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A NOVEL GENERALIZATION OF THE MITTAG-LEFFLER FUNCTION: PROPERTIES AND INTEGRAL TRANSFORMS

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Abstract

In this paper, we introduce a new and more general form of the Mittag-Leffler function. This new generalization contains extra parameters that give the function greater flexibility and a richer analytical structure. After defining this generalized function, we study several of its basic and essential properties in detail. These include its convergence behaviour, different types of integral representations, and several important transforms such as the Laplace transform, Beta transform, Mellin transforms, and Whittaker transforms.
We also show how this new generalized Mittag-Leffler function is related to many other known special functions in the literature. In addition, we examine how this function can be used to solve various fractional differential and integral equations. Overall, the results of this work extend many existing forms of the Mittag-Leffler function and bring them together into a unified framework. This provides a strong base for further theoretical research as well as practical applications in areas such as mathematical physics and engineering.
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1 Introduction

The classical Mittag-Leffler function, depending on a single parameter, was first formulated by the Swedish mathematician Gösta Mittag-Leffler [8, 1903]. It is viewed as an important extension of the exponential function. Originally defined through a single-parameter power series, this function has now become a fundamental part of fractional calculus, especially for modelling physical, biological, and engineering processes that involve memory or hereditary effects.

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha\in\mathbb{C},\ \Re(\alpha)>0. \tag{1.1}$$

This series converges for $z\in\mathbb{C}$ i.e., it is an entire function of order $\rho=\frac{1}{\Re(\alpha)}$. Thus, for $\Re(\alpha)>0$, the function is entire in the complex plane. This function reduces to the exponential function for $\alpha=1$, and remains entire for all $z\in\mathbb{C}$, with order $\rho=\frac{1}{\Re(\alpha)}$. Wiman [20, 1905] proposed a two-parameter generalization of the Mittag-Leffler function, enhancing its flexibility in the analysis and solution of fractional differential equations:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)};\quad \Re(\alpha)>0,\quad \Re(\beta)>0;\quad \alpha,\beta\in\mathbb{C}. \tag{1.2}$$

This function too is entire in the complex plane and has been extensively studied for its role in linear and nonlinear fractional dynamics.

A major advancement in the generalization of the Mittag-Leffler function was achieved by Prabhakar [10, 1971], through the introduction of a three-parameter form involving the Pochhammer symbol $(\gamma)_n$

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}; \quad \alpha,\beta,\gamma \in \mathbb{C}; \quad \Re(\alpha),\Re(\beta),\Re(\gamma) > 0. \tag{1.3}$$

where $(\gamma)_n$ is the Pochhammer symbol Rainville [11, 1960].

$$(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}; \quad (\gamma)_0 = 1. \tag{1.4}$$

Shukla and Prajapati [15, 2007] introduced a more generalized form of the Mittag-Leffler function $E^{\gamma,q}_{\alpha,\beta}(z)$, defined for $\alpha,\beta,\gamma \in \mathbb{C}$; $\Re(\alpha),\Re(\beta),\Re(\gamma) > 0$; $q \in (0,1) \bigcup \mathbb{N}$ as follows

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)\, n!}. \tag{1.5}$$

Salim [13, 2009] defined the function $E^{\gamma,\delta}_{\alpha,\beta}(z)$ for $\alpha,\beta,\gamma,\delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$ as

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)(\delta)_n}. \tag{1.6}$$

Salim and Faraz [14, 2012] proposed a new form of the Mittag-Leffler function, which is defined as follows:

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}, \tag{1.7}$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, and $\Re(\delta) > 0$.
Khan and Ahmed [3, 2013] introduced two new generalized form of *MLF* defined as follows:

$$E^{\gamma,q}_{\alpha,\beta,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\delta)_n}; \quad \alpha,\beta,\gamma,\delta \in \mathbb{C}, \tag{1.8}$$

$$E^{\mu,\rho,\gamma,q}_{\alpha,\beta,\nu,\sigma,\delta,p}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)(\nu)_{\sigma} n(\delta)_{pn}}, \tag{1.9}$$

where $\alpha,\beta,\gamma,\mu,\nu,\rho,\sigma \in \mathbb{C}$; $p,q \in (0,1) \bigcup \mathbb{N}$; $q \leq \Re(\alpha)+p$ and $min\,[\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\delta),\Re(\mu),\Re(\nu),\Re(\rho),\Re(\sigma)] > 0$. This generalization allows for more precise control over convergence and growth behaviour and finds applications in fractional kinetics, anomalous relaxation, and fractional diffusion systems.
In this paper, we introduce and study a new definition of a generalized Mittag-Leffler function, which is given as follows:

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{r} (a_i)_{nq_i} z^n}{\Gamma(\alpha n+\beta) \prod\limits_{i=1}^{s} (b_i)_{np_i}}, \tag{1.10}$$

where $\alpha,\beta \in \mathbb{C}$; $a_i \in \mathbb{C}\,(\forall\, i = 1,2,..,r)$; $b_i \in \mathbb{C}\,(\forall\, i = 1,2,..,s)$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(a_i) > 0$, $\Re(b_i) > 0$ and $p_i\,(\forall i = 1,2,..,s) \in (0,1) \bigcup \mathbb{N}$; $q_i\,(\forall i = 1,2,..,r) \in (0,1) \bigcup \mathbb{N}$.
This definition extends all the previously introduced functions given in equations (1.1) to (1.9).

- Put $\beta = 1$, $r = 1 = s$, $a_1 = b_1$ and $p_1 = q_1$ in equation (1.10) it reduces to classical *MLF* $E_\alpha(z)$ equation (1.1).
- Put $r = 1 = s$, $a_1 = b_1$ and $p_1 = q_1$ in equation (1.10) it reduces to Wiman [20] *MLF* $E_{\alpha,\beta}(z)$ equation (1.2).
- Put $r = 1 = s$, and $p_1 = q_1 = 1 = b_1$in equation (1.10) it reduces to *MLF* defined by Prabhakar [10] $E^{\gamma}_{\alpha,\beta}(z)$ equation (1.3).
- Put $r = 1 = s$, and $p_1 = 1 = b_1$in equation (1.10) it reduces to *MLF* defined by Shukla and Prajapati [15] $E^{\gamma,q}_{\alpha,\beta}(z)$ equation (1.5).
- Put $r = 1 = s$, and $p_1 = 1 = q_1$in equation (1.10) it reduces to *MLF* defined by Salim [13] equation (1.6).
- Put $r = 1 = s$, in equation (1.10) it reduces to *MLF* defined by Salim [14] equation (1.7).
- Put $r = 2 = s$, in equation (1.10) it reduces to *MLF* defined by Khan and Ahmed [3] equation (1.9).

Throughout this study, we make use of the following well-known results to examine different properties and related formulas of the function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$.
For distribution of non-zero zeros of generalized Mittag-Leffler function, we may refer to Kumar and Pathan [4] and for generalized Mittag-Leffler function of two variables, we may refer to Kumar and Pathan [5], while for multivariable and multiparameter generalized Mittag-Leffler function, we may refer to Pathan and Kumar [6].

Beta Transforms: Euler (Beta) transforms of function $f(x)$ is defined as Sneddon [16, 1979].

$$B\{f(x);a,b\} = \int_0^1 x^{a-1}(1-x)^{b-1} f(x)dx. \tag{1.11}$$

Laplace Transforms: The Laplace transform of the function $f(x)$ can be found in the works of Rainville [11, 1960] and Sneddon [16, 1979].

$$L\{f(x);s\} = \int_0^\infty e^{-sx} f(x)dx. \tag{1.12}$$

Mellin -Transforms: The Mellin transform and its inverse for the function $f(x)$ are listed in the *Table of Mellin Transforms* [9, 1974].

$$M\{f(x);s\} = f*(s) = \int_0^\infty x^{s-1} f(x)\,dx \qquad \mathrm{Re}(s) > 0, \tag{1.13}$$

and the inverse Mellin-transform is given by

$$f(x) = M^{-1}\{f*(s);x\} = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} f*(s)\,x^{-s}dx. \tag{1.14}$$

Whittaker transform: Whittaker transform defined by Whittaker and Watson [19, 1962] as

$$\int_0^\infty e^{-\frac{t}{2}} t^{\nu-1} W_{\lambda,\rho}(t)\,dt = \frac{\Gamma\left(\frac{1}{2}+\rho+\nu\right)\Gamma\left(\frac{1}{2}-\rho+\nu\right)}{\Gamma(1-\lambda+\nu)}, \tag{1.15}$$

where Re $(\rho \pm \nu) > \frac{-1}{2}$ and $W_{\lambda,\rho}(t)$ is the Whittaker confluent hypergeometric function.

Wright Hypergeometric function: The Wright generalized hypergeometric function, as presented by Srivastava and Manocha [18, 1984] is defined as follows:

$${}_p\Psi_q\left[\begin{matrix}(a_i,A_i)_{(1,p)};\\(b_i,B_i)_{(1,q)};\end{matrix} z\right] = \sum_{n=0}^\infty \frac{\prod\limits_{i=1}^p \Gamma(a_i+nA_i)}{\prod\limits_{i=1}^q \Gamma(b_i+nB_i)}\frac{z^n}{n!}. \tag{1.16}$$

H-function: Fox's H-function, as formulated by Srivastava [17, 1972], is defined as follows:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p;A_p)\\(b_q;B_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int\limits_L \frac{\prod\limits_{j=1}^m \Gamma(b_j-B_js)\prod\limits_{j=1}^n \Gamma(1-a_j+A_js)}{\prod\limits_{j=m+1}^q \Gamma(1-b_j+B_js)\prod\limits_{j=n+1}^p \Gamma(a_j-A_js)} z^s ds. \tag{1.17}$$

2 Basic Properties of the Function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$:

If $\alpha,\ \beta \in \mathbb{C}$; $a_i \in \mathbb{C}\,(\forall\, i=1,2,..,r)$, $b_i \in \mathbb{C}\,(\forall\, i=1,2,..,s)$; $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(a_i)>0$, $\Re(b_i)>0$ and $p_i\,(\forall i=1,2,..,s) \in (0,1)\bigcup\mathbb{N}$; $q_i\,(\forall i=1,2,..,r)\in(0,1)\bigcup\mathbb{N}$ then,

$$E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z) = \beta E_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z), \tag{2.1}$$

$$E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z) = \frac{1}{\Gamma(\beta)} + \frac{\prod\limits_{i=1}^r (a_i)_{q_i}}{\prod\limits_{i=1}^s (b_i)_{p_i}} zE_{\alpha,\alpha+\beta;b_1+p_1,p_1;\ldots;b_s+p_s,p_s}^{a_1+q_1,q_1;\ldots;a_r+q_r,q_r}(z), \tag{2.2}$$

$$\begin{aligned}&E_{\alpha,\beta-\alpha;b_1,p_1;..;b_s,p_s}^{a_1,q_1;a_2,q_2;\ldots;a_r,q_r}(z) - E_{\alpha,\beta-\alpha;b_1,p_1;..;b_s,p_s}^{a_1-1,q_1;a_2,q_2;\ldots;a_r,q_r}(z)\\&= zq_1\frac{\prod\limits_{i=1}^r (a_i)_{q_i}}{\prod\limits_{i=1}^s (b_i)_{p_i}}\sum_{m=0}^\infty \frac{(m+1)(a_1+q_1)_{mq_1-1}\prod\limits_{i=2}^r (a_i+q_i)_{mq_i} z^m}{\prod\limits_{i=1}^s (b_i+p_i)_{mp_i}\Gamma(\alpha m+\beta)}.\end{aligned} \tag{2.3}$$

***Proof* (2.1):**

$$\text{R.H.S. of (2.1)} \quad = \sum_{n=0}^{\infty} \frac{\beta z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} + \sum_{n=0}^{\infty} \frac{\alpha n z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}}$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha n+\beta) z^n}{\Gamma(\alpha n+\beta+1)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} \qquad = \text{L.H.S. of (2.1).}$$

In particular

$$E_{\alpha,\beta;b_1,p_1}^{a_1,q_1}(z) = \beta E_{\alpha,\beta+1;b_1,p_1}^{a_1,q_1}(z) + z\alpha \frac{d}{dz} E_{\alpha,\beta+1;b_1,p_1}^{a_1,q_1}(z). \tag{2.4}$$

***Proof* (2.2):**

$$\text{L.H.S. of (2.2)} \quad = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}} = \frac{1}{\Gamma(\beta)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{nq_i}}{\prod_{i=1}^{s} (b_i)_{np_i}}$$

$$= \frac{1}{\Gamma(\beta)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r} (a_i)_{q_i} (a_i+q_i)_{(n-1)q_i}}{\prod_{i=1}^{s} (b_i)_{p_i} (b_i+p_i)_{(n-1)p_i}}$$

$$= \frac{1}{\Gamma(\beta)} + \frac{z\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m+(\alpha+\beta))} \frac{\prod_{i=1}^{r} (a_i+q_i)_{mq_i}}{\prod_{i=1}^{s} (b_i+p_i)_{mp_i}}, \qquad \text{putting } n-1=m$$

$$= \frac{1}{\Gamma(\beta)} + \frac{\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} z E_{\alpha,\alpha+\beta;b_1+p_1,p_1;\ldots;b_s+p_s,p_s}^{a_1+q_1,q_1;\ldots;a_r+q_r,q_r}(z) \qquad = \text{R.H.S of (2.2).}$$

In particular

$$E_{\alpha,\beta;b_1,p_1}^{a_1,q_1}(z) = \frac{1}{\Gamma(\beta)} + \frac{(a_1)_{q_1}}{b_1} z E_{\alpha,\alpha+\beta;b_1+1,p_1}^{a_1+q_1,q_1}(z). \tag{2.5}$$

***Proof* (2.3):**

$$\text{L.H.S. of (2.3)} \quad = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)} - \sum_{n=0}^{\infty} \frac{(a_1-1)_{nq_1} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)}$$

$$= \sum_{n=0}^{\infty} \frac{\left\{(a_1)_{nq_1} - (a_1-1)_{nq_1}\right\} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)} = \sum_{n=1}^{\infty} \frac{nq_1 (a_1)_{nq_1-1} \prod_{i=2}^{r} (a_i)_{nq_i} z^n}{\prod_{i=1}^{s} (b_i)_{np_i} \Gamma(\alpha n+\beta-\alpha)},$$

$$= \sum_{m=0}^{\infty} \frac{(m+1) q_1 (a_1)_{mq_1+q_1-1} \prod_{i=2}^{r} (a_i)_{(m+1)q_i} z^{m+1}}{\Gamma(\alpha m+\beta) \prod_{i=1}^{s} (b_i)_{(m+1)p_i}}, \quad \text{putting } n-1=m$$

$$= \sum_{m=0}^{\infty} \frac{z(m+1) q_1 (a_1)_{q_1} (a_1+q_1)_{mq_1-1} \prod_{i=2}^{r} (a_i)_{q_i} (a_i+q_i)_{mq_i} z^m}{\prod_{i=1}^{s} (b_i)_{p_i} (b_i+p_i)_{mp_i} \Gamma(\alpha m+\beta)}$$

$$= zq_1 \frac{\prod_{i=1}^{r} (a_i)_{q_i}}{\prod_{i=1}^{s} (b_i)_{p_i}} \sum_{m=0}^{\infty} \frac{(m+1)(a_1+q_1)_{mq_1-1} \prod_{i=2}^{r} (a_i+q_i)_{mq_i} z^m}{\prod_{i=1}^{s} (b_i+p_i)_{mp_i} \Gamma(\alpha m+\beta)} \qquad = \text{R.H.S. of (2.3).}$$

In particular

$$E_{\alpha,\beta-\alpha;b_1}^{a_1,q_1}(z) - E_{\alpha,\beta-\alpha;b_1}^{a_1-1;q_1} = q_1 z \frac{(a_1)_{q_1}}{b_1} \sum_{m=0}^{\infty} \frac{(m+1)(a_1)_{mq_1-1} z^m}{\Gamma(\alpha m+\beta)(b_1+1)_m}. \tag{2.6}$$

3 Basic Derivatives Properties of the function $E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z)$:

If $\alpha, \beta \in \mathbb{C}$; $a_i \in \mathbb{C}(\forall\, i = 1, 2, .., r)$; $b_i \in \mathbb{C}(\forall\, i = 1, 2, .., s)$; $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(a_i) > 0$, $\Re(b_i) > 0$ and $p_i(\forall i = 1, 2, .., s) \in (0, 1) \bigcup \mathbb{N}$;$q_i(\forall i = 1, 2, .., r) \in (0, 1) \bigcup \mathbb{N}$, also $m \in \mathbb{N}$ then,

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z) = \frac{(1)_m \prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}} E_{(\alpha,\alpha m+\beta),(b_i+mp_i;p_i);(1,1)}^{(a_i+mq_i,q_i);(m+1,1)}(z), \tag{3.1}$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha}), \tag{3.2}$$

$$\left(\frac{d}{dz}\right)^m E_{\frac{m}{k},\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}\left(z^{\frac{m}{k}}\right) = \frac{\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)} \sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\,\Gamma\left(\frac{nm}{k}+1\right) z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)\,\Gamma\left(\frac{nm}{k}-m+1\right)}. \tag{3.3}$$

***Proof* (3.1):**

$$\text{L.H.S. of (3.1)} = \left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(z) = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i} z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=m}^{\infty} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i} n(n-1)\ldots(n-m+1) z^{n-m}}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}} = \sum_{n=m}^{\infty} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i} n! z^{n-m}}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}(n-m)!},$$

$$= \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{r}(a_i)_{(k+m)q_i}(1)_{m+k} z^k}{\Gamma(\alpha(k+m)+\beta)\prod\limits_{i=1}^{s}(b_i)_{(k+m)p_i}(1)_k}, \quad \text{putting } n-m=k$$

$$= \frac{(1)_m \prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}} \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{r}(a_i+mq_i)_{kq_i}(1+m)_k z^k}{\Gamma(\alpha k+m\alpha+\beta)\prod\limits_{i=1}^{s}(b_i+mp_i)_{kp_i}(1)_k}$$

$$= \frac{(1)_m \prod\limits_{i=1}^{r}(a_i)_{mq_i}}{\prod\limits_{i=1}^{s}(b_i)_{mp_i}} E_{(\alpha,\alpha m+\beta),(b_i+mp_i;p_i);(1,1)}^{(a_i+mq_i,q_i);(m+1,1)}(z) \qquad = \text{R.H.S. of (3.1)}.$$

In particular

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta;b_1}^{a_1,q_1}(z) = \frac{(1)_m (a_1)_{mq_1}}{(b_1)_m} E_{(\alpha,\alpha m+\beta),(b_1+m;1);(1,1)}^{(a_1+mq_1,q_1);(m+1,1)}(z). \tag{3.4}$$

***Proof* (3.2):**

$$\text{L.H.S. of (3.2)} = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{w^n z^{n\alpha+\beta-1}}{\Gamma(\alpha n+\beta)} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=0}^{\infty} \frac{(n\alpha+\beta-1)(n\alpha+\beta-2)\ldots(n\alpha+\beta-m)\, w^n z^{n\alpha+\beta-m-1}}{(n\alpha+\beta-1)(n\alpha+\beta-2)\ldots(n\alpha+\beta-m)\,\Gamma(\alpha n+\beta-m)} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= z^{\beta-m-1} E_{\alpha,\beta-m;b_1,p_1;\ldots;b_s,p_s}^{a_1,q_1;\ldots;a_r,q_r}(wz^{\alpha}) \qquad = \text{R.H.S. of (3.2)}.$$

In particular

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta;b_1,1}^{a_1,q_1}(wz^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m;b_1,1}^{a_1,q_1}(wz^{\alpha}). \tag{3.5}$$

***Proof* (3.3):**

$$\text{L.H.S. of (3.3)} = \left(\frac{d}{dz}\right)^m \sum_{n=0}^{\infty} \frac{z^{\frac{nm}{k}}}{\Gamma\left(\frac{nm}{k}+\beta\right)} \frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=1}^{\infty} \frac{\frac{nm}{k}\left(\frac{nm}{k}-1\right)\left(\frac{nm}{k}-2\right)...\left(\frac{nm}{k}-m+1\right)}{\Gamma\left(\frac{nm}{k}+\beta\right)} \frac{\prod_{i=1}^{r}(a_i)_{q_i}(a_i+q_i)_{nq_i-q_i} z^{\left(\frac{nm}{k}-m\right)}}{\prod_{i=1}^{s}(b_i)_{p_i}(b_i+p_i)_{np_i-p_i}}$$

$$= \frac{\prod_{i=1}^{s}\Gamma(b_i)}{\prod_{i=1}^{r}\Gamma(a_i)} \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma\left(\frac{nm}{k}+1\right) z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\prod_{i=1}^{s}\Gamma(b_i+np_i)\Gamma\left(\frac{nm}{k}-m+1\right)} \qquad = \text{R.H. S. of (3.3).}$$

In particular

$$\left(\frac{d}{dz}\right)^{m} E^{a_1,1}_{\frac{m}{k},\beta}\left(z^{\frac{m}{k}}\right) = \sum_{n=1}^{\infty} \frac{(a_1)_n\Gamma\left(\frac{nm}{k}+1\right) z^{\left(\frac{nm}{k}-m\right)}}{\Gamma\left(\frac{nm}{k}+\beta\right)\Gamma\left(\frac{nm}{k}-m+1\right)}. \tag{3.6}$$

4 Basic Integral Properties of the function $E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(z)$

If $\alpha,\beta \in \mathbb{C}$; $a_i \in \mathbb{C}(\forall\, i=1,2,..,r)$, $b_i \in \mathbb{C}(\forall\, i=1,2,..,s)$; $\Re(\alpha)>0,\Re(\beta)>0,\Re(a_i)>0$, $\Re(b_i)>0$ and $p_i(\forall i=1,2,..,s) \in (0,1)\bigcup \mathbb{N}$; $q_i(\forall i=1,2,..,r) \in (0,1)\bigcup \mathbb{N}$, then,

$$\frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(zx^{\alpha})\,dx = E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+\mu;b_1,p_1;...;b_s,p_s}(z), \tag{4.1}$$

$$\frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1} E^{a_i,q_i}_{\alpha,\beta,b_i,p_i}\left[\lambda(s-t)^{\alpha}\right] ds = (x-t)^{\eta+\beta-1} E^{a_i,q_i}_{\alpha,\beta+\eta,b_i,p_i}\left[\lambda(x-t)^{\alpha}\right], \tag{4.2}$$

$$\int_0^z t^{\beta-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(wt^{\alpha})\,dt = z^{\beta} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+1;b_1,p_1;...;b_s,p_s}(wz^{\alpha}). \tag{4.3}$$

***Proof* (4.1):**

$$\text{L.H.S. of (4.1)} \qquad = \frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta;b_1,p_1;...;b_s,p_s}(zx^{\alpha})\,dx$$

$$= \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}} \int_0^1 x^{n\alpha+\beta-1}(1-x)^{\mu-1}dx$$

$$= \frac{1}{\Gamma(\mu)}\sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}} \frac{\Gamma(n\alpha+\beta)\Gamma(\mu)}{\Gamma(n\alpha+\beta+\mu)} = E^{a_1,q_1;...;a_r,q_r}_{\alpha,\beta+\mu;b_1,p_1;...;b_s,p_s}(z) \qquad = \text{R.H.S. of (4.1).}$$

In particular

$$\frac{1}{\Gamma(\mu)}\int_0^1 x^{\beta-1}(1-x)^{\mu-1} E^{a_1}_{\alpha,\beta;b_1}(zx^{\alpha})\,dx = E^{a_1}_{\alpha,\beta+\mu;b_1}(z). \tag{4.4}$$

***Proof* (4.2):**

$$\text{L.H.S. of (4.2)} \qquad = \frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1} E^{a_i,q_i}_{\alpha,\beta,b_i,p_i}\left[\lambda(s-t)^{\alpha}\right] ds.$$

Put $\quad u=\frac{s-t}{x-t}, \quad 1-u=\frac{x-s}{x-t}, \quad ds=(x-t)\,du.$ Therefore,

$$\text{L.H.S. of (4.2)} = \frac{1}{\Gamma(\eta)}\int_0^1 (1-u)^{\eta-1}(x-t)^{\eta-1}u^{\beta-1}(x-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{\lambda^n u^{n\alpha}(x-t)^{n\alpha}}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}(x-t)\,du$$

$$= \frac{1}{\Gamma(\eta)}\int_0^1 (1-u)^{\eta-1}(x-t)^{\eta+n\alpha+\beta-1}u^{\beta+n\alpha-1}\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\lambda^n du$$

$$= \frac{(x-t)^{\eta+\beta-1}}{\Gamma(\eta)}\frac{\Gamma(\eta)\Gamma(\beta+n\alpha)}{\Gamma(\eta+\beta+n\alpha)}\sum_{n=0}^{\infty}\frac{(x-t)^{n\alpha}}{\Gamma(\alpha n+\beta)}\frac{\prod_{i=1}^{r}(a_i)_{nq_i}}{\prod_{i=1}^{s}(b_i)_{np_i}}\lambda^n$$

$$= (x-t)^{\eta+\beta-1} E^{a_i,q_i}_{\alpha,\beta+\eta,b_i,p_i}\left[\lambda(x-t)^{\alpha}\right] \qquad = \text{R.H.S. of (4.2).}$$

In particular

$$\frac{1}{\Gamma(\eta)}\int_t^x (x-s)^{\eta-1}(s-t)^{\beta-1} E^{a_1}_{\alpha,\beta,b_1}\left[\lambda(s-t)^{\alpha}\right] ds = (x-t)^{\eta+\beta-1} E^{a_1}_{\alpha,\beta+\eta,b_1}\left[\lambda(x-t)^{\alpha}\right]. \tag{4.5}$$

***Proof* (4.3):**

$$\text{L.H.S. of (4.3)} \quad = \int_0^z t^{\beta-1} E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}\left(wt^{\alpha}\right) dt = \int_0^z t^{\beta+n\alpha-1}\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(\alpha n+\beta)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}}dt$$

$$= \sum_{n=0}^{\infty}\frac{z^{\beta+n\alpha}w^n}{(\beta+n\alpha)\Gamma(\alpha n+\beta)}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}}{\prod\limits_{i=1}^{s}(b_i)_{np_i}} = z^{\beta}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta+1;b_1,p_1;\ldots;b_s,p_s}\left(wz^{\alpha}\right) = \text{R.H.S. of (4.3).}$$

In particular

$$\int_0^z t^{\beta-1}E^{a_1}_{\alpha,\beta;b_1}\left(wt^{\alpha}\right)dt = z^{\beta}E^{a_1}_{\alpha,\beta+1;b_1}\left(wz^{\alpha}\right). \tag{4.6}$$

5 Relation of $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z)$ with other function

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \frac{1}{\Gamma(\beta)}\,{}_{q_1+\ldots+q_r+1}F_{k+p_1+..+p_s}\left[\begin{matrix}\Delta(q_i,a_i)_{(1,r)},(1,1)\\ \Delta(k,\beta)\,\Delta(p_i,b_i)_{(1,s)}\end{matrix};\ \frac{q_1^{q_1}q_2^{q_2}\ldots q_r^{q_r}z}{k^kp_1^{p_1}p_2^{p_2}\ldots p_s^{p_s}}\right]. \tag{5.1}$$

***Proof*:** Using (1.10) with $\alpha = k$, $p_i, q_i \in \mathbb{N}$ then we have

$$\text{L.H.S. of (5.1)} = \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(kn+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}} = \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(a_1)_{nq_1}(a_2)_{nq_2}\ldots(a_r)_{nq_r}(1)_n}{(\beta)_{kn}(b_1)_{np_1}(b_2)_{np_2}\ldots(b_s)_{np_s}}\frac{z^n}{n!},$$

$$= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod\limits_{j=1}^{q_1}\left(\frac{a_1+j-1}{q_1}\right)_n\prod\limits_{j=1}^{q_2}\left(\frac{a_2+j-1}{q_2}\right)_n\ldots\prod\limits_{j=1}^{q_r}\left(\frac{a_r+j-1}{q_r}\right)_n(1)_n}{\prod\limits_{j=1}^{k}\left(\frac{\beta+j-1}{k}\right)\prod\limits_{j=1}^{p_1}\left(\frac{b_1+j-1}{p_1}\right)_n\prod\limits_{j=1}^{p_2}\left(\frac{b_2+j-1}{p_2}\right)_n\ldots\prod\limits_{j=1}^{p_s}\left(\frac{b_s+j-1}{p_s}\right)_n}\frac{\left(\frac{q_1^{q_1}q_2^{q_2}..q_r^{q_r}z}{k^kp_1^{p_1}p_2^{p_2}..p_s^{p_s}}\right)^n}{n!}$$

$$= \frac{1}{\Gamma(\beta)}\,{}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}\left[\begin{matrix}\Delta(q_i,a_i)_{(1,r)},(1,1)\\ \Delta(k,\beta)\,\Delta(p_i,b_i)_{(1,s)}\end{matrix};\ \frac{q_1^{q_1}q_2^{q_2}..q_r^{q_r}z}{k^kp_1^{p_1}p_2^{p_2}..p_s^{p_s}}\right].$$

where $\Delta(q_r,a_r)$ is q_r-tuple, $\frac{a_r}{q_r}, \frac{a_r+1}{q_r}, \ldots, \frac{a_r+q_r-1}{q_r}$,and in the same way, $\Delta(p_s,b_s)$ is p_s-tuple, $\frac{b_s}{p_s}, \frac{b_s+1}{p_s}, \ldots, \frac{b_s+p_s-1}{p_s}$. All remaining terms are defined as tuples in a similar manner. This expression provides the required hypergeometric representation.

The convergence condition for generalized function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$:

- If $(q_1+\ldots+q_r+1) \le (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ convergence for finite z.
- If $(q_1+\ldots+q_r+1) = (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ convergence when $|z|<1$ and divergence when $|z|>1$, moreover it is absolutely convergent on the circle $|z|=1$, if
$$\mathrm{Re}\left(\sum_{j=0}^{k}\frac{\beta+j-1}{k}\sum_{j=0}^{p_1}\frac{b_1+j-1}{p_1}\ldots\sum_{j=0}^{p_s}\frac{b_s+j-1}{p_s} - \sum_{j=0}^{q_1}\frac{a_1+j-1}{q_1}\sum_{j=0}^{q_2}\frac{a_2+j-1}{q_2}\ldots\sum_{j=0}^{q_r}\frac{a_r+j-1}{q_r}\right) > 0.$$
- If $(q_1+\ldots+q_r+1) > (k+p_1+\ldots+p_s)$, the function ${}_{q_1+\ldots+q_r+1}F_{k+p_1+\ldots+p_s}$ is divergence for finite $|z| \neq 0$

6 Integral Transforms

In this section, we study several important integral transforms of the generalized MittagLeffler function, including the Laplace transform, the Beta (Euler) transform, and the Whittaker transform of the generalized form *MLF* $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z)$

6.1 Mellin-Barnes integral representation of $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(z)$

If $\alpha \in R^{+};\beta \in \mathbb{C};\ a_i \in \mathbb{C}\,(\forall\, i = 1,2,..,r),\ b_i \in \mathbb{C}\,(\forall\, i = 1,2,..,s);\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \Re(a_i) > 0,\ \Re(b_i) > 0;\ p_i, q_i > 0$ and $p_i\,(\forall i = 1,2,..,s);\ q_i\,(\forall i = 1,2,..,r) \in (0,1)\bigcup \mathbb{N}$ then the function $E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(z)$ is represented by Mellin-Barnes integral as

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(z) = \frac{\prod\limits_{i=1}^{s}\Gamma(b_i)}{2\pi i\prod\limits_{i=1}^{r}\Gamma(a_i)}\int\limits_{L}\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}ds. \tag{6.1}$$

Here, $|\arg(z)| < 1$; the integration contour runs from $-i\,\infty$ to $i\,\infty$, and is suitably so as to separate the poles of the integrand at $s = -n\,\forall\, n \in \mathbb{N}_0$ (on the left) from those at $s = \frac{a_i+n}{q_i} \in \mathbb{N}_0$ (also on the right)

***Proof*:** We shall evaluate the integral on the right-hand side of (6.1) by applying the method of residues at the poles $s = 0,-1,-2,...$

$$\frac{\prod\limits_{i=1}^{r}\Gamma(a_i)}{\prod\limits_{i=1}^{s}\Gamma(b_i)}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(z) = \frac{\prod\limits_{i=1}^{r}\Gamma(a_i)}{\prod\limits_{i=1}^{s}\Gamma(b_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\,z^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)} = \sum_{n=0}^{\infty}\frac{(-1)^{-s}\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}$$

$$= \sum_{n=0}^{\infty}\lim_{s\to -n}\frac{\pi(s+n)}{\sin(s\pi)}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)} = \sum_{n=0}^{\infty}\operatorname*{Re\,s}_{s=-n}\left[\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}\right]$$

$$= \frac{1}{2\pi i}\int\limits_{L}\frac{\Gamma(1-s)\Gamma(s)\prod\limits_{i=1}^{r}\Gamma(a_i - sq_i)(-z)^{-s}}{\Gamma(\beta - s\alpha)\prod\limits_{i=1}^{s}\Gamma(b_i - sp_i)}ds.$$

This is complete proof.

6.2 Beta Transform

Using (1.11)

$$\int_0^1 x^{\rho-1}(1-x)^{\sigma-1}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(x^{\lambda}z)dx = \frac{\Gamma(\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma(n\lambda+\rho)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)}\frac{z^n}{n!}. \tag{6.2}$$

***Proof*:**

$$\text{L.H.S.} \quad = \int_0^1 x^{\rho-1}(1-x)^{\sigma-1}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}(x^{\lambda}z)^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}dx$$

$$= \sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}\Gamma(n\lambda+\rho)\Gamma(\sigma)\,z^n}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}(b_i)_{np_i}} = \frac{\Gamma(\sigma)\Gamma(\rho)}{\Gamma(\sigma+\rho)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}(\rho)_{n\lambda}z^n}{\Gamma(\alpha n+\beta)(\rho+\sigma)_{n\lambda}\prod\limits_{i=1}^{s}(b_i)_{np_i}}$$

$$= \frac{\Gamma(\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i)}{\prod\limits_{i=1}^{r}\Gamma(a_i)}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}\Gamma(a_i+nq_i)\Gamma(n\lambda+\rho)\Gamma(n+1)}{\Gamma(\alpha n+\beta)\Gamma(n\lambda+\rho+\sigma)\prod\limits_{i=1}^{s}\Gamma(b_i+np_i)}\frac{z^n}{n!} \qquad = \text{R.H.S. of (6.2).}$$

6.3 Laplace Transform

$$\int_0^{\infty} z^{\rho-1}e^{-sz}E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(xz^{\lambda})\,dz = \frac{\Gamma(\rho)}{s^{\rho}}E^{a_1,q_1;\ldots;a_r,q_r;\rho,\lambda}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}\left(\frac{x}{s^{\lambda}}\right). \tag{6.3}$$

***Proof*:**

$$\text{L.H.S.} \quad = \int_0^{\infty} z^{\rho-1}e^{-sz}\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{r}(a_i)_{nq_i}(xz^{\lambda})^n}{\Gamma(\alpha n+\beta)\prod\limits_{i=1}^{s}(b_i)_{np_i}}dz$$

$$= \int_0^\infty z^{\lambda n+\rho-1} e^{-sz} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} x^n}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} dz = \sum_{n=0}^{\infty} \frac{x^n \prod_{i=1}^{r} (a_i)_{nq_i}}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} \frac{(\rho+\lambda n-1)!}{s^{\rho+\lambda n}}$$

$$= \frac{\Gamma(\rho)}{s^\rho} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} (\rho)_{\lambda n}}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} \left(\frac{x}{s^\lambda}\right)^n = \frac{\Gamma(\rho)}{s^\rho} E^{a_1,q_1;\ldots;a_r,q_r;\rho,\lambda}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s} \left(\frac{x}{s^\lambda}\right) \qquad = \text{R.H.S. of (6.3)}.$$

6.4 Mellin Transform

Using (1.13) we have

$$M\left[E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-wz);s\right] = f*(s) = \int_0^\infty z^{s-1} E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-wz) dz.$$

From equation (6.1) we get

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(-wz) = \frac{\prod_{i=1}^{s} \Gamma(b_i)}{2\pi i \prod_{i=1}^{r} \Gamma(a_i)} \int_L \frac{\Gamma(1-s)\Gamma(s) \prod_{i=1}^{r} \Gamma(a_i - sq_i) w^{-s} z^{-s}}{\Gamma(\beta - s\alpha) \prod_{i=1}^{s} \Gamma(b_i - sp_i)} ds,$$

$$E^{a_i,q_i}_{\alpha,\beta;b_i,p_i}(-z) = M^{-1}\{f*(s);z\} = \frac{\prod_{i=1}^{s} \Gamma(b_i)}{2\pi i \prod_{i=1}^{r} \Gamma(a_i)} \int_L f*(s) z^{-s} ds,$$

$$f*(s) = \frac{\Gamma(1-s)\Gamma(s) \prod_{i=1}^{r} \Gamma(a_i - sq_i)(w)^{-s}}{\Gamma(\beta - s\alpha) \prod_{i=1}^{s} \Gamma(b_i - sp_i)}. \tag{6.4}$$

6.5 Whittaker Transform

Using (1.15)

$$\int_0^\infty e^{-\frac{\phi t}{2}} t^{\xi-1} W_{\lambda,\rho}(\phi t) E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;..;b_s,p_s}(\omega t^\eta) dt = \frac{\prod_{i=1}^{s} \Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r} \Gamma(a_i)} {}_{r+3}\Psi_{s+2}\left[\begin{matrix}(a_i,q_i)_{(1,r)}; \left(\frac{1}{2}\pm\rho+\xi,\eta\right)(1,1); \\ (\beta,\alpha);(b_i,p_i)_{(1,s)}(1-\lambda+\xi,\eta);\end{matrix} \frac{\omega}{\phi^\eta}\right]. \tag{6.5}$$

***Proof*:** Substituting $\phi t = \nu$

$$\text{L.H.S. of (6.5)} = \int_0^\infty e^{-\frac{\nu}{2}} \left(\frac{\nu}{\phi}\right)^{\xi-1} W_{\lambda,\rho}(\nu) \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} (\omega)^n}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} (b_i)_{np_i}} \left(\frac{\nu}{\phi}\right)^{n\eta} \frac{d\nu}{\phi},$$

$$= \frac{\prod_{i=1}^{s} \Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r} \Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i+nq_i)}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} \Gamma(b_i+np_i)} \left(\frac{\omega}{\phi^\eta}\right)^n \int_0^\infty e^{-\frac{\nu}{2}} (\nu)^{n\eta+\xi-1} W_{\lambda,\rho}(\nu) d\nu,$$

$$= \frac{\prod_{i=1}^{s} \Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r} \Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i+nq_i) \Gamma\left(\frac{1}{2}\pm\rho+n\eta+\xi\right) \Gamma(n+1)}{\Gamma(\alpha n+\beta) \prod_{i=1}^{s} \Gamma(b_i+np_i) \Gamma(1-\lambda+n\eta+\xi)} \frac{\left(\frac{\omega}{\phi^\eta}\right)^n}{n!},$$

$$= \frac{\prod_{i=1}^{s} \Gamma(b_i)}{\phi^\xi \prod_{i=1}^{r} \Gamma(a_i)} {}_{r+3}\Psi_{s+2}\left[\begin{matrix}(a_i,q_i)_{(1,r)}; \left(\frac{1}{2}\pm\rho+\xi,\eta\right)(1,1); \\ (\beta,\alpha);(b_i,p_i)_{(1,s)}(1-\lambda+\xi,\eta);\end{matrix} \frac{\omega}{\phi^\eta}\right] \frac{\left(\frac{\omega}{\phi^\eta}\right)^n}{n!} \qquad = \text{R.H.S. of (6.5)}.$$

7 Relationship with some known function

7.1 Relationship with Wright hypergeometric function

If the condition be satisfied, then (1.10) can be written as

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \frac{\Gamma(b_i)}{\Gamma(a_i)} {}_{r+1}\Psi_{s+1}\left[\begin{matrix}(q_i,a_i)_{(1,r)};(1,1); \\ (\beta,\alpha);(p_i,b_i)_{(1,s)};\end{matrix} z\right]. \tag{7.1}$$

***Proof*:**

$$\text{L.H.S. of (7.1)} \quad = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{nq_i} z^n}{\Gamma(\alpha n + \beta) \prod_{i=1}^{s} (b_i)_{np_i}} = \frac{\Gamma(b_i)}{\Gamma(a_i)} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i + nq_i) \Gamma(n+1) z^n}{\Gamma(\alpha n + \beta) \prod_{i=1}^{s} \Gamma(b_i + np_i) n!}$$

$$= \frac{\Gamma(b_i)}{\Gamma(a_i)} {}_{r+1}\Psi_{s+1} \left[\begin{matrix} (q_i, a_i)_{(1,r)}; (1,1); \\ (\beta, \alpha); (p_i, b_i)_{(1,s)}; \end{matrix} z \right] \qquad = \text{R.H.S. of (7.1).}$$

7.2 Relationship with Foxs H-function

Using (6.1) and (1.17), we establish the relationship

$$E^{a_1,q_1;\ldots;a_r,q_r}_{\alpha,\beta;b_1,p_1;\ldots;b_s,p_s}(z) = \frac{\prod_{i=1}^{s} \Gamma(b_i)}{2\pi i \prod_{i=1}^{r} \Gamma(a_i)} H^{1,r+1}_{r+1,s+2} \left[z \middle| \begin{matrix} (0,1); (1-a_i, q_i)_{(1,r)}; \\ (0,1); (1-\beta, \alpha)(1-b, p_i)_{(1,s)} \end{matrix} \right]. \tag{7.2}$$

8 Conclusion

The generalized Mittag-Leffler function can be used in many scientific problems because it can describe systems that have memory or exhibit slow and fast-varying behaviour. It naturally appears in the solutions of fractional differential equations, which are used to model real-life processes, such as anomalous diffusion in physics, viscoelastic materials, electrical circuits with non-ideal capacitors, and biological systems with long-term memory.

References

[1] C. Fox, The G and H Functions as Symmetrical Fourier Kernels, *Transactions of the American Mathematical Society*, **98**(3) (1961), 395429. https://doi.org/10.2307/1993339

[2] H.J. Haubold, A. M. Mathai and R. K. Saxena, Mittag-Leffler Functions and Their Applications, *Journal of Applied Mathematics*, **3** (2011), 1-51. https://doi.org/10.1155/2011/298628

[3] A.M. Khan and S. Ahmed, *On some properties of the generalized Mittag-Leffler function*, Springer Plus 2, **337** (2013), https://doi.org/10.1186/2193-1801-2-337

[4] H. Kumar and M. A. Pathan, On the distribution of non-zero zeros of generalized Mittag-Leffler functions, *Int. J. Eng. Res. Appl.,* **6**(2016), 66-71.

[5] H. Kumar, Real zeros of generalized Mittag-Leffler function of two variables through integral transforms, *Jñānābha*, **48** (2) (2018), 32-45.

[6] M. A. Pathan and H. Kumar, Generalized multivariable Cauchy residue theorem and non-zero zeros of multivariable and multiparameter generalized Mittag-Leffler functions, *Southeast Asian bulletin of mathematics,* **43** (2019), 733-749.

[7] A.M. Mathai, R.K. Saxena and H.J. Haubold, *The H-Function Theory and Applications*, Springer New York, Dordrecht Heidelberg, London, 2010.

[8] G. Mittag-Leffler, Sur la Nouvelle Fonction E(x), *Comptes Rendus de l Academie des Sciences Paris*, **137** (1903), 554-558.

[9] Fritz Oberthettinger, *Tables of Mellin Transforms*, Springer-Verlag Berlin, Heidelberg, New York, 1974.

[10] T.R. Prabhakar, A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, *Yokohama Mathematical Journal*, **19** (1971), 7-15.

[11] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Pub Co., Bronx, New York, 1971.

[12] M. Saigo and A. A. Kilbas, On Mittag-Leffler type function and applications, *Integral Transforms and Special Functions*, **7**(12) (1998), 97-112, https://doi.org/10.1080/10652469808819189

[13] T.O. Salim, Some properties relating to the generalized Mittag-Leffler function *Advances and Applications in Mathematical Analysis*, **4** (2009), 21-30.

[14] T.O. Salim, and Ahmad W. Faraz, A generalization of integral operator associated with fractional calculus Mittag-Leffler function, *Journal of Fractional Calculus and Applications*, **3**(5) (2012), 1-13,DOI: 10.21608/jfca.2012.283765

[15] A.K. Shukla and J.C. Prajapati, On a generalised Mittag-Leffler function and its properties, *Journal of Mathematical Analysis and Applications,* **336**(12) (2007), 797-811.

[16] I.N. Sneddon, *The use of integral Transforms*, Tata McGraw-Hill, New Delhi 1979.

[17] H.M. Srivastava, A contour integral involving Foxs H-function, *Indian Journal of Mathematics,* **14**(1) (1972), 6.
[18] H.M. Srivastava and H.L. Manocha, *A treatise on generating functions* United Kingdom: Ellis Horwood (Halsted Press), 1984.
[19] E.T. Whittaker and G.N. Watson, *A course of modern analysis* Cambridge University, 1962.
[20] A. Wiman, Über den Fundamentalsatz in der Teorie der Funktionen $E_a(x)$, *Acta Math.,* **29**(1905), 191-201.