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CONSTANT OF MOTION FOR THE KEPLER PROBLEM IN A UNIFORM FIELD CONNECTED WITH WHITNEY NUMBERS AND BERNOULLI POLYNOMIALS

J. D. Bulnes[1], H. Kumar[2], H. N. Núñez-Yépez[3], A. L. Salas-Brito[4] and J. López-Bonilla[5]

[1]Departamento de Ciencias Exatas e Tecnologia, Universidade Federal do Amapá, Rod. Juscelino Kubitschek, Jardin Marco Zero, Macapá, AP, Brasil-68 903-419

[2]Department of Mathematics D.A.-V, Post Graduate College, Kanpur, Uttar Pradesh, India-208001

[3]Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apdo. Postal 55-534, Iztapalapa, CDMX-México-CP 09340

[4]Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana-Azcapotzalco, CDMX, México-CP 02200

[5]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista, CDMX, México-CP 07738

Email: bulnes@uifap.br, palhemant2007@rediffmail.com; hemantkumar_kn03@csjmu.ac.in, nyhn@xanum.uam.mx, asb@correo.azc.uam.mx, jlopezb@ipn.mx

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Abstract

We consider the Kepler problem with an additional constant force, and we employ the Laplace-Runge-Lenz vector to construct a constant of motion. Finally, an interesting relation of this constant motion with Whitney numbers and Bernoulli polynomials is derived for computation of the Kepler problem.

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1 Introduction

The Lagrangian for the Kepler problem has the expression:

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{k}{r}, \quad r = \sqrt{x^2 + y^2}, \quad \vec{r} = x\hat{\imath} + y\hat{\jmath}, \tag{1.1}$$

because the orbit is plane due the conservation of the angular momentum whose magnitude is given by:

$$l = m(x\dot{y} - y\dot{x}) = mr^2\dot{\varphi}. \tag{1.2}$$

If now due to (1.1), in L we apply the following transformation [17, pages 99-100]:

$$\tilde{x} = x + \varepsilon_1, \quad \tilde{y} = y + \varepsilon_2, \quad \tilde{t} = t, \tag{1.3}$$

where ε_1 and ε_2 are constants $\ll 1$, using (1.2) and (1.3), we obtain:

$$\tilde{L} - L = -\varepsilon_1 \frac{kx}{r^3} - \varepsilon_2 \frac{ky}{r^3} = -\varepsilon_1 \frac{k}{r^2}\cos\varphi - \varepsilon_2 \frac{k}{r^2}\sin\varphi = -\varepsilon_1 \frac{mk}{l}\dot{\varphi}\cos\varphi - \varepsilon_2 \frac{mk}{l}\dot{\varphi}\sin\varphi,$$

that is:

$$\tilde{L} = L + \frac{d}{dt}\left[\frac{mk}{l}\left(-\varepsilon_1 \sin\varphi + \varepsilon_2 \cos\varphi\right)\right], \tag{1.4}$$

hence (1.4) is a dynamical symmetry of L [17], and therefore it is possible to use the Noether's theorem, in the Lanczos approach [8, 9, 10, 11, 22] to deduce the conservation law associated with (1.3) [2].

The Lanczos technique asks to employ in (1.1) the transformation (1.3) with $\varepsilon_1(t)$ and $\varepsilon_2(t)$ as new degrees of freedom, then:

$$\tilde{L} - L = \dot{\varepsilon}_1 m\dot{x} - \varepsilon_1 \frac{kx}{r^3} + \dot{\varepsilon}_2 m\dot{y} - \varepsilon_2 \frac{ky}{r^3}, \tag{1.5}$$

and the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{\varepsilon}_j}\right) - \frac{\partial \tilde{L}}{\partial \varepsilon_j} = 0, \quad j = 1, 2, \tag{1.6}$$

imply the relations:

$$\frac{d}{dt}(m\dot{x}) + \frac{kx}{r^3} = \frac{d}{dt}\left(m\dot{x} + \frac{mk}{l}\sin\varphi\right) = 0, \quad \frac{d}{dt}(m\dot{y}) + \frac{ky}{r^3} = \frac{d}{dt}\left(m\dot{y} - \frac{mk}{l}\cos\varphi\right) = 0,$$

(1.5) and (1.6) generate the following two constants of motion:

$$ml\dot{x} + mk\sin\varphi = -c_2, \quad ml\dot{y} - mk\cos\varphi = c_1. \tag{1.7}$$

It is easy see that (1.7) express the conservation of the Laplace-Runge-Lenz vector [2, 6, 7, 21], in fact:

$$\vec{p}\times\vec{L} = m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})\times l\hat{k} = ml(\dot{y}\hat{\imath} - \dot{x}\hat{\jmath}), \quad -mk\frac{\vec{r}}{r} = -mk(\cos\varphi\hat{\imath} + \sin\varphi\hat{\jmath}),$$

therefore:

$$\begin{aligned}\vec{A} := \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r} =& (ml\dot{y} - mk\cos\varphi)\hat{\imath} - (ml\dot{x} + mk\sin\varphi)\hat{\jmath}\\ =& \ c_1\hat{\imath} + c_2\hat{\jmath}.\end{aligned} \tag{1.8}$$

The Lanczos method gives a simple process to apply the Noether's theorem [3, 5, 16, 18, 19], thus in this case it was possible to determine the conservation law associated with the dynamical symmetry (1.4) for the Kepler problem.

The Kepler problem with an additional constant force $\vec{F}$ is considered in Sec. 2 to show the existence of the constant of motion:

$$\Omega := \vec{A}\cdot\vec{F} + \frac{m}{2}\left[F^2r^2 - (\vec{r}\cdot\vec{F})^2\right], \tag{1.9}$$

which coincides with the quantity obtained in [4, 12] via another approach.

2 Kepler problem with a constant force

In this situation the equations of motion are given by:

$$\frac{d\vec{p}}{dt} = -\frac{k}{r^3}\vec{r} + \vec{F}, \quad \frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}, \tag{2.1}$$

and we know that $\vec{A}$ is a constant vector if $\vec{F} = \vec{0}$. If $\vec{F} \neq \vec{0}$ then it is natural first to investigate the evolution of the Laplace-Runge-Lenz vector [14], thus from (1.8) and (2.1):

$$\frac{d}{dt}\vec{A} = 2(\vec{p}\cdot\vec{F})\vec{r} - (\vec{r}\cdot\vec{F})\vec{p} - (\vec{r}\cdot\vec{p})\vec{F}, \tag{2.2}$$

The (2.2) implies the relation:

$$\frac{d}{dt}(\vec{A}\cdot\vec{F}) = (\vec{r}\cdot\vec{F})(\vec{p}\cdot\vec{F}) - F^2(\vec{p}\cdot\vec{r}), \quad F = |\vec{F}|. \tag{2.3}$$

On the other hand, we have the interesting property:

$$\frac{d}{dt}\frac{m}{2}\left[(\vec{r}\cdot\vec{F})^2 - r^2F^2\right] = (\vec{r}\cdot\vec{F})(\vec{p}\cdot\vec{F}) - mF^2r\dot{r}, \quad \vec{r}\cdot\vec{p} = mr\dot{r} \tag{2.4}$$

whose application in (2.3) generates the constant (1.9):

$$\Omega = \vec{A}\cdot\vec{F} + \frac{m}{2}\left[F^2r^2 - (\vec{r}\cdot\vec{F})^2\right] = \vec{A}\cdot\vec{F} + \frac{m}{2}F^2r^2\sin^2\beta, \quad \beta = \sphericalangle(\vec{r},\vec{F}), \tag{2.5}$$

deduced in [16, 17]. If we employ $\vec{F} = F\hat{k}$ and we observe that the energy is given by:

$$H = \frac{p^2}{2m} - \frac{k}{r} - Fz, \tag{2.6}$$

then (2.5) it is equivalent to:

$$\left(2H + \frac{k}{r}\right)z - mr\dot{r}\dot{z} + \frac{F}{2}\left(3z^2 + r^2\right) = \text{ Constant of motion.} \tag{2.7}$$

To find the symmetry associated with the conservation of Ω, is an open problem.

3 Computation of Equation (2.7) due to Whitney numbers and Bernoulli polynomials

If in (2.7), we consider the constant of motion is λ, λ is finite and greater than 0 , and then we write that in the form

$$\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2+r^2\right)\right\}t=\lambda t. \tag{3.1}$$

Clearly, the coefficients of t in (3.1) give the equation (2.7).

Now to compute the equation (2.7) due to Whitney numbers and Bernoulli polynomials, we introduce r-Whitney numbers $w_{m,r}(n,p)$ and $W_{m,r}(n,p)$ of first kind and second kind, respectively, given by [1, 15]

$$m^n(x)_n=\sum_{p=0}^{n}w_{m,r}(n,p)(mx+r)^p;\ \text{where, the basis } =\{(mx+r)^p:p=0,\dots,n\}, \tag{3.2}$$

and

$$(mx+r)^n=\sum_{p=0}^{n}W_{m,r}(n,p)m^p x_p;\ \text{where, the basis } =\{m^p x_p:p=0,\dots,n\}. \tag{3.3}$$

The exponential generating functions of the numbers $w_{m,r}(n,p)$ given in (3.2) and $W_{m,r}(n,p)$ given in (3.3), are represented by [15]

$$\sum_{n\geq p}w_{m,r}(n,p)\frac{t^n}{n!}=\frac{1}{p!}\left(\frac{\ln(1+mt)}{m}\right)^p(1+mt)^{-\frac{r}{m}}, \tag{3.4}$$

and

$$\sum_{n\geq p}W_{m,r}(n,p)\frac{t^n}{n!}=\frac{1}{p!}\left(\frac{\text{ext}[mt]-1}{m}\right)^p\text{ext}[rt], \tag{3.5}$$

respectively.

Further, the generalized Bernoulli polynomials are defined by the exponential generating function [13, 20]

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!}=\left(\frac{t}{\text{ext}[t]-1}\right)^{\alpha}\exp[xt]. \tag{3.6}$$

In the formula (3.6), putting $x=0$ we get a generating formula for generalized Bernoulli numbers of order $\alpha, B_n^{(\alpha)}$, also called Nörlund polynomials, given by

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}\frac{t^n}{n!}=\left(\frac{t}{\text{ext}[t]-1}\right)^{\alpha},\ \text{provided that } B_n^{(1)}=B_n. \tag{3.7}$$

The recurrence relation for $B_n^{(\alpha)}$ is given by

$$B_n^{(\alpha+1)}=\left(1-\frac{n}{\alpha}\right)B_n^{(\alpha)}-nB_{n-1}^{(\alpha)}. \tag{3.8}$$

The summation of generalized Bernoulli numbers of order $\alpha, B_n^{(\alpha)}$, is given by

$$B_n^{(\alpha)}=\sum_{q=0}^{n}(-1)^q\frac{q!}{(q+n)!}\binom{\alpha+n}{n-q}\binom{\alpha+q-1}{q}\sum_{j=0}^{q}(-1)^{q-j}\binom{q}{j}j^{q+n}. \tag{3.9}$$

Theorem 3.1. *Let in the Eqn.* (3.1), *for* $\mu\neq 0, \lambda=\left(\frac{r-s}{\mu}\right)$ *such that* $r-s>0$, *and* $p\geq 1$, *then following relation is occurred*

$$\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\frac{r-s}{\mu}-\frac{Fr^2}{2}\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)=\sum_{q=0}^{n}\frac{1}{\mu^n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q). \tag{3.10}$$

Proof. Consider the Eqn. (3.1) and $\lambda=\left(\frac{r-s}{\mu}\right)$ such that $r-s>0$ to get that in the form

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2+r^2\right)\right\}t\right]=\exp\left[\left(\frac{r-s}{\mu}\right)t\right]. \tag{3.11}$$

Then, the Eqn. (3.11) may be written by

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2 \quad +r^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p$$

$$=\left(\frac{1}{\exp[t]-1}\right)^p\exp\left[\left(\frac{r}{\mu}\right)t\right]\left(\frac{t^p}{\exp\left[\left(\frac{s}{\mu}\right)t\right]}\right). \tag{3.12}$$

Further rearranging the equation (3.12) to get that given by

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z} \quad +\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\left(\frac{1}{\left(\frac{\exp[t]-1}{\mu}\right)}\right)^p\exp\left[r\left(\frac{t}{\mu}\right)\right]\left(\frac{1}{\mu^p p!}\frac{\left(\ln\left(1+\mu\left(\frac{\exp[t]-1}{\mu}\right)\right)\right)^p}{\left(1+\mu\left(\frac{\exp[t]-1}{\mu}\right)\right)^{\frac{s}{\mu}}}\right). \tag{3.13}$$

In the formula (3.13), on defining (3.4) we find

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\exp\left[r\left(\frac{t}{\mu}\right)\right]\sum_{q=p}^{\infty}w_{\mu,s}(q,p)\frac{\left(\frac{\exp[t]-1}{\mu}\right)^{q-p}}{q!}. \tag{3.14}$$

Again, in Eqn. (3.14), making some manipulations to find that

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z} \quad +\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,S}(q+p,p)\frac{q!}{(q+p)!}\exp\left[r\left(\frac{t}{\mu}\right)\right]\left(\frac{\exp\left[\mu\frac{t}{\mu}\right]-1}{\mu}\right)^q\frac{1}{q!} \tag{3.15}$$

Now in the equation (3.15) on defining the Whitney number of second kind, we find

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\left(\frac{t}{\exp[t]-1}\right)^p\exp\left[\frac{Fr^2}{2}t\right]$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,s}(q+p,p)\frac{q!}{(q+p)!}\sum_{n\geq q}W_{\mu,r}(n,q)\frac{t^n}{n!}. \tag{3.16}$$

Again, in the equation (3.16) on applying the formula (3.6), we get

$$\exp\left[\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}t\right]\sum_{n=0}^{\infty}B_n^{(p)}\left(\frac{Fr^2}{2}\right)\frac{t^n}{n!}$$

$$=p!\sum_{q=0}^{\infty}w_{\mu,s}(q+p,p)\frac{q!}{(q+p)!}\sum_{n\geq q}W_{\mu,r}(n,q)\frac{\frac{t^n}{\mu^n}}{n!}. \tag{3.17}$$

Then in the equation (3.17), on making series manipulations we find

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(3z^2\right)\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{1}{\mu^n}\sum_{q=0}^{n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q)\frac{t^n}{n!}. \tag{3.18}$$

Again then, equating both sides coefficients of $\frac{t^n}{n!}, \forall n\geq 0$ and $p\geq 1$ in equation (3.18), we get

$$\sum_{q=0}^{n}\frac{n!}{(n-q)!q!}\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\frac{F}{2}\left(\ \left(3z^2\right)\right\}^q B_{n-q}^{(p)}\left(\frac{Fr^2}{2}\right)$$

$$=\sum_{q=0}^{n}\frac{1}{\mu^n}\frac{p!q!}{(q+p)!}w_{\mu,s}(q+p,p)W_{\mu,r}(n,q). \tag{3.19}$$

Finally, making an appeal to the equations (3.1) and (3.19) with statement of the Theorem 3.1, we derive the result (3.10). □

Corollary 3.1. *In the Theorem* 3.1 *putting* $n = 1, p = 1$ *and* $F = 0$*, there exists a first integral of motion for a dynamical system in the form of Whitney numbers*

$$\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\right\}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]+\frac{1}{2}. \tag{3.20}$$

Proof. Making an appeal to the Theorem 3.1 and putting $n = 1, p = 1$ and $F = 0$, we get

$$B_1^{(1)}(0)+\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}+\ \frac{F}{2}\left(3z^2\right)\right\}B_0^{(1)}(0)$$
$$=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]. \tag{3.21}$$

In Eqn. (3.7) on setting $\alpha = 1$, and again on using (3.8) and (3.9), we get the values of the generalized Bernoulli's numbers as

$$B_0^{(1)}(0)=1, B_1^{(1)}(0)=-\frac{1}{2}, B_1^{(2)}(0)=-1,\ldots,$$

and then putting these values in (3.21) we find

$$-\frac{1}{2}+\left\{\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}\right\}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)\right]. \tag{3.22}$$

The Eqn. (3.22) gives the dynamical system (3.20) in the form of the Whitney numbers. □

4 Conclusion

The expression

$$\left(2H+\frac{k}{r}\right)z-mr\dot{r}\dot{z}=\frac{1}{\mu}\left[w_{\mu,s}(1,1)W_{\mu,r}(1,0)+\frac{1}{2}w_{\mu,s}(2,1)W_{\mu,r}(1,1)+\frac{\mu}{2}\right]$$

represents a first integral of motion for a dynamical system, commonly arising in the study of classical mechanics, such as the motion of a particle in a central force field (e.g., Kepler's problem) or specific coupled oscillator systems. This equation establishes a conserved relationship between the particle's spatial coordinates and axis, its velocities and the constants of the system, such as total energy and mass.

References

[1] H. Belbachir and I. E. Bousbaa, Translated Whitney and r-Whitney numbers: a combinatorial approach, *J. Integer Seq.,* **16**(8) (2013), article 13.8.6.

[2] J. D. Bulnes, H. Kumar, T. Kim and J. López-Bonilla, Laplace-Runge-Lenz vector via Noether's theorem, *Advanced Studies in Contemporary Maths.,* **36**(1) (2026), 53-62.

[3] Nina Byers, Emmy Noether's discovery of the deep connection between symmetries and conservation laws, *Proc. Symp. Heritage E. Noether, Bar-Ilan University, Tel Aviv, Israel,* Dec. 2-4, 1996. arXiv:physics/9807044v2 [physics.hist-ph] 23 Sep 1998.

[4] O. J. Campuzano-Cardona, H. N. Nez-Yépez, A. L. Salas-Brito and G. I. Sánchez-Ortiz, Constant of motion for the hydrogen atom in an external field, *Eur. J. Phys.,* **16**(5) (1995), 220-222.

[5] J. Hanc, S. Tuleja and M. Hancova, Symmetries and conservation laws: Consequences of Noether's theorem, *Am. J. Phys.,* **72**(4) (2004), 428-435.

[6] L. N. Hand and J. D. Finch, *Analytical mechanics,* Cambridge University Press, 1998.

[7] H. Kaplan, The Runge-Lenz vector as an "extra" constant of the motion, *Am. J. Phys.,* **54**(2) (1986), 157-161.

[8] P. Lam-Estrada, J. López-Bonilla and R. López-Vázquez, Lanczos approach to Noether's theorem, *Bull. Soc. for Mathematical Services and Standards (India),* **3**(3) (2014), 1-4.

[9] P. Lam-Estrada, J. López-Bonilla and R. López-Vázquez, On the Noether's theorem, *Prespacetime J.,* **6**(4) (2015), 322-325.

[10] C. Lanczos, *The variational principles of mechanics,* University of Toronto Press, 1970, Chap. 11.

[11] C. Lanczos, Emmy Noether and the calculus of variations, *Bull. Inst. Math. Appl.,* **9**(8) (1973), 253-258.

[12] J. López-Bonilla, J. Morales and G. Ovando, On a constant of motion for the Stark problem in its classical version, *Eur. J. Phys.,* **19**(4) (1998), 401-402.

[13] Y. L. Luke, *The Special Functions and Their Approximations,* Vol. I, Academic Press, New York, London, 1969.

[14] R. P. Martínez y Romero, H. N. Núñez-Yépez and A. L. Salas-Brito, Two quantum extra symmetries and associated potentials, *J. Math. Phys.,* **67** (2026), 032102.
[15] M. Mihoubi and M. Tiachachat, Some applications of the r-Whitney numbers, *C. R. Acad. Sci. Paris,* Ser. I, **352** (2014), 965-969.
[16] J. Read and N. J. *Teh [Eds.], The Philosophy and Physics of Noether's Theorems,* Cambridge University Press, 2022.
[17] A. Rojo and A. Bloch, *The principle of least action. History and Physics,* Cambridge University Press, 2018.
[18] D. E. Rowe, Emmy Noether and her theorems, *Ann. Phys. (Berlin),* **536** (2024), 2300479.
[19] W. Sarlet and F. Cantrijn, Generalizations of Noether's theorem in classical mechanics, *SIAM Review,* **23**(4) (1981), 467-494.
[20] H. M. Srivastava, An explicit formula for the generalized Bernoulli polynomials, *J. Math. Anal. Appl.,* **130** (1988), 509-513.
[21] G. F. Torres del Castillo, *An introduction to Hamiltonian mechanics,* Birkhuser, Boston, MA, 2018.
[22] J. Yaljá Montiel-Pérez, J. López-Bonilla, R. López-Vázquez and S. Vidal-Beltrán, Some applications of Noether's theorem, *World Scientific News,* **106** (2018), 69-80.