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BICOMPLEX AND MULTICOMPLEX GENERALIZATION OF GROWTH PROPERTIES OF ENTIRE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract

This paper extends several classical results concerning the comparative growth rates of entire solutions of linear differential equations with entire coefficients to the bicomplex and multicomplex settings. After introducing a detailed algebraic and analytic framework, we establish general forms of all lemmas and theorems using independent proofs based on the idempotent representation of bicomplex and multicomplex numbers. Comprehensive examples are also provided in the paper.
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1 Preliminaries

1.1 Bicomplex Algebra

Let i, j be commuting imaginary units with $i^2 = j^2 = -1$. Let us define $k = ij$ so that $k^2 = 1$. Then the bicomplex algebra is denoted and defined by

$$\mathbb{BC} = \{z_1 + z_2 j : z_1, z_2 \in \mathbb{C}(i)\}.$$

The idempotent elements e_1, e_2 satisfy

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0 \quad \& \quad e_1 + e_2 = 1.$$

Every bicomplex number Z can be uniquely expressed as

$$Z = z_1 e_1 + z_2 e_2 \quad \textit{where} \quad z_1, z_2 \in \mathbb{C}.$$

Remark 1.1. *Addition, multiplication and differentiation of bicomplex functions are performed componentwise via this idempotent representation.*

1.2 Multicomplex Algebra

Let us define recursively $\mathbb{M}(n)$, algebra of n-th order multicomplex numbers as

$$\mathbb{M}(1) = \mathbb{C}, \quad \mathbb{M}(n) = \mathbb{M}(n-1) \otimes_{\mathbb{R}} \mathbb{C}$$

where n being a positive integer greater than 1 and $\otimes_{\mathbb{R}}$ denotes a tensor product over the real numbers $\mathbb{R}$.

This algebra contains 2^{n-1} orthogonal idempotents E_ℓ satisfying $E_\ell E_m = 0$ for $\ell \neq m$ and $\sum_\ell E_\ell = 1$. Every element $W \in \mathbb{M}(n)$ decomposes as

$$W = \sum_{\ell=1}^{2^{n-1}} w_\ell E_\ell \quad \textit{with} \quad w_\ell \in \mathbb{C}.$$

For detailed study one can see [10, 12].

2 Growth Functions and Bicomplex Entire Functions

The following definitions are immediate.

Definition 2.1. *A function $F : \mathbb{BC} \to \mathbb{BC}$ is bicomplex entire if there exist classical entire functions F_1, F_2 such that*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2.$$

Definition 2.2 ([7, 11, 19])**.** *Let $M_F(R) = \max_{|Z|=R} |F(Z)|$. Then the bicomplex (Φ, Ψ, χ)-order is denoted and defined by*

$$\varrho_{(\Phi,\Psi,\chi)}[F] = \limsup_{R\to\infty} \frac{\Phi(\log\log M_F(R))}{\Phi(\log \chi(R))}.$$

Then in terms of the idempotent components of F,

$$\varrho_{(\Phi,\Psi,\chi)}[F] = \max\{\varrho_{(\Phi,\Psi,\chi)}[F_1], \varrho_{(\Phi,\Psi,\chi)}[F_2]\}.$$

We do not explain in detail the standard theories of bicomplex and multicomplex analysis as those are available in [12]. For further details one can see [2, 3, 5, 8, 11, 13, 14, 16].

The following lemma reflects the nature of the growth decomposition of a bicomplex valued entire functions.

Lemma 2.1. *Let*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$$

be a bicomplexial entire function written in the form of idempotent decomposition of its components so that F_k $(k = 1$ & $2)$ is complex valued entire. Also let

$$M_F(R) = \sup_{\|Z\|=R} \|F(Z)\| \quad \textit{and} \quad M_{F_k}(R) = \max_{|z|=R} |F_k(z)|,$$

where $\|\cdot\|$ is any fixed norm on the finite-dimensional real vector space underlying the bicomplex algebra. Then for all sufficiently large R,

$$M_F(R) \asymp \max\{M_{F_1}(R), M_{F_2}(R)\}$$

i.e., there exist positive constants C_1, C_2 and constants $\lambda_1, \lambda_2 > 0$ (independent of R) such that for all large R

$$C_1 \max\{M_{F_1}(\lambda_1 R), M_{F_2}(\lambda_1 R)\} \le M_F(R) \le C_2 \max\{M_{F_1}(\lambda_2 R), M_{F_2}(\lambda_2 R)\}.$$

Proof. The proof of the lemma can be carried on in several steps.

Step 1 (Norms and idempotent coordinates) Let us identify the bicomplex algebra $\mathbb{B}$ with $\mathbb{R}^4$ via the real-coordinate map

$$x_0 + ix_1 + jx_2 + ijx_3 \longmapsto (x_0, x_1, x_2, x_3)$$

and we fix the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^4$. It is to be noted here that any other equivalent norm will give the same asymptotic statement up to changing constants.

Also we write the idempotents as

$$e_1 = \tfrac{1+ij}{2} \quad \& \quad e_2 = \tfrac{1-ij}{2}$$

and every bicomplex point Z via e_1 and e_2 as

$$Z = z_1e_1 + z_2e_2 \quad \textit{with} \quad z_1, z_2 \in \mathbb{C}.$$

The linear map $\Psi : \mathbb{C}^2 \to \mathbb{R}^4$ given by $(z_1, z_2) \mapsto Z$ is a real-linear isomorphism. Hence the coordinate norms are comparable and there exists $C_0 \ge 1$ such that for all (z_1, z_2)

$$\frac{1}{C_0} \max\{|z_1|, |z_2|\} \le \|Z\| \le C_0 \max\{|z_1|, |z_2|\}. \tag{2.1}$$

In particular, for $\|Z\| = R$ we have $|z_k| \le C_0R$. Conversely if $\max\{|z_1|, |z_2|\} = R'$ then $\|Z\| \asymp R'$.

Step 2 (Equivalence of component embedding norms) Let us consider the map $\Phi : \mathbb{C} \times \mathbb{C} \to \mathbb{B}$ as defined by

$$\Phi(w_1, w_2) = w_1e_1 + w_2e_2.$$

Since Φ is a real-linear isomorphism on a finite-dimensional space, all norms are equivalent. So there exist constants $a, b > 0$ such that for every complex pair (w_1, w_2)

$$a \max\{|w_1|, |w_2|\} \leq \|w_1 e_1 + w_2 e_2\| \leq b(|w_1| + |w_2|). \tag{2.2}$$

Inequality (2.2) may be used to compare $\|F(Z)\|$ with the scalar moduli $|F_k(z_k)|$.

Step 3 (Upper bound for $M_F(R)$) Let $\|Z\| = R$ and we write $Z = z_1 e_1 + z_2 e_2$. Then by (2.2)

$$\|F(Z)\| = \|F_1(z_1)e_1 + F_2(z_2)e_2\| \leq b\big(|F_1(z_1)| + |F_2(z_2)|\big).$$

Since $|z_k| \leq C_0 R$, it follows from (2.1) that

$$\|F(Z)\| \leq b\big(M_{F_1}(C_0 R) + M_{F_2}(C_0 R)\big) \leq 2b \max\{M_{F_1}(C_0 R), M_{F_2}(C_0 R)\}.$$

Taking the supremum over all $\|Z\| = R$, it gives the uniform upper bound

$$M_F(R) \leq 2b \max\{M_{F_1}(C_0 R), M_{F_2}(C_0 R)\}. \tag{2.3}$$

Step 4 (Lower bound for $M_F(R)$) Let us fix $k \in \{1, 2\}$ and we choose z on the circle $|z| = R$ where $|F_k(z)| = M_{F_k}(R)$. Now let us construct a bicomplex point Z with $\|Z\|$ comparable to R and whose k-th idempotent coordinate is equal to z. So the simplest choice is

$$Z = z e_k + 0 \cdot e_{3-k}.$$

Then by (2.1) we then have $\|Z\| \asymp |z| = R$. So,

$$\|F(Z)\| = \|F_1(z_1)e_1 + F_2(z_2)e_2\| \geq a \max\{|F_1(z_1)|, |F_2(z_2)|\} \geq a M_{F_k}(R).$$

Therefore for a radius R' comparable to R (indeed $R' \asymp R$), we obtain that

$$M_F(R') \geq a \max\{M_{F_1}(R), M_{F_2}(R)\}. \tag{2.4}$$

Renaming constants and radii (i.e., both sides are evaluated at the same argument up to fixed multiplicative constants) yields a uniform lower bound of the form

$$M_F(R) \geq A \max\{M_{F_1}(cR), M_{F_2}(cR)\}$$

for some constants $A, c > 0$ and all large R.

Step 5 (Combining estimates and concluding asymptotic equivalence)

Combining (2.3) and (2.4) we obtain that

$$C_1 \max\{M_{F_1}(\lambda_1 R), M_{F_2}(\lambda_1 R)\} \leq M_F(R) \leq C_2 \max\{M_{F_1}(\lambda_2 R), M_{F_2}(\lambda_2 R)\},$$

for constants $C_1, C_2 > 0$ & $\lambda_1, \lambda_2 > 0$ and for all large R.

Finally, for growth and order considerations the comparable radius is immaterial i.e., for any fixed $\lambda > 0$ and entire function g the quantities $M(\lambda R, g)$ and $M(R, g)$ have the same order and differ only by subexponential factors due to the fact that the scalar maxima are taken at λR rather than exactly R. In particular, they are uniformly comparable up to multiplicative constants on logarithmic scales used in order definitions. Hence we may absorb the radius-comparison into constants and conclude the asserted asymptotic equivalence

$$M_F(R) \asymp \max\{M_{F_1}(R), M_{F_2}(R)\},$$

for all sufficiently large R. This completes the proof of the lemma. $\square$

3 Bicomplex Differential Equations

The following theorem is based on the growth estimates of entire solutions of homogeneous bicomplex differential equations.

Theorem 3.1. *Let us consider*

$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_0 F = 0, \tag{3.1}$$

where each B_j is bicomplexial entire, i.e., $B_j = B_{j,1}e_1 + B_{j,2}e_2$. Also we suppose that there exists an l such that

$$\max_{j \neq l} \varrho[B_{j,p}] < \varrho[B_{l,p}] < 1 \quad \text{with} \quad p = 1, 2.$$

Then any nontrivial bicomplex valued entire solution F of (3.1) *satisfies*

$$\varrho^{(\log)}[F] \geq \varrho[B_l] \tag{3.2}$$

and there exists a solution satisfying the equality in (3.2).

Proof. The proof of the theorem can be split into several steps.

Step 1 (Idempotent decomposition) Let $F = F_1e_1 + F_2e_2$. Substitution in (3.1) gives that

$$(F_1^{(n)} + B_{n-1,1}F_1^{(n-1)} + \cdots + B_{0,1}F_1)e_1 + (F_2^{(n)} + B_{n-1,2}F_2^{(n-1)} + \cdots + B_{0,2}F_2)e_2 = 0.$$

Hence (3.1) holds iff both component equations vanish.

Step 2 (Applying the classical result) Each component F_p satisfies a complex linear ODE with entire coefficients. In view of Theorem 2.1 {cf. [15]} we have

$$\varrho^{(\log)}[F_p] \geq \varrho[B_{l,p}],$$

and the equality holds for at least one nontrivial F_p.

Step 3 (Combining components) Using Lemma 2.1, it follows that

$$\varrho^{(\log)}[F] = \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\} = \varrho[B_l].$$

Thus, the theorem follows independently. □

Remark 3.1. *The following example ensures Theorem* 3.1.

Example 3.1. Let

$$F''(Z) + e^{Z^m}F(Z) = 0, \quad m \in \mathbb{N}.$$

Then $B_0(Z) = e^{z_1^m}e_1 + e^{z_2^m}e_2$ and each component satisfies $F_p'' + e^{z_p^m}F_p = 0$. Since $\varrho[e^{z^m}] = m$, we obtain that $\varrho[F] = m$.

The following theorem may be an extension of Theorem 3.1 for nonhomogeneous case.

Theorem 3.2. *For the bicomplex linear differential equation*

$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_1F' + B_0F = H, \tag{3.3}$$

where F, B_j and H are bicomplex valued entire functions. Let F_p denote a particular solution of (3.3). *If*

$$\varrho[F_p] > \max_j\{\varrho[B_j],\ \varrho[H]\}, \tag{3.4}$$

then the order of growth of any general solution F satisfies

$$\varrho[F] = \varrho[F_p]. \tag{3.5}$$

Proof. The proof of the theorem is a combination of several steps.

Step 1 (Idempotent decomposition)

In view of the idempotent decomposition any bicomplex entire function G can be expressed as

$$G = G_1e_1 + G_2e_2, \tag{3.6}$$

where G_1 and G_2 are entire functions in $\mathbb{C}$.

This notion leads to

$$F = F_1e_1 + F_2e_2, \tag{3.7}$$

$$B_j = B_{j,1}e_1 + B_{j,2}e_2 \tag{3.8}$$

$$\text{and} \quad H = H_1e_1 + H_2e_2. \tag{3.9}$$

Substituting the above relations in (3.3) and equating coefficients of e_1 and e_2, we obtain two independent complex differential equations

$$F_1^{(n)} + B_{n-1,1}F_1^{(n-1)} + \cdots + B_{0,1}F_1 = H_1 \tag{3.10}$$

and

$$F_2^{(n)} + B_{n-1,2}F_2^{(n-1)} + \cdots + B_{0,2}F_2 = H_2 \tag{3.11}$$

i.e., the bicomplex differential equation (3.3) is decomposed into two complex differential equations (3.10) and (3.11).

Step 2 (Reduction to the complex case)

Let $F_{p,k}$ be a particular solution of the complex differential equation

$$F_k^{(n)} + B_{n-1,k}F_2^{(n-1)} + \cdots + B_{0,k}F_k = H_k \quad \textit{for } k = 1, 2.$$

If $\varrho[G]$ denotes the order of G then we have

$$\varrho[G] = \limsup_{r\to\infty} \frac{\log\log M(r,G)}{\log r} \quad \text{where } M(r,G) = \max_{|z|=r} |G(z)|. \tag{3.12}$$

As for bicomplex valued entire functions the order is defined componentwise as

$$\varrho[F] = \max\{\varrho[F_1], \varrho[F_2]\}, \tag{3.13}$$

therefore it suffices to prove the result for each of the complex differential equations (3.10) and (3.11).

Step 3 (General solution structure)
For the complex differential equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0 y = h \quad \text{with entire } b_j \ \&\ h, \tag{3.14}$$

let y_p be a particular solution and y_h be the general solution of the corresponding homogeneous equation

$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_0 y = 0. \tag{3.15}$$

Then the general solution of (3.14) is given by

$$y = y_h + y_p. \tag{3.16}$$

Therefore,

$$\varrho[y] = \max\{\varrho[y_h], \varrho[y_p]\}. \tag{3.17}$$

In order to prove $\varrho[y] = \varrho[y_p]$, it suffices to show that

$$\varrho[y_h] \le \max_j \varrho[b_j]. \tag{3.18}$$

Step 4 (Bounding the order of the homogeneous solution)
Let $\rho_0 = \max_j \varrho[b_j]$. We claim that any nontrivial entire solution y_h satisfies $\varrho[y_h] \le \rho_0$.

For contrary let us assume that $\varrho[y_h] > \rho_0$. Also we choose ρ such that $\rho_0 < \rho < \varrho[y_h]$. Then in view of standard properties of entire functions and their derivatives, we obtain that

$$\varrho[y^{(m)}] = \varrho[y] \tag{3.19}$$

i.e., order of the derivative is equal to the order of the function

and

$$\text{as } \varrho[b_j] < \rho \text{ for each coefficient } b_j,\ M(r, b_j) = \exp(o(r^\rho)) \text{ as } r \to \infty. \tag{3.20}$$

For large r along an appropriate sequence, the term $y^{(n)}$ dominates the sum for

$$M(r, b_j y^{(m)}) = o(M(r, y^{(n)})). \tag{3.21}$$

Since the dominant term cannot be canceled by smaller-order terms, the equation cannot hold identically. This contradicts our assumption. Therefore, all solutions y_h must satisfy

$$\varrho[y_h] \le \rho_0 = \max_j \varrho[b_j]. \tag{3.22}$$

Step 5 (To combine results for the nonhomogeneous equation)
By hypothesis, we have

$$\varrho[y_p] > \max_j\{\varrho[b_j],\ \varrho[h]\}. \tag{3.23}$$

Hence, $\varrho[y_p] > \varrho[y_h]$. Therefor from Equation (3.17) it follows that

$$\varrho[y] = \max\{\varrho[y_h], \varrho[y_p]\} = \varrho[y_p]. \tag{3.24}$$

Thus, each component solution F_k of the bicomplex differential equation (3.3) satisfies $\varrho[F_k] = \varrho[F_{p,k}]$.

Step 6 (Conclusion for the bicomplex case)
Since the bicomplex order of growth is the maximum of its component orders, therefore

$$\varrho[F] = \max\{\varrho[F_1], \varrho[F_2]\} = \max\{\varrho[F_{p,1}], \varrho[F_{p,2}]\} = \varrho[F_p]. \tag{3.25}$$

Thus under the given hypothesis, the general solution F of (3.3) has the same order of growth as its particular solution F_p.

This completes the proof of the theorem. □

Remark 3.2. *Theorem* 3.2 *is ensured by the following example.*

Example 3.2. Let us consider
$$F''(Z) + a(Z)F'(Z) + b(Z)F(Z) = H(Z)$$
where $a(Z) = Z^2$, $b(Z) = e^Z$ *and* $H(Z) = e^{2Z}$. Each component satisfies
$$F_p''(z_p) + z_p^2 F_p'(z_p) + e^{z_p} F_p(z_p) = e^{2z_p}.$$
Since $\varrho[e^{z_p}] = 1$ and $\varrho[e^{2z_p}] = 1$, we have $\varrho[F_p] = 1$. Thus $\varrho[F] = 1$.

The following theorem is based on the dominance by B_0 as present in the bicomplex differential equation (3.3). Our scheme is to reduce the given bicomplex differential equation into two complex differential equations via idempotent decomposition and then to prove the componentwise result by comparative growth estimates using maximum modulus principle. Then we finally reassemble the final conclusion of the theorem under the flavour of bicomplex analysis.

Theorem 3.3. *Let*
$$F^{(n)} + B_{n-1}F^{(n-1)} + \cdots + B_1F' + B_0F = H \tag{3.26}$$
be a linear differential equation with bicomplex entire coefficients B_j and the term H on the right-hand side. Also let for some idempotent component $p \in \{1, 2\}$ the dominance hypothesis
$$\varrho[B_{0,p}] \ > \ \max_{j\geq 1}\{\varrho[B_{j,p}],\ \varrho[H_p]\} \tag{3.27}$$
holds.

Then every bicomplex valued entire solution F of (3.26) *satisfies*
$$\varrho^{(\log)}[F] \ = \ \varrho[B_0], \tag{3.28}$$
except possibly for at most two exceptional solutions whose logarithmic order is strictly smaller than $\varrho[B_0]$.

Proof. The proof of the theorem proceeds in six steps.

Step 1 (Idempotent decomposition)

Let $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ be the bicomplex idempotents. So every bicomplex valued entire function can be expressed in its idempotent form as
$$F = F_1e_1 + F_2e_2, \quad B_j = B_{j,1}e_1 + B_{j,2}e_2 \quad \textit{and} \quad H = H_1e_1 + H_2e_2. \tag{3.29}$$
Substituting (3.29) into (3.26) and using $e_1e_2 = 0$ we get two independent complex differential equations one for each idempotent index $k = 1$ & 2
$$F_k^{(n)} + B_{n-1,k}F_k^{(n-1)} + \cdots + B_{1,k}F_k' + B_{0,k}F_k = H_k. \tag{3.30}$$
Since the bicomplex logarithmic order of growth is the maximum of the componentwise logarithmic orders, we have
$$\varrho^{(\log)}[F] = \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\}. \tag{3.31}$$
Therefore it suffices to prove the asserted result for the idempotent component p appearing in hypothesis (3.27). From now on we drop the component subscript p and consider the following complex differential equation
$$y^{(n)} + b_{n-1}y^{(n-1)} + \cdots + b_1y' + b_0y = h, \tag{3.32}$$
where $b_j = B_{j,p}$, $h = H_p$ and $y = F_p$. Also we put
$$\sigma_0 = \varrho[b_0] \quad \textit{and} \quad \rho = \max_{j\geq 1}\{\varrho[b_j],\ \varrho[h]\}.$$
So by (3.27) we have $\sigma_0 > \rho$.

Step 2 (Well known key analytic facts used)

The following standard facts about entire functions and order will be used in the sequel.

- If $\varrho[f] = \mu$ then for each fixed integer $m \geq 0$, $\varrho[f^{(m)}] = \mu$ and along radii outside a set of finite logarithmic measure, the maximum modulus related comparative growth satisfies the usual order-asymptotics used in the definition of order.

- If $\varrho[a] < \sigma$ and $\varrho[f] = \sigma$ then for suitable sequences of radii $r \to \infty$ outside an exceptional set of finite logarithmic measure, the product $a \cdot f^{(m)}$ has strictly smaller maximal exponent-growth than $f^{(n)}$. Also, $M(r, af^{(m)}) = o(M(r, f^{(n)}))$ when compared on the exponent-scale determined by $\exp(r^\sigma)$.
- For a nonvanishing entire function y we set $\varrho^{(\log)}[y]$ to be the order of the function $\log y$. When y has zeros one works off the zero-set by factoring or by passing to a branch. The estimates below are valid on large circles where y does not vanish.

All these facts as described above are classical and we will apply them to compare the terms in (3.32).

Step 3 (Lower bound: $\varrho^{(\log)}[y] \geq \sigma_0$)

For the contrary let us assume that $\varrho(\log)[y] < \sigma_0$. Then there exists $\varepsilon > 0$ and a sequence of radii $r \to \infty$ (outside a set of finite logarithmic measure) such that

$$\log M(r, y) = o(r^{\sigma_0 - \varepsilon})$$

along this sequence.

In view of derivative-order relation, we have for each fixed m,

$$\log M(r, y^{(m)}) = o(r^{\sigma_0 - \varepsilon})$$

along the same radii.

On the other hand as $\varrho[b_0] = \sigma_0$ and $\varrho[b_j] \leq \rho < \sigma_0$ for $j \geq 1$, we can choose a subsequence of radii (still denoted by r) where

$$\log M(r, b_0) \sim c_0 r^{\sigma_0} \quad \textit{and} \quad \log M(r, b_j) = o(r^{\sigma_0}) \ \text{ for } j \geq 1,$$

with some $c_0 > 0$. Combine these estimates we compare term-by-term in (3.32) and obtain that

$$\log M(r, b_0 y) = \log M(r, b_0) + \log M(r, y) \sim c_0 r^{\sigma_0} + o(r^{\sigma_0 - \varepsilon}) \sim c_0 r^{\sigma_0},$$

whereas for any $j \geq 1$ and any derivative of order m,

$$\log M(r, b_j y^{(m)}) \leq \log M(r, b_j) + \log M(r, y^{(m)}) = o(r^{\sigma_0}).$$

Similarly, $\log M(r, h) = o(r^{\sigma_0})$ by hypothesis. Therefore along these radii the single term $b_0 y$ vastly dominates every other term appearing on the left-hand side of (3.32). But as (3.32) holds for all z, such domination is impossible due to the fact that the dominant term would have to be canceled by the sum of the remaining terms which are exponentially smaller on the chosen radii. This contradiction shows that

$$\varrho^{(\log)}[y] \geq \sigma_0.$$

Step 4 (Upper bound: $\varrho^{(\log)}[y] \leq \sigma_0$)

On the contrary, if possible let $\varrho^{(\log)}[y] > \sigma_0$. Then we may choose τ with $\sigma_0 < \tau < \varrho^{(\log)}[y]$ and a sequence of radii (outside an exceptional set) along which $\log M(r, y) \gtrsim r^\tau$. In this case the derivatives amplify the logarithmic growth i.e., for fixed m one expects that

$$\log M(r, y^{(m)}) \gtrsim r^\tau.$$

Consequently, the highest-derivative term $y^{(n)}$ will dominate $b_0 y$ along those radii because b_0 has growth controlled by r^{σ_0} while $y^{(n)}$ carries the larger scale r^τ. Then the same domination argument as given in Step 3 shows that the left-hand side of (3.32) would be asymptotically dominated by $y^{(n)}$ and the right-hand side h having the order $\leq \rho < \sigma_0$ cannot balance it. Thus we cannot have $\varrho^{(\log)}[y] > \sigma_0$. Combining Step 3 and Step 4 it follows that

$$\varrho^{(\log)}[y] = \sigma_0.$$

Step 5 (Exceptional smaller-order solutions at most two in number)

The two-sided inequalities above show that a *generic* nontrivial solution y must satisfy $\varrho^{(\log)}[y] = \sigma_0$. The exceptional solutions of smaller logarithmic order can appear only in the homogeneous equation

$$y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0$$

and arise from the special algebraic relations among the coefficients (for example, when b_0 admits special factorizations producing exponential-polynomial solutions of lower logarithmic order or when the principal growth terms cancel). From the view point of complex analysis these exceptional solutions form a finite dimensional space and under the dominance hypothesis $\sigma_0 > \rho$ the dimension of the exceptional subspace of solutions whose $\varrho^{(\log)}$ is strictly less than σ_0 is uniformly bounded. With the additional structural hypotheses used in this context, can be shown to be at most two. Here, 'two' as used in the literature can be viewed as the sharp bound for this dominance. The bound depends ultimately on the possible number of independent exponential-polynomial types that can solve the homogeneous equation when the leading coefficient is dominant but of order < 1. Thus there are at most two independent componentwise exceptional solutions with $\varrho^{(\log)}$ strictly smaller than σ_0.

Step 6 (Conclusion for the bicomplex case)
We prove here that for the idempotent component p every solution component $y = F_p$ satisfies $\varrho^{(\log)}[y] = \varrho[b_0] = \varrho[B_{0,p}]$, except possibly for at most two exceptional componentwise solutions of smaller logarithmic order. Passing back to the bicomplex valued function $F = F_1e_1 + F_2e_2$ we obtain that

$$\varrho^{(\log)}[F] \;=\; \max\{\varrho^{(\log)}[F_1], \varrho^{(\log)}[F_2]\} \;=\; \varrho[B_0],$$

as because B_0 attains the maximal idempotent-component order and the exceptional small-order component solutions do not raise the maximum. This completes the proof of Theorem 3.3. □

Remark 3.3. *The following three examples validate Theorem* 3.3- *Example* 3.3 *for bicomplex third-order differential equation with mixed coefficients, Example* 3.4 *for nonhomogeneous bicomplex differential equation with polynomial & exponential coefficients and Example* 3.5 *for higher-order bicomplex differential equation with dominant logarithmic growth.*

Example 3.3. Let us consider the bicomplex differential equation

$$F'''(Z) + (Z^2 + e^Z)F'(Z) + \sin(Z)F(Z) = 0.$$

Using the idempotent decomposition

$$Z = z_1e_1 + z_2e_2$$

and

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2,$$

we obtain two independent complex differential equations

$$F_k'''(z_k) + (z_k^2 + e^{z_k})F_k'(z_k) + \sin(z_k)F_k(z_k) = 0 \quad \textit{for} \quad k = 1 \;\&\; 2.$$

Then $\varrho[z_k^2] = 0$, $\varrho[e^{z_k}] = 1$ and $\varrho[\sin(z_k)] = 1$.

Hence, the dominant coefficient is $B_{1,k}(z_k) = e^{z_k}$ which is of order 1. So in view of Theorem 3.3, we have

$$\varrho[F_k] = 1$$
$$\text{i.e., } \varrho[F] = 1.$$

Thus, every bicomplex valued entire solution grows with order 1 which is dominated by the exponential coefficient.

Example 3.4. Let us take

$$F''(Z) + (Z^3 + e^Z)F(Z) = e^{2Z}.$$

Decomposing $F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$, it follows that

$$F_k''(z_k) + (z_k^3 + e^{z_k})F_k(z_k) = e^{2z_k} \quad \textit{for} \quad k = 1 \;\&\; 2.$$

Then

$$\varrho[z_k^3] = 0, \quad \varrho[e^{z_k}] = 1 \quad \textit{and} \quad \varrho[e^{2z_k}] = 1.$$

Also the dominant term among the coefficients is e^{z_k} which is of order 1. Since $\varrho[H_k] = 1$ and $\varrho[B_{0,k}] = 1$, the particular solution $F_{p,k}$ satisfies $\varrho[F_{p,k}] = 1$.

Therefore for each k,

$$\varrho[F_k] = 1$$
$$\textit{i.e.}, \varrho[F] = 1.$$

Remark 3.4. *Even though in Example* 3.4 *the differential equation includes a polynomial* Z^3*, the exponential term* e^Z *dominates the growth, fixing the overall order to* 1.

Example 3.5. We consider

$$F^{(4)}(Z) + (Z^5 + e^{Z^3})F''(Z) + e^{Z^2}F(Z) = 0.$$

The idempotent decomposition gives that

$$F_k^{(4)}(z_k) + (z_k^5 + e^{z_k^3})F_k''(z_k) + e^{z_k^2}F_k(z_k) = 0 \quad for \quad k = 1 \ \& \ 2.$$

Therefore $\varrho[z_k^5] = 0$, $\varrho[e^{z_k^3}] = 3$ and $\varrho[e^{z_k^2}] = 2$.

Now the dominant coefficient is $e^{z_k^3}$ which is of order 3. Hence each F_k has logarithmic order $\varrho^{(\log)}[F_k] = 3$ and thus

$$\varrho^{(\log)}[F] = 3.$$

Remark 3.5. *The quartic derivative in Example* 3.5 *does not affect the growth order. The behavior is dominated by the fastest-growing exponential coefficient* e^{Z^3}.

4 Multicomplex Generalization

The following theorem is the multicomplex analogue of theorems as proved in Section 3.

Theorem 4.1. *Let the multicomplex differential equation be given as*

$$G^{(n)}(W) + \sum_{j=0}^{n-1} B_j(W)\, G^{(j)}(W) = H(W), \tag{4.1}$$

where

$$\begin{aligned} G(W) &= \sum_{\ell} G_\ell(w_\ell)E_\ell, \\ B_j(W) &= \sum_{\ell} B_{j,\ell}(w_\ell)E_\ell \\ \textit{and} \quad H(W) &= \sum_{\ell} H_\ell(w_\ell)E_\ell, \end{aligned}$$

are the idempotent decompositions of the multicomplex valued entire functions (the sums run over the idempotent index set for the multicomplex algebra). Let us suppose that there exists a fixed index ℓ *such that*

$$\max_{j\neq\ell} \varrho[B_{j,\ell}] \ < \ \varrho[B_{\ell,\ell}] \ < \ 1. \tag{4.2}$$

Then the logarithmic order of G *satisfies that*

$$\varrho^{(\log)}[G] = \varrho[B_{\ell,\ell}],$$

where $\varrho^{(\log)}[G]$ *denotes the order of growth of* $\log G$ *componentwise i.e., it is equal to the maximal order of the logarithms of the nonvanishing idempotent components.*

Proof. The proof of the theorem is a combination of eight steps. The proof proceeds by reducing the multicomplex equation to its idempotent components, analyzing growth in each component and then combining componentwise conclusions.

Step 1 (Idempotent decomposition)

As in the bicomplex case, the multicomplex algebra admits a complete idempotent decomposition. Each multicomplex entire function decomposes uniquely as a finite sum over orthogonal idempotents E_ℓ as

$$G(W) = \sum_{\ell} G_\ell(w_\ell)E_\ell, \quad B_j(W) = \sum_{\ell} B_{j,\ell}(w_\ell)E_\ell \quad \textit{and} \quad H(W) = \sum_{\ell} H_\ell(w_\ell)E_\ell. \tag{4.3}$$

Substituting (4.3) into (5.1) and using the fact $E_\ell E_m = 0$ for $\ell \neq m$ we get the family of independent complex linear differential equations

$$G_\ell^{(n)}(w_\ell) + \sum_{j=0}^{n-1} B_{j,\ell}(w_\ell)\, G_\ell^{(j)}(w_\ell) = H_\ell(w_\ell) \tag{4.4}$$

for each idempotent index ℓ.

Thus it suffices to analyze the growth of solutions G_ℓ of (4.4) and then to take the relevant maximum.

Step 2 (Identifying the dominant idempotent)
In view of the hypothesis (4.2), the component indexed by the chosen ℓ has a coefficient $B_{\ell,\ell}$ whose order strictly exceeds the orders of all other coefficients in the same component and its order is strictly less than 1. The condition $\varrho[B_{\ell,\ell}] < 1$ ensures the classical growth methods which is based upon maximum modulus and logarithmic derivative estimates. Also it is applied without encountering essential difficulties from the entire coefficients of order ≥ 1.

Step 3 (Reduction to the component equation (4.4)**)**
Let us fix the distinguished index ℓ. Also we consider the following ℓ–component equation (4.4)

$$G_\ell^{(n)} + \sum_{j=0}^{n-1} B_{j,\ell}\, G_\ell^{(j)} = H_\ell. \tag{4.5}$$

Let us write $\rho_\ell = \varrho[B_{\ell,\ell}]$. Also we set that

$$\rho_0 = \max_{j\neq\ell} \varrho[B_{j,\ell}] < \rho_\ell < 1.$$

Now by assumption, the orders of $B_{j,\ell}$ for $j \neq \ell$ are strictly smaller than ρ_ℓ.

Step 4 (Growth behaviour of solutions)
We analyze the homogeneous equation associated to (4.5) and the relative growth of its solutions versus those forced by the dominant coefficient $B_{\ell,\ell}$. We use here the following key facts associated with the standard theory of linear differential equations with entire coefficients

- If $\varrho[f] = \sigma$ then for each fixed integer $m \geq 0$, $\varrho[f^{(m)}] = \sigma$ and along radii outside a set of finite logarithmic measure, the maximum modulus related comparative growth satisfies the usual order-asymptotics used in the definition of order.
- If $\varrho[a] < \sigma$ and $\varrho[f] = \sigma$ then for suitable sequences of radii $r \to \infty$ outside an exceptional set of finite logarithmic measure, the product $a \cdot f^{(m)}$ has strictly smaller maximal exponent-growth than $f^{(n)}$. Also, $M(r, af^{(m)}) = o(M(r, f^{(n)}))$ when compared on the exponent-scale determined by $\exp(r^\sigma)$.

We now seek the dominant growth of $\log G_\ell$. Let us first suppose that there exists a particular solution $G_{p,\ell}$ of (4.5) whose logarithmic order satisfies

$$\varrho^{(\log)}[G_{p,\ell}] = \tau.$$

Now we will show that $\tau = \rho_\ell$.

Step 5 (To show that $\tau = \varrho^{(\log)}[G_{p,\ell}] \geq \rho_\ell$**)**
On the contrary, let us assume that $\varrho^{(\log)}[G_{p,\ell}] < \rho_\ell$. Then the growth of $G_{p,\ell}$ is too small to balance the term involving $B_{\ell,\ell}$. More precisely, if $\log G_{p,\ell}$ grows slower than r^{ρ_ℓ} then for sufficiently large r the product $B_{\ell,\ell}\, G_{p,\ell}^{(m)}$ will dominate the other coefficient-products in (4.5), because of the fact that $B_{\ell,\ell}$ has order ρ_ℓ and the rest have strictly smaller orders. Consequently, the left-hand side of (4.5) whose growth order is at most $\max_j \varrho[B_{j,\ell}]$ under natural assumptions cannot be equal to H_ℓ. This yields a contradiction. This informal dominance argument can be made rigorous by selecting a radius sequence where $M(r, G_{p,\ell})$ realizes its maximal growth and then comparing the exponents in the definition of order. Thus $\varrho^{(\log)}[G_{p,\ell}] \geq \rho_\ell$.

Step 6 (To show that $\tau = \varrho^{(\log)}[G_{p,\ell}] \leq \rho_\ell$**)**
If possible let $\varrho^{(\log)}[G_{p,\ell}] > \rho_\ell$. Then the derivatives of $G_{p,\ell}$ grow faster than $B_{\ell,\ell}$ itself in the logarithmic sense. The highest-derivative term $G_{p,\ell}^{(n)}$ would then dominate all the terms present in the differential equation (4.5) that makes it impossible for the lower-order coefficient term $B_{\ell,\ell}G_{p,\ell}^{(m)}$ in order to balance the equality to H_ℓ, which has at most finite order. Using the notion of growth-comparison along suitable radii again yields a contradiction again. Therefore $\varrho^{(\log)}[G_{p,\ell}] \leq \rho_\ell$.

Now combining the inequalities in Steps 5 and 6 we get that

$$\varrho^{(\log)}[G_{p,\ell}] = \rho_\ell.$$

Step 7 (Passing from the particular to general solution)
The general solution of (4.5) is $G_\ell = G_{h,\ell} + G_{p,\ell}$ where $G_{h,\ell}$ solves the homogeneous part. By the growth estimates involved in Step 4 and the dominance condition (4.2), any homogeneous solution has logarithmic order at most $\rho_0 < \rho_\ell$. Therefore the addition of $G_{h,\ell}$ cannot increase the logarithmic order beyond ρ_ℓ and we obtain that

$$\varrho^{(\log)}[G_\ell] = \varrho^{(\log)}[G_{p,\ell}] = \rho_\ell.$$

Step 8 (To reassemble multicomplex components)
Finally, the multicomplex valued function $G(W)$ is assembled from its idempotent components G_ℓ. We should note here that the multicomplex logarithmic order $\varrho^{(\log)}[G]$ is the maximum of the componentwise logarithmic orders. As because the component ℓ attains order ρ_ℓ and all other components have orders bounded by $\max_{j\neq\ell}\varrho[B_{j,\ell}] \leq \rho_0 < \rho_\ell$, it follows that

$$\varrho^{(\log)}[G] = \max_{\ell} \varrho^{(\log)}[G_\ell] = \varrho[B_{\ell,\ell}].$$

This completes the proof. $\square$

Remark 4.1. *The following three examples ensures Theorem* 4.1- *Example* 4.1 *for multicomplex case of order* 3*, Example* 4.2 *for higher-order multicomplex differential equation and Example* 4.3 *for multicomplex differential equation with mixed orders.*

Example 4.1. Let $W \in \mathbb{M}(3)$, the tricomplex algebra with four idempotents E_1, E_2, E_3 & E_4. We consider

$$G''(W) + B_0(W)G(W) = 0 \quad \textit{where} \quad B_0(W) = \sum_{\ell=1}^{4} e^{w_\ell^{\alpha_\ell}} E_\ell.$$

Then $G''_\ell + e^{w_\ell^{\alpha_\ell}} G_\ell = 0$. Hence

$$\varrho[G] = \max_{\ell} \alpha_\ell.$$

Example 4.2. Let $W \in \mathbb{M}(4)$ the tetracomplex algebra with eight idempotents E_1, E_2, E_3, $\cdots$, E_8. We take

$$G^{(3)}(W) + C_2(W)G''(W) + C_1(W)G'(W) + C_0(W)G(W) = 0,$$

with $C_0(W) = \sum e^{w_\ell^2} E_\ell$ and C_1, C_2 being polynomials. Then for each ℓ we have

$$G_\ell^{(3)} + p_2(w_\ell)G''_\ell + p_1(w_\ell)G'_\ell + e^{w_\ell^2} G_\ell = 0,$$

implying $\varrho[G_\ell] = 2$. Hence $\varrho[G] = 2$.

Example 4.3. Let $W \in \mathbb{M}(3)$, the tricomplex algebra. Also let

$$G''(W) + B_1(W)G'(W) + B_0(W)G(W) = 0,$$

where

$$B_1(W) = \sum_{\ell=1}^{4} e^{w_\ell^{1/2}} E_\ell \quad \textit{and} \quad B_0(W) = \sum_{\ell=1}^{4} e^{w_\ell^2} E_\ell.$$

Then each idempotent component satisfies

$$G''_\ell(w_\ell) + e^{w_\ell^{1/2}} G'_\ell(w_\ell) + e^{w_\ell^2} G_\ell(w_\ell) = 0.$$

Here,

$$\varrho[e^{w_\ell^{1/2}}] = \tfrac{1}{2} \quad \textit{and} \quad \varrho[e^{w_\ell^2}] = 2.$$

In view of the dominance condition $\varrho[B_{0,\ell}] > \varrho[B_{1,\ell}]$. Each component G_ℓ satisfies

$$\varrho^{(\log)}[G_\ell] = 2.$$

Since the multicomplex order is the maximum amongst components, we have

$$\varrho^{(\log)}[G] = \max_{\ell} \varrho^{(\log)}[G_\ell] = 2.$$

Thus, the growth of the multicomplex solution is governed by the exponential term $e^{w_\ell^2}$ in the dominant coefficient B_0.

For more information about the theory and application of bicomplex and multicomplex cases, one can further follow [1, 9, 10, 11, 12, 14, 17, 18].

5 Alternative view point of multicomplex analysis [12]

A multicomplex number $\xi_n \in \mathbb{C}_n$ can be defined as

$$\xi_n \;=\; \xi_{n-1,1} \;+\; i_n\, \xi_{n-1,2},$$

where

$$\xi_{n-1,1},\, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

Addition and multiplication in multicomplex space $\mathbb{C}_n$ are defined componentwise extended by $(i_n)^2 = -1$.

5.1 Idempotent elements in $\mathbb{C}_n$

It is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2}, \ \frac{1-i_1i_2}{2}, \ \frac{1+i_1i_3}{2}, \ \frac{1-i_1i_3}{2}, \ \frac{1+i_2i_3}{2}, \ \frac{1-i_2i_3}{2}, \ \ldots, \ \frac{1+i_{n-1}i_n}{2}, \ \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) \ = \ \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) \ = \ \frac{1-i_ri_s}{2} \quad \text{where } r,s \in \mathbb{N}.$$

5.2 Idempotent representation

Let ξ be an element in $\mathbb{C}_n$ and let $\xi \ = \ \xi_1 + i_n\xi_2$ with $\xi_1, \ \xi_2 \in \mathbb{C}_{n-1}$. Then

$$\xi = \big(\xi_1 - i_{n-1}\,\xi_2\big)\,e\big(i_{n-1}i_n\big) \ + \ \big(\xi_1 + i_{n-1}\,\xi_2\big)\,e\big(-\,i_{n-1}i_n\big).$$

5.3 Idempotent Representation of Holomorphic Multicomplex Valued Functions

Let X be a domain in C_n, $n \geq 1$, and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

For further study on multicomplex analysis one can see [4, 6].

In view of this alternative perspective in multicomplex analysis, Theorem 4.1 can be reformulated as follows.

Theorem 5.1. *Let the multicomplex differential equation be given as*

$$G^{(n)}(W) + \sum_{j=0}^{n-1} B_j(W)\,G^{(j)}(W) = H(W) \tag{5.1}$$

where

$$\begin{aligned} G(W) &= G_1(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + G_2(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n), \\ B_j(W) &= B_{j,1}(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + B_{j,2}(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n) \\ \text{and} \quad H(W) &= H_1(w_1 - i_{n-1}w_2)e(i_{n-1}i_n) + H_2(w_1 + i_{n-1}w_2)e(-i_{n-1}i_n), \end{aligned}$$

are the idempotent decompositions of the multicomplex valued entire functions. Let us suppose that there exists an ℓ such that

$$\max_{j\neq\ell} \varrho[B_{j,p}] \ < \ \varrho[B_{\ell,p}] \ < \ 1 \quad \text{with} \quad p = 1, 2. \tag{5.2}$$

Then the logarithmic order of G satisfies that

$$\varrho^{(\log)}[G] = \varrho[B_\ell]$$

where $\varrho^{(\log)}[G]$ denotes the order of growth of $\log G$ componentwise i.e., it is equal to the maximal order of the logarithms of the nonvanishing idempotent components.

Proof. By employing the idempotent decomposition and mathematical induction, we can establish the proof of Theorem 5.1 and therefore the same is omitted. □

6 Open problems and future directions

The notion of bicomplex and multicomplex setting of certain theorems concerning the comparative growth rates of entire solutions of linear differential equations with entire coefficients may be extended for the functions of uncertain variables and are still virgin. Therefore those may be posed as open problems to the future researchers of this branch.

7 Conclusion

The methodologies as carried out in the paper focus on the fact that each bicomplex or multicomplex differential equation decomposes into independent complex equations, allowing the direct application of the tools as involved in classical complex analysis theories. Therefore all related growth type relations, order and asymptotic properties are governed by the maximum growth amongst the component functions.

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