ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

***Jñānābha*, Vol. 56(1) (2026), 76-80**

LACUNARY POLYNOMIAL INTERPOLATION BY g-SPLINE: $(0, 1, 2, 5)$ CASE

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(Received: May 01, 2024*; In format: July* 09, 2025*; Revised: April* 02, 2026*; Accepted: April* 08, 2026*)*

DOI: https://doi.org/10.58250/jnanabha.2026.56109

Abstract

The author focuses on Lacunary Polynomial Interpolation *(LPI)*. This study proves the existence and uniqueness of the g-spline of degree six for *LPI*, which interpolates $(0, 1, 2, 5)$ data. In addition to verifying the interpolation of data, error bounds associated with this method are derived and analyzed.

2020 Mathematical Sciences Classification: 41A15, 65D05 and 65D07.
Keywords and Phrases: LPI, Splines, g-splines, Error bounds.

1 Introduction

LPI is an extension of Hermite interpolation. It includes matching of values and derivatives at certain points but does not claim that these points to be successive. Thus, whenever observations provide scattered and irregular data about a function and its derivatives then *LPI* arises. Several innovative concepts, tools and applications have been developed for *LPI* [7]. The theory of spline plays a significant role in avoiding the complications occurred by polynomials of a higher degree and also to achieve accurate results. *LPI* is generalized by g-spline.

Schoenberg [17] initiated the study of g-splines for *LPI*. He presented that under certain conditions the interpolatory g-splines exist and are unique. Numerous authors presented global procedures for solution of *LPI* by splines [8, 9, 18, 20]. Fawzy [2, 3, 4, 5, 6] described certain local procedures to solve *LPI* by splines.

By applying these procedures some authors considered certain class of g-splines for solution of a particular *LPI* [1, 10, 19]. We Studied *LPI* in different directions [11, 12, 13, 14, 15, 16].

Here, the author focuses on a specialized form of *LPI* and deals with gaps in the data provided (knowing the value and first, second and fifth derivatives, but not the third and fourth derivatives). The research investigates a specific mathematical structure to solve this problem. This ensures that for a specific set of data points, there is one and only one polynomial curve of this type that fits perfectly. The study verifies that sixth-degree spline successfully interpolates the $(0, 1, 2, 5)$ data set. The paper provides a rigorous investigation into error bounds. By establishing these bounds, the author demonstrates the accuracy and stability of using high-degree g-splines for non-consecutive data points, providing a reliable framework for approximation in complex computational problems.

The considered g-spline is defined as follows:
$\Delta : 0 = x_0 < x_1 <, \dots, < x_n = 1$, be any subdivision of $[0, 1]$ with $h_k = x_{(k+1)} - x_k; k = 0, 1, \dots, n-1$ and S_Δ be the class of splines satisfying the following properties:

$$S_\Delta \in C^1[0, 1], \tag{1.1}$$

$$S_\Delta \in \Pi_6, \tag{1.2}$$

in each interval $\left[x_k, x_{(k+1)}\right], k = 0, 1, \dots, n-1$.

In the next section, we investigate the existence and uniqueness of the above g-splines which interpolate $(0, 1, 2, 5)$ data and investigate the error bounds for considered spline when, $f(x) \in C^6[0, 1]$.

2 Existence, Uniqueness and Error Bounds

The spline interpolant for $(0,1,2,5)$- data and for $f(x) \in C^6[0,1]$, defined above is given by:

$$S_\Delta = \sum_{j=0}^{6} (x - x_k) \frac{S_k^{(j)}}{j!}; x \in [x_k, x_{k+1}],$$

where $k = 0, 1, \ldots, n-1$.
We constructed each $S_k^{(j)}$ as the following:

$$S_k = f_k, \tag{2.1}$$

$$S_k^{(1)} = f_k^{(1)}, \tag{2.2}$$

$$S_k^{(2)} = f_k^{(2)}, \tag{2.3}$$

$$S_k^{(5)} = f_k^{(5)}, \tag{2.4}$$

where $k = 0, 1, \ldots, n-1$.
Now

$$S_k = f_k + (x-x_k) f_k^{(1)} + \frac{(x-x_k)^2}{2!} f_k^{(2)} + \frac{(x-x_k)^3}{3!} S_k^{(3)} + \frac{(x-x_k)^4}{4!} S_k^{(4)} + \frac{(x-x_k)^5}{5!} f_k^{(5)} + \frac{(x-x_k)^6}{6!} S_k^{(6)}.$$

From the interpolatory conditions and continuity of S_Δ, we get

$$f_{k+1} = f_k + h_k f_k^{(1)} + \frac{h_k^2}{2!} f_k^{(2)} + \frac{h_k^3}{3!} S_k^{(3)} + \frac{h_k^4}{4!} S_k^{(4)} + \frac{h_k^5}{5!} f_k^{(5)} + \frac{h_k^6}{6!} S_k^{(6)}, \tag{2.5}$$

$$f_{k+1}^{(1)} = f_k^{(1)} + h_k f_k^{(2)} + \frac{h_k^2}{2!} S_k^{(3)} + \frac{h_k^3}{3!} S_k^{(4)} + \frac{h_k^4}{4!} f_k^{(5)} + \frac{h_k^5}{5!} S_k^{(6)}, \tag{2.6}$$

$$f_{k+1}^{(2)} = f_k^{(2)} + h_k S_k^{(3)} + \frac{h_k^2}{2!} S_k^{(4)} + \frac{h_k^3}{3!} f_k^{(5)} + \frac{h_k^4}{4!} S_k^{(6)}, \tag{2.7}$$

$$f_{k+1}^{(5)} = f_k^{(5)} + h_k S_k^{(6)}, \tag{2.8}$$

$$S_k^{(6)} = \frac{f_{k+1}^{(5)} - f_k^{(5)}}{h_k}. \tag{2.9}$$

From equations (2.5) and (2.6), we get

$$S_k^{(3)} = \frac{24}{h_k^3} \left[\mathbb{D}_k - \frac{h_k}{4} \mathbb{E}_k \right], \tag{2.10}$$

and

$$S_k^{(4)} = -\frac{72}{h_k^4} \left[\mathbb{D}_k - \frac{h_k}{3} \mathbb{E}_k \right], \tag{2.11}$$

where

$$\mathbb{D}_k = f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)}, \tag{2.12}$$

and

$$\mathbb{E}_k = f_{k+1}^{(1)} - f_k^{(1)} - h_k f_k^{(2)} - \frac{h_k^4}{4!} f_k^{(5)} - \frac{h_k^5}{5!} S_k^{(6)}. \tag{2.13}$$

The selection of $S_k^{(j)}; j = 3, 4$ agreed on the following basis:

$$D_{\mathcal{L}}^{(j)} S_k = D_{\mathcal{R}}^{(j)} S_k; j = 0, 1, 2, 5; k = 0, 1, \ldots, n-1,$$

where $D_{\mathcal{L}}$ and $D_{\mathcal{R}}$ are denote left derivative and right derivative respectively.

Thus

$$S_{\Delta} \in C^{(0,1,2,5)}[x_0,x_n] = \{f \in C[x_0,x_n]\,; D_{\mathcal{R}}^{(j)} f \in C[x_0,x_n] : j = 0,1,2,5\}.$$

Hence there exists unique spline $S_{\Delta}(x)$ defined by equations (2.9) - (2.13) satisfying the interpolatory conditions (2.1) - (2.4).

Theorem 2.1. *If S_{Δ} be spline solving the $(0,1,2,5)$-LPI as defined above for which $f(x) \in C^6[0,1]$ then for all $x \in [0,1]$ the following inequality fulfilled:*

$$\left|S_k^{(j)} - f_k^{(j)}\right| \le c_{k,j} h^{6-j} w_6(f,h)\,; j = 0,1,\ldots,6$$

where $w_6(f,h)$ is the modulus of continuity of $f(x)$ and the constants $c_{k,j}$; $j = 0,1,\ldots,6$ are given by followings:

$$\begin{gathered} c_{k,0} = \frac{1}{36}, \quad c_{k,1} = \frac{1}{10}, \\ c_{k,2} = \frac{11}{40}, \quad c_{k,3} = \frac{1}{12}, \\ c_{k,4} = \frac{3}{10}, \quad c_{k,5} = 1, \\ c_{k,6} = 1. \end{gathered}$$

Proof. From equation (2.9), we get

$$\left|S_k^{(6)} - f_k^{(6)}\right| \le w_6(f,h). \tag{2.14}$$

From equation (2.11), we derive

$$\left|S_k^{(4)} - f_k^{(4)}\right| = \left|-\frac{72}{h_k^4}\left[\mathbb{D}_k - \frac{h_k}{3}\mathbb{E}_k\right] - f_k^{(4)}\right|,$$

From equation (2.12) and (2.13), we obtain

$$\begin{aligned} |S_k^{(4)} - f_k^{(4)}| = |-\frac{72}{h_k^4}[&f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)} \\ &- \frac{h_k}{3}\{f_{k+1}^{(1)} - f_k^{(1)} - h_k f_k^{(2)} - \frac{h_k^4}{4!} f_k^{(5)} - \frac{h_k^5}{5!} S_k^{(6)}\}] - f_k^{(4)}|. \end{aligned} \tag{2.15}$$

Now

$$f_{k+1} = f_k + h_k f_k^{(1)} + \frac{h_k^2}{2!} f_k^{(2)} + \frac{h_k^3}{3!} f_k^{(3)} + \frac{h_k^4}{4!} f_k^{(4)} + \frac{h_k^5}{5!} f_k^{(5)} + \frac{h_k^6}{6!} f_k^{(6)}(t_{k,1}), \tag{2.16}$$

where $x_k < t_{k,1} < x_{k+1}$,

$$f_{k+1}^{(1)} = f_k^{(1)} + h_k f_k^{(2)} + \frac{h_k^2}{2!} f_k^{(3)} + \frac{h_k^3}{3!} f_k^{(4)} + \frac{h_k^4}{4!} f_k^{(5)} + \frac{h_k^5}{5!} f_k^{(6)}(t_{k,2}), \tag{2.17}$$

where $x_k < t_{k,2} < x_{k+1}$.
From equations (2.15), (2.16) and (2.17), we derive

$$\begin{aligned} \left|S_k^{(4)} - f_k^{(4)}\right| &\le \frac{1}{10} h_k^2 w_6(f,h_k) + \frac{2}{10} h_k^2 w_6(f,h_k), \\ &\le \frac{3}{10} h_k^2 w_6(f,h_k). \end{aligned} \tag{2.18}$$

From equation (2.10), we get

$$\left|S_k^{(3)} - f_k^{(3)}\right| = \left|\frac{24}{h_k^3}\left[\mathbb{D}_k - \frac{h_k}{4}\mathbb{E}_k\right] - f_k^{(3)}\right|.$$

From equation (2.12) and (2.13), we obtain

$$\left|S_k^{(3)} - f_k^{(3)}\right| = \left|\frac{24}{h_k^3}\left[f_{k+1} - f_k - h_k f_k^{(1)} - \frac{h_k^2}{2!} f_k^{(2)} - \frac{h_k^5}{5!} f_k^{(5)} - \frac{h_k^6}{6!} S_k^{(6)}\right.\right.$$

$$- \frac{h_k}{4}\{f_{k+1}^{(1)} - f_k^{(1)} - h_k f_k^{(2)} - \frac{h_k^4}{4!} f_k^{(5)} - \frac{h_k^5}{5!} S_k^{(6)}\}\Big] - f_k^{(3)}\Big|. \quad (2.19)$$

From equations (2.16), (2.17) and (**??**), we establish

$$\left|S_k^{(3)} - f_k^{(3)}\right| \leq \frac{1}{30} h_k^3 w_6(f, h_k) + \frac{1}{20} h_k^3 w_6(f, h_k),$$
$$\leq \frac{1}{12} h_k^3 w_6(f, h_k). \quad (2.20)$$

Now

$$\left|S_k^{(5)} - f_k^{(5)}\right| = \left|f_k^{(5)} + h_k S_k^{(6)} - \{f_k^{(5)} + h_k f_k^{(6)}(t_{k,1})\}\right|,$$

where $x_k < t_{k,2} < x_{k+1}$.

$$\left|S_k^{(5)} - f_k^{(5)}\right| \leq h_k w_6(f, h_k).$$

$$\left|S_k^{(2)} - f_k^{(2)}\right| = \left|f_k^{(2)} + h_k S_k^{(3)} + \frac{h_k^2}{2!} S_k^{(4)} + \frac{h_k^3}{3!} f_k^{(5)} + \frac{h_k^4}{4!} S_k^{(6)} - \{f_k^{(2)} + h_k f_k^{(3)} + \frac{h_k^2}{2!} f_k^{(4)} + \frac{h_k^3}{3!} f_k^{(5)} + \frac{h_k^4}{4!} f_k^{(6)}(t_{k,1})\}\right|,$$

where $x_k < t_{k,1} < x_{k+1}$.
From equations (2.14), (2.18) and (2.20), we derive

$$\left|S_k^{(2)} - f_k^{(2)}\right| \leq \frac{11}{40} h_k^4 w_6(f, h_k).$$

From equations (2.14), (2.18) and (2.20), we also get

$$\left|S_k^{(1)} - f_k^{(1)}\right| \leq \frac{1}{10} h_k^5 w_6(f, h_k).$$

Applying same technique as above and from equations (2.14), (2.18) and (2.20), we finally establish

$$|S_k - f_k| \leq \frac{1}{36} h_k^6 w_6(f, h_k),$$

where

$$h = \underset{k}{Max} \quad h_k.$$

□

3 Conclusion

A g-spline of degree six for $(0, 1, 2, 5)$-type *LPI* exists uniquely. The study establishes the precise limits of potential error for the resulting g-spline, specifically evaluating these boundaries under the condition $f(x) \in C^6[0, 1]$.

References

[1] I. R. Ahmed, (0, 1, 4) lacunary interpolation by spline, *Anna. Sectio. Computa. Tom.*, **VI** (1984), 125-129.

[2] Th. Fawzy, Note on lacunary interpolation by splines II: (0, 2) interpolation, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **6** (1985), 117-123.

[3] Th. Fawzy, (0, 1, 3) Lacunary interpolation by G-splines, *Annales Univ. Sci., Budapest, Section Maths.*, **XXXIX** (1986), 63-67.

[4] Th. Fawzy, (0, 2, 4) Lacunary interpolation with g-splines, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **7** (1987), 75-78.

[5] Th. Fawzy, Lacunary interpolation by g-splines: (0, 2, 3) Case, *Studia Sci. Math. Hung.*, **23** (1988), 335-338.

[6] Th. Fawzy and F. Holail, A spline solution for the (0, m) lacunary interpolation problem, *Ann. Univ. Sci. Budapest. Sect. Comput.*, **13** (1992), 157-168.

[7] G. G. Lorentz, S. D. Riemenschneider, and K. Jetter, *Birkhoff interpolation*, Addison Wesley, Mass., U. K., 1983.

[8] A. Meir and A. Sharma, Lacunary interpolation by splines, *Siam J. Num. Anal.*, **10** (3) (1973), 433-442.

[9] R. S. Mishra and K. K. Mathur, Lacunary interpolation by splines: (0, 2, 3) and (0, 2, 4) Cases, *Acta Math. Acad. Sci. Hung.*, **36** (1980), 251-260.

[10] A. K. Pathak and P. Tiwari, A Comparative study on g-splines of (0,1,2,5)-LPI, *Bulletin of Calcutta Mathematical Society*, **113** (5) (2021), 461-468.

[11] A. K. Pathak and P. Tiwari, Birkhoff interpolation on non-uniformly distributed nodes on unit circle in complex plane, *Int. J. Research and Analytical Review*, **5** (3), (2018), 1699-1703.

[12] A. K. Pathak and P. Tiwari, Revisiting some Pl type Birkhoff interpolation problems, *J. of Indian Mathematical Society*, **86** (1-2), (2019), 118-125.

[13] A. K. Pathak and P. Tiwari, Incomplete Pl type interpolation on non-uniformly distributed nodes, *Journal of Scientific Research*, **64** (2), (2020), 212-215.

[14] A. K. Pathak and P. Tiwari, A new class of Pl type interpolation, *Applied Mathematics E-Notes*, **19** (2019), 413-418.

[15] A. K. Pathak and P. Tiwari, Incomplete Pl type interpolation on zeros of polynomials with complex coefficients, *Examples and Counterexamples*, **5** (2024), 100132.

[16] A. K. Pathak and P. Tiwari, Tiwari, A New Class of Pl type Interpolation in Complex Plane-II, *Jñānābha*, **53** (2) (2023), 260-264.

[17] I. J. Schoenberg, On the Ahlberg-Nilson extension of spline interpolation: The g-splines and their optimal properties, *J. Math. Anal. Appl.*, **21** (1968), 207-231.

[18] B. K. Swartz and R. S. Varga, A note on lacunary interpolation by splines, *Siam J. Num. Anal.*, **10** (1) (1973), 443-447.

[19] R. Srivastava, Quintic lacunary interpolation through g-splines, *Int. J. of World Res.*, **1** (XX) (2015), 24-30.

[20] A. K. Varma, Lacunary interpolation by splines, *Acta. Math. Acad. Sci. Hungar*, **31** (3-4) (1978), 183-192.