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NEW CLASS OF GENERALIZED HERMITE POLYNOMIALS INVOLVING CUBIC HERMITE POLYNOMIALS

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Abstract

The great success of the theory of an exponential generating function for generalized special polynomials has stimulated the development of a corresponding theory of Hermite polynomials of one and more variables. We use this type of a standard generating function to derive certain results for example, Rodrigues' formula, integral representation, matrix representations and differential equations not attempted by any researcher so far. Application of Rodrigues' formula and the integral representation attempted herein produce first few polynomials of cubic Hermite polynomials and some inequalities.

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1 Introduction

The quantum physics is analysed by classical Hermite polynomials, $H_n(x)\ \forall n = 0, 1, 2, 3, \ldots$; ([1], see also in [10, Section 105, p.189] and [11, Eqn. (16), p.74]) and defined through Rodrigues' formula

$$H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n \left\{\exp\left(-x^2\right)\right\}, D_x^n \equiv \frac{d^n}{dx^n}. \tag{1.1}$$

Then in 1934, Bell [2] presented a generalization of (1.1) as a class of polynomials in form of exponential function defined by

$$\xi_n(x,t;r) = \exp\left(-xt^r\right) D_t^n \exp\left(xt^r\right), \xi_{-n-1}(x,t;r) = 0; D_t^n \equiv \frac{d^n}{dt^n}. \tag{1.2}$$

Here in (1.2), r is a fixed integer $> 0, n$ is an arbitrary integer ≥ 0, while x and t are independent variables.

Later on, Gould and Hopper ([6, p.52], see also in [11, Eqn. (12), p. 77]) introduced operational formulas connected with generalization of (1.1) in the form

$$H_n^r(x,a;p) = (-1)^n x^{-a} \exp\left(px^r\right) D_x^n \left\{x^a \exp\left(-px^r\right)\right\}. \tag{1.3}$$

M. Lahiri [7] studied the generalized Hermite polynomials $H_{n,m,v}(x)$, $\forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, $v > 0, x$ and t being independent variables, through generating function

$$\sum_{n=0}^{\infty} H_{n,m,v}(x)\frac{t^n}{n!} = \exp\left(vxt - t^m\right). \tag{1.4}$$

Motivated by above results, we introduce a standard exponential generating function of a class of poly-Hermite polynomials denoted by $\mathcal{H}_n^m(x)\ \forall x \in \mathbb{R}, n = 0, 1, 2, \ldots; m$ being a positive integer, x and t being independent variables, defined by

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp\left((-1)^{m+1}\binom{m}{0}t^m + (-1)^m\binom{m}{1}t^{m-1}x + (-1)^{m-1}\binom{m}{2}t^{m-2}x^2\right.$$
$$\left.+(-1)^{m-2}\binom{m}{3}t^{m-3}x^3 + \cdots + \binom{m}{m-1}tx^{m-1}\right)$$
$$= \exp\left(-\left[\binom{m}{0}(-t)^m + \binom{m}{1}(-t)^{m-1}x + \binom{m}{2}(-t)^{m-2}x^2\right.\right.$$
$$\left.\left.+\binom{m}{3}(-t)^{m-3}x^3 + \cdots + \binom{m}{m-1}(-t)x^{m-1} + x^m\right]\right)\exp(x^m)$$

where,

$$\binom{m}{r} = \begin{cases} \frac{m(m-1)\ldots(m-r+1)}{r!}, 0 \le r \le m; \\ 0, r > m. \end{cases} \tag{1.5}$$

Thus more suitably a new class of standard generating function (1.5) can be written as

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp(x^m)\exp(-(x-t)^m). \tag{1.6}$$

This class of functions is "extraordinary" because it generalizes the classical Hermite polynomials (where $m = 2$) to higher-order exponents. For $m = 2$, the formula (1.6) reduces to the standard Rodrigues' formula (1.1) as

$$\mathcal{H}_n^2(x) = H_n(x) = (-1)^n \exp\left(x^2\right) D_x^n\left\{\exp\left(-x^2\right)\right\}, D_x^n \equiv \frac{d^n}{dx^n} \tag{1.7}$$

whereas for $m > 2$, these polynomials often appear in problems involving non-Gaussian statistics, higher order differential equations, and quantum optics.

2 Rodrigues' formula

Theorem 2.1. *For all $n \ge 0$, the standard exponential generating function* (1.6) *satisfies the Rodrigues ' formula*

$$\mathcal{H}_n^m(x) = (-1)^n \exp(x^m)\frac{\partial^n}{\partial x^n}\exp(-x^m). \tag{2.1}$$

Proof. Considering the standard exponential generating function (1.6) and in its right hand side on using the Maclaurin's Theorem to find that

$$\sum_{n=0}^{\infty} \mathcal{H}_n^m(x)\frac{t^n}{n!} = \exp(x^m)\sum_{n=0}^{\infty}\frac{\partial^n}{\partial t^n}\exp(-(x-t)^m)\bigg|_{t=0}\frac{t^n}{n!}$$
$$= \exp(x^m)\sum_{n=0}^{\infty}(-1)^n\frac{\partial^n}{\partial v^n}\exp(-v^m)\bigg|_{v=x}\frac{t^n}{n!} \tag{2.2}$$

Now equating the coefficients of t^n in both sides of the Eqn. (2.2), we get the Rodrigues' formula (2.1). □

Specially, putting $m = 2$ in the formula (1.5) and (1.6), we get following known generating functions for classical Hermite polynomials, $H_n(x)\ \forall n = 0, 1, 2, 3, \ldots$, (1.1) as

$$\sum_{n=0}^{\infty} \mathcal{H}_n^2(x)\frac{t^n}{n!} = \exp\left(-t^2 + 2tx\right)$$
$$= \exp\left(x^2\right)\exp\left(-(x-t)^2\right) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{2.3}$$

Proposition 2.1. *For x and t being independent variables, the Eqns.* (1.5) *or* (1.6) *and* (2.1)*, become cubic polynomials, $\mathcal{H}_n^3(x)\ \forall n \ge 0$, and there exists a standard generating function in the form*

$$\sum_{n=0}^{\infty} \mathcal{H}_n^3(x)\frac{t^n}{n!} = \exp\left(t^3 - 3xt^2 + 3x^2t\right) = \exp\left(x^3\right)\exp\left(-(x-t)^3\right), \tag{2.4}$$

and connect to Rodrigues' formula given by

$$\mathcal{H}_n^3(x) = (-1)^n e^{x^3} \frac{\partial^n}{\partial x^n} \left(e^{-x^3} \right). \tag{2.5}$$

Proof. By applying equations (1.5) and (1.6) along with Theorem 2.1, and setting m = 3, we yield formulas (2.4) and (2.5) of Proposition 2.1. □

Theorem 2.2. *The standard generating function* (2.4) *creates the series formula of a new class of cubic generalized Hermite polynomials*

$$\mathcal{H}_n^3(x) = n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^{\left[\frac{n-2m}{3}\right]} \frac{3^{n-2m-3k} \left(x^2\right)^{n-2m-3k}}{(n-2m-3k)!k!m!}, n = 0, 1, 2, 3, \ldots. \tag{2.6}$$

Proof. In the generating function (2.4) on making an appeal to series rearrangement techniques, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{H}_n^3(x) \frac{t^n}{n!} &= \exp\left(t^3 + 3x^2 t\right) \exp\left(-3xt^2\right) \\ &= \sum_{n=0}^{\infty} n! \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{k=0}^{\left[\frac{n-2m}{3}\right]} \frac{3^{n-2m-3k} \left(x^2\right)^{n-2m-3k}}{(n-2m-3k)!k!m!} \frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Equating like powers of t in both sides of (2.7), we obtain the series formula (2.6) of the cubic Hermite polynomials, $\mathcal{H}_n^3(x)\ \forall n \geq 0$. □

Also, we derive first few generalized Hermite polynomials and integral representation via Rodrigues' formula (2.5) and the standard generating function (2.4). We present some of its analytic properties to find its matrix representation and some inequalities not attained till now.

Again then, we introduce a multivariable analogue of the generating function (2.4). This multivariable analogue of generating function will become helpful for exploring new ideas of future researches in the field of special functions and other branches of mathematical sciences.

3 Applications of the Rodrigues' formula (2.5)

Applying the Rodrigues' formula (2.5), we derive following first few cubic Hermite polynomials, $\mathcal{H}_n^3(x)\ \forall n \geq 0$ in the form

$$\begin{aligned}&\mathcal{H}_0^3(x) = 1, \mathcal{H}_1^3(x) = 3x^2, \mathcal{H}_2^3(x) = 9x^4 - 6x, \mathcal{H}_3^3(x) = 27x^6 - 54x^3 + 6, \\ &\mathcal{H}_4^3(x) = 81x^8 - 324x^5 + 180x^2.\end{aligned} \tag{3.1}$$

The linear advection-type operators, commonly found in the study of hyperbolic partial differential equations, transport equations or specialized numerical analysis. Therefore, we present following inequalities.

Theorem 3.1. *For all $x, t \in \mathbb{R}$ such that $|t| \leq 1$, if by the formula* (2.4)*, it is defined as*

$$F(x,t) = \exp\left(t^3 - 3xt^2 + 3x^2 t\right) \text{ and } g(x,t) = t^3 - 3xt^2 + 3x^2 t. \tag{3.2}$$

Then there exist following inequalities

$$\left\| \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) F(x,t) \right\| \leq \left\| \mathcal{H}_1^3(x) \right\| \|F(x,t)\|. \tag{3.3}$$

$$\frac{\left\| \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) F(x,t) \right\|}{\left\| \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) g(x,t) \right\|} \leq \|F(x,t)\|. \tag{3.4}$$

$$\left\| \left\{ \frac{1}{x^2} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \right\}^n F(x,t) \right\| \leq 3^n \|F(x,t)\|. \tag{3.5}$$

Proof. Considering (3.2), we get

$$\frac{\partial}{\partial t} F(x,t) = \left(3t^2 - 6xt + 3x^2\right) \exp\left(t^3 - 3xt^2 + 3x^2 t\right) = F(x,t) \frac{\partial}{\partial t} g(x,t). \tag{3.6}$$

Also, we find

$$\frac{\partial}{\partial x} F(x,t) = \left(-3t^2 + 6xt\right) \exp\left(t^3 - 3xt^2 + 3x^2 t\right) = F(x,t) \frac{\partial}{\partial x} g(x,t). \tag{3.7}$$

On making an application of first and third relations of (3.6) and (3.7) respectively and then adding them, we find

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)F(x,t)=3x^2F(x,t)=F(x,t)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)g(x,t). \tag{3.8}$$

Then, using the relations of (3.1) and (3.8), we obtain the results (3.3), (3.4) and (3.5). □

It is noted that the generating function (2.4) is also applicable to obtain the integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$.

4 Integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$

In this section to obtain integral and matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, by the formula (2.2) for $m=3$, we consider that

$$\mathcal{H}_n^3(x)=\frac{\partial^n}{\partial t^n}\exp\left(t^3-3xt^2+3x^2t\right)\Big|_{t=0} \tag{4.1}$$

and make an appeal to the formula of complex analysis

$$\Psi(x)=\frac{\partial^n}{\partial t^n}F(x,t)\Big|_{t=0}=\frac{n!}{2\pi i}\int_C\frac{F(x,u)}{u^{n+1}}du, \tag{4.2}$$

where, $i=\sqrt{(-1)}, C$ is a contour such that $|u|\leq 1$;
then, applying (4.1) and (4.2), we write

$$\mathcal{H}_n^3(x)=\frac{n!}{2\pi i}\int_C\frac{\exp\left(u^3-3xu^2+3x^2u\right)}{u^{n+1}}du, u=e^{i\theta}, 0\leq\theta\leq 2\pi. \tag{4.3}$$

Theorem 4.1. *The real integral representation of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, exists and given by*

$$\mathcal{H}_n^3(x)=\frac{n!}{\pi}\int_0^{\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta \tag{4.4}$$

Proof. For $u=e^{i\theta}, 0\leq\theta\leq 2\pi$, the formula (4.3) is transformed into

$$\begin{aligned}\mathcal{H}_n^3(x)=&\frac{n!}{2\pi}\int_0^{2\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta\\&+\frac{i(n!)}{2\pi}\int_0^{2\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\sin\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta\end{aligned} \tag{4.5}$$

In the definite integral (4.4), we have

$$\begin{aligned}&\sin\left(\sin(6\pi-3\theta)-3x\sin(4\pi-2\theta)+3x^2\sin(2\pi-\theta)-n(2\pi-\theta)\right)\\&=-\sin\left(\sin(3\theta)-3x\sin(2\theta)+3x^2\sin(\theta)-n\theta\right)\end{aligned} \tag{4.6}$$

Therefore, by (4.6), the integral (4.5) becomes in the real definite integral, given by

$$\begin{aligned}\mathcal{H}_n^3(x)=&\frac{n!}{\pi}\int_0^{\pi}\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\\&\times\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta.\end{aligned} \tag{4.7}$$

Hence, the Theorem 4.1 is followed. □

Theorem 4.2. *If all conditions of the Theorem 3.1 are followed, then the following matrix representations of $\mathcal{H}_n^3(x)$ $\forall n=1,2,3,4,\ldots$, hold*

$$\begin{bmatrix}F\\\frac{\partial F}{\partial t}\\\frac{\partial^2 F}{\partial t^2}\\\frac{\partial^3 F}{\partial t^3}\\\frac{\partial^4 F}{\partial t^4}\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&6&0\\0&0&0&0&24\end{bmatrix}\begin{bmatrix}F\\\dot{g}F\\\left((\dot{g})^2+\ddot{g}\right)F\\\left(\frac{1}{6}(\dot{g})^3+\frac{1}{2}\dot{g}\ddot{g}+1\right)F\\\left(\frac{1}{24}(\dot{g})^4+\frac{1}{4}(\dot{g})^2\ddot{g}+\frac{1}{8}(\ddot{g})^2+\dot{g}\right)F\end{bmatrix}, \tag{4.8}$$

with coefficient matrix

$$C=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&6&0\\0&0&0&0&24\end{bmatrix}. \tag{4.9}$$

Also

$$\begin{bmatrix} H_0^3(x) \\ H_1^3(x) \\ H_2^3(x) \\ H_3^3(x) \\ H_4^3(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} F \\ \dot{g}F \\ \left((\dot{g})^2 + \ddot{g}\right) F \\ \left(\frac{1}{6}(\dot{g})^3 + \frac{1}{2}\dot{g}\ddot{g} + 1\right) F \\ \left(\frac{1}{24}(\dot{g})^4 + \frac{1}{4}(\dot{g})^2\ddot{g} + \frac{1}{8}(\ddot{g})^2 + \dot{g}\right) F \end{bmatrix}_{t=0}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 3x^2 \\ 9x^4 - 6x \\ \frac{9}{2}x^6 - 9x^3 + 1 \\ \frac{27}{8}x^8 - \frac{27}{2}x^5 + \frac{15}{2}x^2 \end{bmatrix}. \tag{4.10}$$

Proof. By Theorem 3.1 on considering Eqn. (3.2)

$$F(x,t) = \exp\left(t^3 - 3xt^2 + 3x^2t\right) \text{ and } g(x,t) = t^3 - 3xt^2 + 3x^2t, \text{ to get}$$

$$\frac{\partial}{\partial t}F(x,t) = \dot{g}F, \frac{\partial^2 F}{\partial t^2} = \left((\dot{g})^2 + \ddot{g}\right) F, \frac{\partial^3 F}{\partial t^3} = \left((\dot{g})^3 + 3\dot{g}\ddot{g} + 6\right) F,$$

$$\frac{\partial^4 F}{\partial t^4} = \left((\dot{g})^4 + 6(\dot{g})^2\ddot{g} + 3(\ddot{g})^2 + 24\dot{g}\right) F. \tag{4.11}$$

Employing the equations given in (4.11), we formulate the matrix representation (4.8) with the coefficient matrix (4.9).

Again then, applying (3.1) and (3.2) in the result (4.9), we yield the matrix representation (4.10) for the generalized Hermite polynomials $\mathcal{H}_n^3(x)$ $\forall n = 1, 2, 3, 4, \ldots$. □

5 Inequalities via integral representation of classical and cubic Hermite polynomials

We are familiar with formulae relating to Bernuolli polynomials $B_n(x)$ $\forall n = 0, 1, 2, 3, 4, \ldots$, and the Bernuolli numbers $B_n, n = 0, 1, 2, 3, \ldots$, given by [11, p.85]

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{5.1}$$

and

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \tag{5.2}$$

respectively.

Using these results (5.1), (5.2) and the integral representations of classical and generalized Hermite polynomials to obtain the inequalities:

Theorem 5.1. *The real integral representation of classical Hermite polynomials $H_n(x)$ $\forall n = 1, 2, 3, 4, \ldots$, exists and given by*

$$H_n(x) = \frac{n!}{\pi}\int_0^{\pi} \exp[2x\cos\theta - \cos 2\theta] \times \cos(2x\sin\theta - \sin 2\theta - n\theta)d\theta. \tag{5.3}$$

Then, there exists following inequalities

$$\sum_{n=0}^{m}\left|\frac{m!}{(m-n)!}B_{m-n}2^n x^n\right| + \sum_{n=0}^{\infty}\sum_{j=0}^{n-2}\left|\frac{m!}{(m-j)!}a_j B_{m-j}x^j\right| \leq \sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi}\left|B_m(2x\cos\theta)\exp[-\cos 2\theta]\right| d\theta. \tag{5.4}$$

provided that the constants $a_{-j} = 0$ $\forall j \in \mathbb{N}$ and $a_0 \neq 0$.

Also, there exists

$$a_0 \leq I_0(1) - 1 \Rightarrow a_0 \leq 0.266 \tag{5.5}$$

Proof. In the formula (5.3) on replacing x by xt and then in its both sides multiplying by $\frac{t}{e^t-1}$ and again making an appeal to (5.1) and (5.2) to get

$$\begin{aligned}&\sum_{m=0}^{\infty} B_m H_n(xt)\frac{t^m}{m!}\\ &=\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta-\sin 2\theta-n\theta)d\theta,\end{aligned}$$

then we have

$$\begin{aligned}&\sum_{m=0}^{\infty} B_m\left\{(2xt)^n+P_{n-2}(xt)\right\}\frac{t^m}{m!}\\ &=\sum_{m=0}^{\infty}\frac{t^m}{m!}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta-\sin 2\theta-n\theta)d\theta,\end{aligned}\tag{5.6}$$

where, $P_{n-2}(x)=a_0+a_1x+\cdots+a_{n-2}x^{n-2}=\sum_{j=0}^{n-2}a_jx^j$,
provided that $a_{-j}=0\ \forall j\in\mathbb{N}, a_0\neq 0$.

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}B_m2^nx^n\frac{t^{m+n}}{m!}+\sum_{m=0}^{\infty}B_m\sum_{n=0}^{\infty}\sum_{j=0}^{n-2}a_jx^j\frac{t^{m+j}}{m!}\\ &=\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi} B_m(2x\cos\theta)\exp[-\cos 2\theta]\cos(2xt\sin\theta-\sin 2\theta-n\theta)d\theta.\end{aligned}$$

Therefore, $\forall m=0,1,2,3,4,\ldots$, following inequality holds

$$\begin{aligned}&\sum_{n=0}^{m}\left|\frac{m!}{(m-n)!}B_{m-n}2^nx^n\right|+\sum_{n=0}^{\infty}\sum_{j=0}^{n-2}\left|\frac{m!}{(m-j)!}a_jB_{m-j}x^j\right|\\ &\le\sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi}\left|B_m(2x\cos\theta)\exp[-\cos 2\theta]\right|d\theta.\end{aligned}\tag{5.7}$$

For example when $m=0, n=0$ and $j=0$, so that we get

$$B_0+a_0B_0\le I_0(1),\tag{5.8}$$

where, $I_0(1)$ is the modified Bessel function of the first kind and $I_0(1)=1.266$.

Also $B_0=1$.

Hence, the Theorem 5.1 is followed. □

Theorem 5.2. *The real integral representation of $\mathcal{H}_n^3(x)\ \forall n=1,2,3,4,\ldots,$ exists in (4.4) as*

$$\begin{aligned}\mathcal{H}_n^3(x)=\frac{n!}{\pi}\int_0^{\pi}&\exp\left[\cos 3\theta-3x\cos 2\theta+3x^2\cos\theta\right]\\ &\times\cos\left(\sin 3\theta-3x\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta.\end{aligned}\tag{5.9}$$

Then following inequalities hold

$$\begin{aligned}&\left|\sum_{n=0}^{\left[\frac{m}{2}\right]}3^nx^{2n}2n!\sum_{k=0}^{[(m-2n)/2]}\binom{m-2n}{2k}\binom{m}{2n}\frac{2k!}{k!}B_kB_{m-2n-2k}\right|\\ &+\left|\sum_{n=0}^{\infty}\sum_{j=0}^{2n-3}a_jx^{2j-3}j!\binom{m}{j}\sum_{k=0}^{[(m-j)/2]}\binom{m-j}{2k}\frac{2k!}{k!}B_kB_{m-j-2k}\right|\\ &\le\sum_{n=0}^{\infty}\frac{n!}{\pi}\int_0^{\pi}\left|\exp[\cos 3\theta]\left\{\sum_{k=0}^{\left[\frac{m}{2}\right]}\binom{m}{2k}\frac{2k!}{k!}B_k\left(3x^2\cos 2\theta\right)B_{m-2k}(-3x\cos 2\theta)\right\}\right|d\theta,\end{aligned}\tag{5.10}$$

provided that the constants $a_{-j}=0\ \forall j\in\mathbb{N}$ and $a_0\neq 0$.

Also

$$a_0\le 0.266.\tag{5.11}$$

Proof. Considering the result (4.4) and multiplying by $\left(\frac{t^2}{e^{t^2}-1}\right)\left(\frac{t}{e^t-1}\right)$ in both of the sides to get

$$\mathcal{H}_n^3(xt)\left(\frac{t^2}{e^{t^2}-1}\right)\left(\frac{t}{e^t-1}\right)=\frac{n!}{\pi}\int_0^\pi \exp[\cos 3\theta]\frac{t\exp(-3xt\cos 2\theta)}{e^t-1}\left\{\frac{t^2\exp\left(3x^2t^2\cos\theta\right)}{e^{t^2}-1}\right\}$$
$$\times\cos\left(\sin 3\theta-3xt\sin 2\theta+3x^2\sin\theta-n\theta\right)d\theta. \tag{5.12}$$

Then making an appeal to the formulae (5.1) and (5.2) in the result (5.11) and then obeying same techniques of the Theorem 5.1, we obtain the inequalities (5.9) and (5.10). □

6 Conclusion

We may remark in passing that this work is the way to search new areas of research in the field of Special Functions, for example in a class of Hermite polynomials consisting of the generalized Hermite polynomials different to the polynomials found in the research work of the researchers ([3], [4], [5], [7], [8], [9] and others). Therefore, in directions of further researches done so far in various fields of science and engineering, we introduce a multivariable analogue of the generating function (2.4) given by

$$F\left(x_1,\ldots,x_k,t\right)=\exp\left(t^3-3\left(x_1+\cdots+x_k\right)t^2+3\left(x_1+\cdots+x_k\right)^2t\right). \tag{6.1}$$

Setting $|x|^2=|x_1+\cdots+x_k|^2$, in the generating function (5.1) to satisfying some of the analytic properties are given by

$$\sum_{q=1}^{k}\Delta_q F\left(x_1,\ldots,x_k,t\right)=3kF\left(x_1,\ldots,x_k,t\right),$$

where,

$$\Delta_q\equiv\frac{1}{|x|^2}\left(\frac{\partial}{\partial x_q}+\frac{\partial}{\partial t}\right);q=1,2,3,\ldots,k. \tag{6.2}$$

Further due to the operator (5.2), there exists a formula

$$\prod_{q=1}^{k}\Delta_q F\left(x_1,\ldots,x_k,t\right)=3^kF\left(x_1,\ldots,x_k,t\right). \tag{6.3}$$

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques: Polynomes d' Hermite,* Gauthier-Villars: Paris, France, 1926.

[2] E. T. Bell, Exponential polynomials, *Ann. of Math.,* **35**(2) (1934), 258-277.

[3] R. C. Singh Chandel, Generalized Hermite polynomials, *Jñānābha,* **Sec. A 2** (July 1972), 19-27.

[4] R. C. Singh Chandel, R. D. Agrawal and H. Kumar, Two variable analogue of Gould and Hopper's polynomials, *Journal of Maulana Azad College of Technology,* **25** (1992), 63-69.

[5] R. C. Singh Chandel, R. D. Agrawal and H. Kumar, A class of polynomials in several variables, *Ganita Sandesh,* **4**(1) (1990), 27-32.

[6] H. W. Gould and A. T. Hopper, Operational formulas connected with two generalization of Hermite polynomials, *Duke Math. J.,* **29** (1962), 51-63.

[7] M. Lahiri, On generalization of Hermite polynomials, *Proc. Amer. Math. Soc.,* **27** (1971), 117-121.

[8] M. A. Pathan, A new class of generalized Hermite-Bernoulli polynomials, *Georgian Math. J.,* **19** (2012), 559-573. DOI 10.1515/gmj-2012-0019

[9] M. A. Pathan and H. Kumar, Generalized three classes of mixed type double Bernoulli-Gegenbauer-Gould and Hopper polynomials, *South East Asian J. of Mathematics and Mathematical Sciences,* **20**(2) (2024), 67-86.
DOI: `https://doi.org/10.56827/SEAJMMS.2024.2002.6`

[10] E. D. Rainville, *Special Functions,* MacMillan, New York, 1960, Chalsea Pub. Co. Bronx, New York, 1971.

[11] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions,* Halsted Press (Ellis Horwood)/Wiley, Chichester/New York, 1984.