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INVERSION OF AN INTEGRAL INVOLVING A PRODUCT OF GENERALIZED HYPERGEOMETRIC FUNCTION AND ALEPH-FUNCTION AS KERNEL

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Abstract

The solution for a convolution integral equation of Fredholm type, whose kernel involves a product of generalized hypergeometric function and κ-Function has been determined by Pathan,Kumar and Chandel (2016), Chandel-Kumar (2020). In the present paper, our aim is to derive an exact solution of the convolution integral equation of Fredholm type due to Chaurasia-Saxena (2008) and Goyal-Salim (1998). A number of results due to Chandel-Kumar (2023) and Srivastave (1985) follow as special cases by specializing and the parameters of the generalized hypergeometric function and the Aleph-Function ($\aleph$-function).

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1 Introduction

The Aleph function, introduced by Südland *et al.* [17] (also see Srivastava [15] and Sachan *et al.* [16]), is defined in terms of Mellin Barnes type integrals [10]:

$$\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x] = \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left\{x \,\middle|\, \begin{matrix} (a_j,\alpha_j)_{1,n}\,; [\tau_i\,(a_{ji},\alpha_{ji})]_{n+1}\,, p_i; r \\ (b_j,\beta_j)_{1,m}\,; [\tau_i\,(b_{ji},\beta_{ji})]_{m+1}\,, q_i; r \end{matrix}\right\}$$
$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{p_i,q_i,\tau_i;r}(s)x^s ds, \tag{1.1}$$

where $\omega = \sqrt{-1}$ and

$$\Omega^{m,n}_{p_i,q_i,\tau_i;r}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\tau_i\sum_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\sum_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)}.$$

For the nature of contour L in (1.1), the conditions of existence, the convergence and other details of this function, see [14].

In this paper we will also use the following notation [6,9]:

$$F[\mu,x] = {}_PF_Q\left[\begin{matrix} A_P \\ B_Q \end{matrix} : \mu x\right] = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{p}(A_j)_r\,\mu^r x^r}{\prod_{j=1}^{Q}(B_j)_r\,(r)!}. \tag{1.2}$$

The main object of this paper is to evaluate an exact solution of the following convolution integral equation of Fredholm type [3,5]:

$$\int_0^\infty y^{-1}u\left(\frac{x}{y}\right)\phi(y)dy = g(x), \quad (x>0), \tag{1.3}$$

where g is a prescribed function, ϕ is an unknown function to be determined and the kernel $u(x)$ is given by

$$u(x) = F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x], \tag{1.4}$$

where $\aleph[*]$ and $F[*]$ are given by equations (1.1) and (1.2) respectively.
Our method of solution of integral eq. (1.3) with kernel $u(x)$ given by (1.4) would depend on the theory of Melline transform defined by

$$\Phi(s) = M\{\phi(x); s\} = \int_0^\infty x^{s-1}\phi(x)dx, \tag{1.5}$$

provided that the integral exists.

2 Mellin Transform of $u(x)$.

In order to solve the integral equation, we shall require the following result contained in
Lemma 2.1: Let $U(s) = \mathcal{M}\{u(x); s\}$ where $u(x)$ is defined by (1.4), then

$$\begin{aligned} U(s) = \sum_{r=0}^{\infty} & \frac{\prod_{j=1}^{P}(A_j)_r \mu^r}{\prod_{j=1}^{Q}(B_j)_r (r)!} \\ & \times \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j(r+s))\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j(r+s))}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}(r+s))\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}(r+s))}, \end{aligned} \tag{2.1}$$

provided that

$$-\min_{1\le j\le m}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right) < \operatorname{Re}(r+s) < \min_{1\le j\le n}\operatorname{Re}\left(\frac{1-a_j}{\alpha_j}\right); r = 0, 1, 2, \ldots.$$

Proof.

$$U(S) = \mathcal{M}\left\{F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]; s\right\} = \int_0^\infty x^{s-1}F[\mu x]\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]dx. \tag{2.2}$$

Expending the generalized hypergeometric function as given by (1.2) changing order of summation and integration, we get

$$U(S) = \sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_r \mu^r}{\prod_{j=1}^{Q}(B_j)_r (r)!}\mathcal{M}\left\{x^r\aleph^{m,n}_{p_i,q_i,\tau_i;r}[x]; s\right\}. \tag{2.3}$$

Now applying the following known formulae [4, p. 307] and [14, p. 15]

$$\mathcal{M}\left\{x^\rho\phi(x); s\right\} = \Phi(s+\rho) \tag{2.4}$$

$$\mathcal{M}\left\{\aleph^{m,n}_{p_i,q_i,\tau_i;r}[ax]\right\} = \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}-\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}+\alpha_{ji}s)}a^{-s} \tag{2.5}$$

provided that

$$-\min_{1\le j\le m}\operatorname{Re}\left(\frac{b_j}{\beta_j}\right) < \operatorname{Re}(s) < \min_{1\le j\le n}\operatorname{Re}\left(\frac{1-a_j}{\alpha_j}\right),$$

we arrive at the desired result. □

3 Solution of the Integral Equation

The solution of the convolution integral equation of Fredholm type (1.3) is given in the

Theorem 3.1. *Let the Mellin transforms $\Phi(s), G(s)$ and $U(s) \neq 0$ of the functions $\Phi(s), G(s)$ and $U(S) \neq 0$ of the functions $\phi(x), g(x)$ and $u(x)$ defined by (1.4) exist and are analytic in some infinite strip $\gamma < \operatorname{Re}(s)\delta$ of the complex s-plane. Also assume that for a fixed $c \in (\gamma, \delta)$, $u^*(x)$ be defined by*

$$u^*(x) = \mathcal{M}^{-1}\left\{U^*(s); x\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}U^*(s)ds \tag{3.1}$$

where,

$$U^*(s) = \left[\rho^k\frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{\infty}\frac{\prod_{j=1}^{P}(A_j)_r \mu^r}{\prod_{j=1}^{Q}(B_j)_r (r)!}\right.$$

$$\times \frac{\prod_{j=1}^{m}\Gamma\{b_j+\beta_j(r+s+\rho k+\sigma)\}\prod_{j=1}^{n}\Gamma\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\{1-b_{ji}-\beta_{ji}(r+s+\rho k+\sigma)\}\prod_{j=n+1}^{p_i}\Gamma\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\}}\Bigg]^{-1}, \tag{3.2}$$

provided that the following sets of conditions hold

(i) $|\arg x| < \frac{1}{2}\Omega\pi$, *where* $\Omega = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ji} > 0$,

(ii) $\rho \neq 0$ *and* k *is a non-negative integer,*

(iii) $-\min_{1\le j\le m}\mathrm{Re}\left(\frac{b_j}{\beta_j}\right) < \mathrm{Re}(r+s+\rho k+\sigma) < \min_{1\le j\le n}\mathrm{Re}\left(\frac{1-a_j}{\alpha_j}\right)$,

(iv) $P \le Q[P = Q+1$ *and* $|\mu| < 1]$, *No one of* $B_1, B_2, \ldots, B_Q$ *is zero or a negative integer.*

Then the integral equation (1.3) *has its solution given by*

$$\phi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_y\right)^k\{y^\sigma g(y)\}\,dy. \tag{3.3}$$

provided that the integral exists.

Proof. Applying the convolution theorem for Mellin transfoems [4, p.307], we find from (1.3), that

$$U(S)\Phi(S) = G(s), \tag{3.4}$$

where $\Phi(s)$ and $G(S)$ are Mellin transforms of $\phi(x)$ and $g(x)$ and $U(s)$ is given in (2.1).

Replacing s by $s+\rho k+\sigma$ in (3.4), we get

$$F(s+\rho k+\sigma) = U^*(s)\left\{\rho^k\left(-\frac{s+\rho k}{\rho}\right)_k\right\}G(s+\rho k+\sigma). \tag{3.5}$$

Now making an appeal to the formula given by Shrivastava [13, p. 14]

$$\mathcal{M}\left\{\left(x^{h+1}D_x\right)^n f(x);s\right\} = h^n\left(-\frac{s+hn}{h}\right)_n F(s+hn)$$

$$(h \neq 0, n \quad \text{a non-negative integer }),$$

we obtain

$$F(s+\rho k+\sigma) = U^*(s)\mathcal{M}\left[\left(y^{\rho+1}D_x\right)^k\{y^\sigma g(y)\};s\right]. \tag{3.6}$$

Further applying (2.4), we derive

$$\mathcal{M}\left\{x^{\rho k+\sigma}\phi(x);s\right\} = \mathcal{M}\left\{\int_0^\infty y^{-1}u^*\left(\frac{x}{y}\right)\left(y^{\rho+1}D_x\right)^k\{y^\rho g(y)\}\,dy;s\right\}. \tag{3.7}$$

Inverting both sides of (3.8) by using Mellin inversion theorem [4, p.307], we establish the desired theorem. □

4 Deductions

(i) If we take $P = 0, Q = 0, A_1 = -N, A_2 = a+N-1, \mu = -1/b$ in equation (1.4) and reduce generalized hypergeometric function to Bessel polynomial, $Y_N(x,a,b)$ [2], we get

Corollary 4.1. *Under the hypothesis of Theorem, the integral equation*

$$\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\phi(y)dy = g(x); \quad (x>0) \tag{4.1}$$

where the kernel

$$u_1(x) = Y_N(x,a,b)\aleph_{p_i,q_i,\tau_i;r}^{m,n}[x] \tag{4.2}$$

has its solution $\phi(x)$ *given by*

$$\emptyset(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u_1\left(\frac{x}{y}\right)\left(y^{\rho+1}Dy\right)^k\{y^\sigma g(y)\}\,dy, \tag{4.3}$$

provided that the integral exists and $u_1^*(x)$ *is the Melline inverse transform of*

$$U^*(s) = \left[\rho^k\frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{N}\frac{(-N,r)(a+N-1,r)(-1/b)^r}{(r)!}\right.$$

$$\left.\times \frac{\prod_{j=1}^{m}\Gamma\{b_j+\beta_j(r+s+\rho k+\sigma)\}\prod_{j=1}^{n}\Gamma\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\}}{\sum_{i=1}^{r}\tau_i\prod_{j=m+1}^{q_i}\Gamma\{1-b_{ji}-\beta_{ji}(r+s+\rho k+\sigma)\}\prod_{j=n+1}^{p_i}\Gamma\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\}}\right]^{-1}, \tag{4.4}$$

provided that set of conditions (i), (ii), and (iii) of the main Theorem are satisfied.

(ii) On taking $p = 3, Q = 2$, *and replacing* $A_1 = -N, A_2 = N+1, A_3 = \eta, B_1 = 1, B_2 = \gamma$; *the generalized hypergeometric function is reduced to Rice's polynomial* [12, p.287] *and we have*

Corollary 4.2. *Under the hypothesis surrounding the Theorem, the integral equation*

$$\int_0^\infty y^{-1} u_1\left(\frac{x}{y}\right)\psi(y)dy = g(x); (x>0) \tag{4.5}$$

where, the kernel

$$u_1(x) = H_N(\eta,\gamma,x)\aleph^{m,n}_{p_i,q_i,\tau_i;}[x] \tag{4.6}$$

has its solution $\psi(x)$ given by

$$\psi(x) = x^{-\rho k-\sigma}\int_0^\infty y^{-1}u_2^*\left(\frac{x}{y}\right)\left(y^{\rho+1}Dy\right)^k\{y^\sigma g(y)\}\,dy, \tag{4.7}$$

provided that the integral exists and $U_1^*(x)$ is the Melline inverse transform of

$$U_2^*(s) = \left[\rho^k \frac{\Gamma\left(\frac{-s}{\rho}\right)}{\Gamma\left\{-\frac{(s+\rho k)}{\rho}\right\}}\sum_{r=0}^{N}\frac{(-N,r)(N+1,r)(\eta,r)}{(1,r)(\gamma,r)(r)!}\right.$$

$$\left.\times\frac{\prod_{j=1}^m \Gamma\{b_j+\beta_j(r+s+\rho k+\sigma)\}\prod_{j=1}^n\Gamma\{1-a_j-\alpha_j(r+s+\rho k+\sigma)\}}{\sum_{i=1}^r \tau_i\prod_{j=m+1}^{q_i}\Gamma\{1-b_{ji}-\beta_{ji}(r++\rho k+\sigma)\}\prod_{j=n+1}^{p_i}\Gamma\{a_{ji}+\alpha_{ji}(r+s+\rho k+\sigma)\}}\right]^{-1},$$

provided that the set of conditions (i), (ii) and (iii) stated with the main Theorem are satisfied.
It may be pointed out here that the generalized hypergeometric function is very general in nature and it reduces into a number of special functions and orthogonal polynomials. Also the H-function is a very general function and has for its particular cases a number of important special functions [8].

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