$$

Dividing both sides of the above equation by $\varphi(t)$, we obtain

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{1}{t} - \pi\frac{\cosh(\pi t)}{\sinh(\pi t)}.$$

Now integrating the above equation with respect to t, we obtain

$$\ln\varphi(t) = c + \ln t - \ln\sinh(\pi t), t > 0,$$

from which by taking the limit of both sides of the above equation, when $t \to 0$, we obtain

$$\lim_{t\to 0}\ln\varphi(t) = c + \lim_{t\to 0}\ln\frac{t}{\sinh(\pi t)} = c + \ln\frac{1}{\pi}.$$

Now, using the characteristic function property in the above equation, that is,

$$\varphi(0) = 1 \Rightarrow \ln\varphi(0) = 0,$$

we obtain

$$0 = c - \ln\pi \Rightarrow c = \ln\pi.$$

Hence, substituting back $c = \ln\pi$ in the above equation, we obtain

$$\begin{aligned}\mathrm{Ln}\,\varphi(t) &= \ln\pi + \ln t - \ln\sinh(\pi t)\\ &= \ln\left(\frac{\pi t}{\sinh(\pi t)}\right).\end{aligned}$$

Thus, exponentiating both sides of the above equation, we obtain

$$\varphi(t) = \frac{\pi t}{\sinh(\pi t)},$$

or

$$\varphi(t) = \pi t\,\mathrm{csch}(\pi t),$$

which is the characteristic function of standard logistic distribution. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Suppose $X_1, X_2, \ldots, X_n$ are i.i.d random variables from standard logistic distribution and $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ be the corresponding order statistics. Let $W_{n-1,n-1} = \max(W_1, W_2, \ldots, W_{n-1})$, where $W_1, W_2, \ldots, W_n$ are i. i.d exponential random variables with* pdf $f(w) = e^{-w}, 0 < w < \infty$. *For an absolutely continuous random variable X, if X and $X_{1,n} + W_{n-1,n-1}$ are identically distributed, then X has standard logistic distribution.*

Proof. Let $\Phi_1(t)$ and $\Phi_2(t)$ be the characteristic functions of $X_{1,n}$ and $W_{n,n}$ respectively. We have

$$\Phi_1(t) = n\int_{-\infty}^{\infty} e^{itx}\left(\frac{e^{-x}}{1+e^{-x}}\right)^{n-1}\frac{e^{-x}}{(1+e^{-x})^2}dx.$$

Let $u = \frac{1}{1+e^{-x}}$, then (Ahsanullah *et al.* [4, 2012])

$$\begin{aligned}\Phi_1(t) &= n\int_0^1 u^{n-1}\left(\frac{u}{1-u}\right)^{it}\left(\frac{1-u}{u}\right)^{n-1} du\\ &= n\int_0^1 u^{it}(1-u)^{n-it-1}du\\ &= nB(1+it, n-it)\\ &= \frac{\Gamma(1+it)\Gamma(n-it)}{\Gamma(n)}, \text{ using the definition of the Beta function, } B(x,y).\end{aligned}$$

Also

$$\Phi_2(t) = (n-1)\int_0^{\infty} e^{itx}\left(1-e^{-x}\right)^{n-2}e^{-x}dx$$

Let $u = 1 - e^{-x}$. Then

$$\begin{aligned}\Phi_2(t) &= (n-1)\int_0^1 u^{n-2}(1-u)^{-it}du \\ &= (n-1)B(1-it, n-1) \\ &= \frac{\Gamma(1-it)\Gamma(n)}{\Gamma(n-it)}.\end{aligned}$$

The characteristic function of $X_{1,n} + W_{n-1,n-1}$ is

$$\begin{aligned}\Phi_1(t)\Phi_2(t) &= \frac{\Gamma(1+it)\Gamma(n-it)}{\Gamma(n)}\frac{\Gamma(1-it)\Gamma(n)}{\Gamma(n-it)} \\ &= \Gamma(1+it)\Gamma(1-it),\end{aligned}$$

which is the characteristic function of the standard logistic distribution. This completes the proof of Theorem 4.2. □

Theorem 4.3. *If an absolutely continuous random variable X has the characteristic function $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$ if and only if the pdf of the random variable has the standard logistic distribution, where $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, -\infty < x < \infty$.*

Proof. We have already seen that the characteristic function $\varphi(t)$ of the standard logistic distribution is $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.

We will prove here the only if case.
Suppose $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.
Using the inverse formulae of the characteristic function, we have

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\Gamma(1-it)\Gamma(1+it)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{\pi t}{\cosh(\pi t)}dt \\ &= \int_{-\infty}^{\infty} e^{-itx}\frac{t}{e^{\pi t}-e^{-\pi t}}dt \\ &= \int_0^{\infty} e^{-itx}\frac{t}{e^{\pi t}-e^{-\pi t}}dt + \int_{-\infty}^{0} e^{-itx}\frac{t}{e^{\pi t}-e^{-\pi t}}dt \\ &= \int_0^{\infty} e^{-itx}\frac{t}{e^{\pi t}-e^{-\pi t}}dt + \int_0^{\infty} e^{itx}\frac{t}{e^{\pi t}-e^{-\pi t}}dt \\ &= \int_0^{\infty} e^{-itx}\frac{te^{-\pi t}}{1-e^{-2\pi t}}dt + \int_0^{\infty} e^{itx}\frac{te^{-\pi t}}{1-e^{-2\pi t}}dt \\ &= \sum_{n=0}^{\infty}\int_0^{\infty} te^{-t(ix+(2n+1)\pi)}dt + \sum_{n=0}^{\infty}\int_0^{\infty} te^{t(ix-(2n+1)\pi)}dt \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{(ix+(2n+1)\pi)^2} + \frac{1}{(ix-(2n+1)\pi)^2}\right) \\ &= \sum_0^{\infty}\frac{2(2n+1)^2\pi^2-2x^2}{((2n+1)^2\pi^2+x^2)^2}.\end{aligned} \tag{4.1}$$

Now, it is well-known that the identity $\frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right) = \sum_0^{\infty}\frac{2(2n+1)^2\pi^2-2x^2}{((2n+1)^2\pi^2+x^2)^2}$ is a valid mathematical identity derived from the partial fraction expansion of the hyperbolic secant function. Hence, (4.1) represents the series representation for $\frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right)$, and thus (4.1) reduces to

$$f(x) = \frac{1}{4}\operatorname{sech}^2(x/2). \tag{4.2}$$

The hyperbolic secant function given in (4.2) can easily be expressed in its exponential form and simplifying, as follows:

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, -\infty < x < \infty,$$

which is the required pdf of the logistic distribution. This completes the proof of Theorem 4.3. $\square$

4.2 Characterizations Based on Domain of Attraction

In this section on characterization by domain of attraction, the following definition and examples are relevant:

Definition of Domain of Attraction: A distribution $F(x)$ is said to lie in the domain of attraction of a limit distribution $G(x)$ if, after appropriate normalization (shifting and scaling), the distribution of the sum, maximum, or minimum of n independent and identically distributed random variables converges in distribution to $G(x)$ as n tends to infinity.

Examples:

- **Normal Distribution:** By the central limit theorem, for many distributions with finite variance, the normalized sum is in the domain of attraction of the normal distribution.
- **Extreme Value Distributions:** The normalized maximum or minimum of many distributions is in the domain of attraction of the Gumbel, Fréchet, or Weibull distributions.
  In what follows, we present some characterization results of logistic distribution based on domain of attraction, as provided in Theorems 4.4, 4.5 and 4.6.

Theorem 4.4. *Suppose $X_1, X_2, \ldots, X_n$ are $n, (n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $S_n = X_1 + X_2 + \ldots, +X_n$. Then, the normalized sum $\left(= \frac{\sqrt{\left(\frac{3}{n}\right)} s_n}{\pi}\right)$ belongs to the domain of the standard normal distribution.*

Proof. If the random variable X has the standard logistic distribution, then its characteristic function $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.

The characteristic function $\Psi_n(t)$ of $\frac{\sqrt{\left(\frac{3}{n}\right)} S_n}{\pi}$ is

$$\Psi n(t) = \left(\Gamma\left(1 - \sqrt{\left(\frac{3}{n}\right)}\frac{it}{\pi}\right)\Gamma\left(1 + \sqrt{\left(\frac{3}{n}\right)}\frac{it}{\pi}\right)\right)^n.$$

Then, using the identity $\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$ and for small z :

$$\Psi_n(t) = \left(\frac{\frac{\sqrt{3}}{\sqrt{n}}t}{\sinh\left(\frac{\sqrt{3}}{\sqrt{n}}t\right)}\right)^n.$$

Substitute $x = \frac{\sqrt{3}}{\sqrt{n}}t$ in the above equation. Then, we have

$$\Psi_n(t) = \left(\frac{x}{\sinh(x)}\right)^n.$$

Expanding $\sinh(x)$ for small x, we obtain

$$\frac{x}{\sinh(x)} = 1 - \frac{x^2}{6} + o\left(x^2\right).$$

Substitute $x = \frac{\sqrt{3}}{\sqrt{n}}t$ in the above equation. Then, we have

$$\Psi_n(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n.$$

Thus, taking the limit of both sides of the above equation as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \Psi_n(t) = \lim_{n\to\infty}\left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n = e^{-t^2/2},$$

which is the characteristic function of the standard normal distribution. Thus $\frac{\sqrt{\left(\frac{3}{n}\right)} S_n}{\pi}$ belongs to the domain of the standard normal distribution. This completes the proof of Theorem 4.4. $\square$

Now, we prove the following two Theorems 4.5 and 4.6 for the domain of attraction of the maximum and the minimum of the logistic random variables. To prove these two theorems, we need the following lemmas.

Lemma 4.1 (Von Mises [26, 1964]). *Let $X_1, X_2, \ldots, X_n$ be $n, (n > 1)$, independent and identically distributed continuous random variables with cdf $F(x)$, pdf $f(x)$, $\bar{F}(x) = 1$ - $F(x)$ and $\beta(f) = \sup\{x \mid F(x) < 1$. If $F(x)$ has derivatives on $[c_0, \beta(f)]$, for some $c_0 > 0$, then, if $\lim_{x \uparrow \beta(F)} \frac{f(x)}{\bar{F}(x)} = c, c > 0$, then F belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. See Von Mises [26]. □

Lemma 4.2 (Ahsanullah and Nevzorov [8, 2001, p 87]). *Let F^{-1} denote the inverse of cdf F. For a continuous random variable, the necessary and sufficient condition that $X_{n,n}$ belongs to the domain of attraction of type 1 extreme value distribution of the maximum is*

$$\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1,$$

and the normalizing constants are $a_n = F^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = F^{-1}\left(1 - \frac{1}{ne}\right) - F^{-1}\left(1 - \frac{1}{n}\right)$.

Proof. See Ahsanullah and Nevzorov [8, 2001, p 87]. □

Lemma 4.3 (Ahsanullah and Nevzorov [8, 2001, p 87]). *For a continuous random variable, the necessary and sufficient condition that $X_{1,n}$ belong to the domain of attraction of type 1 extreme value distribution of the maximum is*

$$\lim_{c \to 0} \frac{F^{-1}(c) - F^{-1}(2c)}{F^{-1}(2c) - F^{-1}(4c)} = 1.$$

The normalizing constants are $a_n = F^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = F^{-1}\left(1 - \frac{1}{ne}\right) - F^{-1}\left(1 - \frac{1}{n}\right)$.

Proof. See Ahsanullah and Nevzorov [8, 2001, p 87]. □

Theorem 4.5. *Theorem 4.5. Suppose $X_1, X_2, \ldots, X_n$ are $n, (n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $X_{n,n} = \max(X_1, X_2, \ldots, X_n)$. Then the normalized $X_{n,n}$ belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. We shall prove the sufficient condition for the domain of attraction of logistic distribution for $X_{n,n}$. It can be shown that, for standard logistic distribution,

$$\lim_{c \to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1,$$

and $a_n = \ln(n)$ and $b_n = 1$.

Thus

$$\begin{aligned} &P\left(X_{n,n} \le (x + \ln n)\right. \\ &= \lim_{n \to \infty} \left(\frac{1}{1 + e^{-(x + \ln n)}}\right)^n = e^{-e^{-x}}. \end{aligned}$$

Hence $X_{n,n}$ belongs to the domain of attraction of Gumbel (type 1 extreme value) distribution of maximum. This completes the proof of the sufficient condition for the domain of attraction of logistic distribution for $X_{n,n}$. □

Theorem 4.6. *Suppose $X_1, X_2, \ldots, X_n$ are n, $(n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $X_{1,n} = \min(X_1, X_2, \ldots, X_n)$. Then the normalized $X_{1,n}$ belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. We shall prove the sufficient condition for the domain of attraction of logistic distribution for $X_{1,n}$. It can be shown that for standard logistic distribution

$$\lim_{c \to 0} \frac{F^{-1}(c) - F^{-1}(2c)}{F^{-1}(2c) - F^{-1}(4c)} = 1,$$

and $a_n = -\ln(n)$ and $b_n = 1$.

Thus

$$
\begin{aligned}
& P\left(X_{1,n} \geq x-\ln n\right) \\
& =1-P\left(X_{n.n}>x-\ln n\right) \\
& =\lim _{n \rightarrow \infty}\left(1+e^{x-\ln n}\right)^{-n} \\
& =e^{-e^x} .
\end{aligned}
$$

Hence

$F_{1,n}(x)=1-e^{-e^x}$, which is the type 1 extreme value distribution of the minimum.

The normalizing constants are the same as that of $X_{n,n}$. Thus

$$
\begin{aligned}
\lim _{n \rightarrow \infty} & P\left(X_{1,n} \leq x+\ln n\right) \\
& =\lim _{n \rightarrow \infty}\left(1+e^{(x+\ln n)}\right)^{-n} \\
& =\lim _{n \rightarrow \infty}\left(1+\frac{1}{n} e^{-x}\right)^{-n} \\
& =e^{-e^{-x}} .
\end{aligned}
$$

Hence $X_{1,n}$ belongs to the domain of attraction of Gumbel (type 1 extreme value) distribution of minimum. This completes the proof of the sufficient condition for the domain of attraction of logistic distribution for $X_{1,n}$. □

4.3 Characterizations by Truncated Moments

In what follows, we will establish our proposed characterization results of logistic distribution by right truncated moment, that is, by taking a relation between the right truncated moment and reversed failure rate function. To prove this, we need the following Lemmas.

Lemma 4.4. *Suppose the random variable X is absolutely continuous with* cdf $F(x)$ *and pdf $f(x)$. Let $a=\inf\{x \mid F(x)>0\}$, and $b=\sup\{x \mid F(x)<1\}$. We also assume that $E(X)$ and $f'(x)$ exists for all $x, x \in(a, b)$. If $E(X \geq x)=h(x) \tau(x)$, where $\tau(x)=\frac{f(x)}{1-F(x)}, h'(x)$ exists and $\int_a^b \frac{u+h'(x)}{h(u)} du$ is finite for all $x, x \in(a, b)$, then $f(x)=c e^{-\int_a^x \frac{u+h/(x)}{h(u)} du}$, where c is determined by the condition $\int_a^b f(x) dx=1$.*

Proof. See Ahsanullah [1, 2017, p.50]. □

Lemma 4.5. *Suppose that X is a standard logistic random variable, with the pdf given by*

$$
f(x)=\frac{e^{-x}}{\left(1+e^{-x}\right)^2}=\frac{e^x}{\left(1+e^x\right)^2}, x \in(-\infty, \infty) .
$$

Then,

$$
\int_x^{\infty} u f(u) d u=\ln \left(1+e^x\right)-\frac{x e^x}{\left(1+e^x\right)} .
$$

Proof. Using the integration by parts formula, and, on simplification, it can easily be seen that the integral

$$
\int_x^{\infty} u \frac{e^u}{\left(1+e^u\right)^2} d u=\ln \left(1+e^x\right)-\frac{x e^x}{\left(1+e^x\right)} .
$$

□

Theorem 4.7. *Suppose that the random variable X is absolutely continuous (with respect to Lebesgue measure) with cdf $F(x)$ and pdf $f(x)$ for $x \in(-\infty, \infty)$. Suppose that the random variable X has finite expectation and $f'(x)$ exists for all $x \in(-\infty, \infty)$. Then X is a standard logistic random variable if and only if $E(X \geq x)=h(x) \tau(x)$, where $\tau(x)=\frac{f(x)}{1-F(x)}$ and*

$$
h(x)=\frac{\left(1+e^x\right)^2}{e^x} \ln \left(1+e^x\right)-x\left(1+e^x\right) .
$$

Proof. The proof of "if" part easily follows from Lemma 4.6. We will prove here the "only if" case.

Suppose that $E(X \geq x) = h(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{1-F(x)}$.

Then, we have

$$\frac{E(X \geq x)}{\tau(x)} = h(x),$$

or

$$\frac{\int_x^\infty uf(u)du}{f(x)} = h(x). \tag{4.3}$$

In the above equation (4.3), using Lemma 4.5, after simplification, we have

$$h(x) = \frac{(1+e^x)^2}{e^x} \ln(1+e^x) - x(1+e^x). \tag{4.4}$$

Differentiating both sides of the above equation with respect to x, we obtain

$$h'(x) = \left(e^x - e^{-x}\right) \ln(1+e^x) - xe^x. \tag{4.5}$$

Now, expressing (4.3) as

$$\int_x^\infty uf(u)du = h(x)f(x),$$

and differentiating both sides of the above equation with respect to x, we obtain

$$-xf(x) = h'(x)f(x) + h(x)f'(x),$$

or

$$\frac{f'(x)}{f(x)} = -\frac{x + h'(x)}{h(x)}. \tag{4.6}$$

In the above equation, using (4.5) and (4.6), after simplification, we obtain

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{(1-e^x)\left[x - \frac{e^x+1}{e^x}\ln(1+e^x)\right]}{(1+e^x)\left[x - \frac{e^x+1}{e^x}\ln(1+e^x)\right]}\\ &= \frac{1-e^x}{1+e^x}.\end{aligned} \tag{4.7}$$

Now, it is easily seen that the expression on the right hand side of the above equation (4.7) is given by

$$\frac{d\ln\left(\frac{e^x}{(1+e^x)^2}\right)}{dx} = \frac{1-e^x}{1+e^x}.$$

Hence, we have

$$\frac{d\ln f(x)}{dx} = \frac{d\ln\left(\frac{e^x}{(1+e^x)^2}\right)}{dx}.$$

On integrating the above equation with respect to x, we obtain

$$\ln f(x) = c + \ln\frac{e^x}{(1+e^x)^2}, \tag{4.8}$$

where c is a constant to be determined.

Exponentiating both sides of the equation (4.8), we obtain

$$f(x) = c\frac{e^x}{(1+e^x)^2}, \tag{4.9}$$

where c is the normalizing constant.

On integrating the equation (4.8) with respect to x from $x = -\infty$ to $x = +\infty$, we easily obtain $\int_{-\infty}^{\infty} \frac{e^x}{(1+e^x)^2}dx = 1$, and hence using the condition $\int_{-\infty}^{\infty} f(x)dx = 1$, we obtain $c = 1$.

Thus, we have

$$f(x) = \frac{e^x}{(1+e^x)^2} = \frac{e^{-x}}{(1+e^{-x})^2},$$

which is the probability density function of the standard logistic distribution. This completes the proof of Theorem 4.7. □

Remark 4.1. *For characterization of logistic distribution based on the left truncated moment, $E(X \mid X \leq x)$, that is, by taking a relation between left truncated moment and failure rate function, see Ahsanullah and Shakil* [3, 2015].

5 Conclusion

Logistic distribution offers several advantages over the normal distribution in certain statistical modeling contexts. Its simple *CDF*, symmetric shape, and applicability in logistic regression make it a powerful tool in statistical science. The characterization of this distribution, as well as methods for determining its suitability for real-world data, is an ongoing area of research. As demonstrated by the works of Khatri, Rao, Shanbhag, Glänzel, and others, the characterization of probability distributions through moments and truncated moments provides valuable insights that aid in understanding and applying the logistic distribution effectively.

In this paper, we have presented an overview of the key properties and characterizations of the logistic distribution, underscoring its significance in model building and statistical inference. Further research into advanced characterization methods will continue to expand the utility of this distribution in various domains of statistical analysis.

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