ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, **Vol. 56(1) (2026), 192-205**

SOME INFERENCES ON CHARACTERIZATIONS OF PROBABILITY DISTRIBUTIONS, WITH SPECIAL REFERENCE TO THE CONTRIBUTIONS OF PROFESSOR C. R. RAO ON THE CHARACTERIZATIONS OF DISTRIBUTIONS

M. Ahsanullah[1], M. Shakil[2] and R. C. Singh Chandel[3]
[1]Professor Emeritus, Rider University, Lawrenceville, NJ, USA-08648
[2]Miami Dade College, Hialeah, FL, USA-33012
[3]Department of Mathematics, D.V. Postgraduate College, Orai, Uttar Pradesh, India - 285001
Email: ahsan@rider.edu, mshakil@mdc.edu, rc_chandel@yahoo.com

(Received: March 25, 2026*; In format: April* 13, 2026*; Revised: June* 25, 2026*; Accepted: June* 30, 2026*)*

DOI: https://doi.org/10.58250/jnanabha.2026.56119

Abstract

The characterization of probability distributions plays a fundamental role in statistical theory and mathematical statistics. Before applying any probabilistic model to observed data, it is essential to ensure that the assumed distribution satisfies the relevant defining properties. Characterization theorems provide powerful tools for identifying probability distributions through functional, moment-based, or distributional properties of random variables.

In this paper, we present several important characterization results for classical probability distributions, with special reference to the contributions of Professor C. R. Rao on the characterization of distributions. We also propose and develop some new characterization results for logistic distribution. Logistic distribution occupies an important place in statistical theory and is often regarded as a close analogue of normal distribution. It arises naturally in several theoretical and applied contexts, including logistic growth models and logistic regression. The present work discusses basic properties and selected characterizations of the logistic distribution, thereby contributing to the existing literature on characterization theory.

2020 Mathematical Sciences Classification: 60E05; 62E10; 62E15; 62G30.
Keywords and Phrases: Characterization; characteristic function; logistic distribution; moments; statistical inference.

1 Introduction

The identification and validation of probability distributions constitute a central theme in probability theory and mathematical statistics. Prior to applying any distributional model to real data, it is necessary to verify that the underlying assumptions of the model are satisfied. The study of characterizations of probability distributions addresses this issue by determining conditions under which a distribution is uniquely specified by certain properties of random variables. As pointed out by Nagaraja [27, 2006], "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". Furthermore, according to Koudou and Ley [19, 2014], "In probability and statistics, a characterization theorem occurs when a given distribution is the only one which satisfies a certain property. Besides their evident mathematical interest per se, characterization theorems also deepen our understanding of the distributions under investigation and sometimes open unexpected paths to innovations which might have been uncovered otherwise". Thus, it is very important to characterize a particular probability distribution subject to certain conditions before applying it to some real-world data.

Professor C. R. Rao occupies a preeminent place in the development of characterization theory for continuous statistical distributions. From the mid-twentieth century onward, as probability and statistics underwent rapid axiomatization and unification, Rao's work helped shape characterization theory into a rigorous and coherent discipline. His contributions clarified why certain distributions, most notably the normal distribution-emerge inevitably from natural probabilistic, geometric, and inferential principles rather than from ad hoc modeling assumptions. One of the early significant contributions in this area was made by Khatri and Rao [18, 1972] in their paper "Functional Equations and Characterization of Probability Laws through Linear Functions of Random Variables." They obtained important characterization results, particularly for the normal distribution, based on properties of linear functions of random variables. Their work stimulated further research and laid the groundwork for subsequent developments in distribution theory.

A major milestone in literature appeared in 1973 with the publication of the monograph Characterization Problems in Mathematical Statistics by Kagan, Linnik, and Rao [16, 1973].

This authoritative work presents a comprehensive collection of characterization results for the normal and several other distributions, including those based on the equality in distribution of linear statistics derived from random samples and on the constancy of regression of one random variable on another. The book continues to serve as a standard reference in the field.

Rao and Shanbhag [31, 1986] achieved further progress, by reviewing and unifying several important results on characterization problems and introduced refined versions of earlier theorems as corollaries of the Lau-Rao-Shanbhag *(LRS)* theorems. This connection led to some useful variants of the Lau-Rao theorem, which have been successfully applied to a variety of characterization problems in statistics (Lau and Rao [22, 1982]).

In recent years, several researchers and authors have investigated the characterizations based on truncated moments by employing simple relationships between two moments truncated from the left at the same point. For details, see Glänzel *et al.* [13, 1984], and Kotz and Shanbhag [21, 1980]. For another approach, which involves both left- and right-truncated moments, often expressed through products involving the reverse hazard rate and suitable functions of the truncation point, the interested readers are referred to Ahsanullah [1, 2017], Ahsanullah *et al.* [5, 2021], Shakil and Ahsanullah [33, 2015], and Shakil *et al.* [32, 2018]. Furthermore, Ahsanullah and Shakil [6, 2024] characterized the continuous uniform distribution using its characteristic function (CF), by analyzing its functional relations and properties. Moreover, for characterizations of logistic distribution through order statistics with independent exponential shifts, see Ahsanullah *et al.* [4, 2012]. These results define the distribution via specific functional forms, aiding in parameter estimation and goodness-of-fit tests.

For further study of characterization problems, the interested reader may consult the following monographs: Kagan *et al.* [16, 1973]; Mathai and Pederzoli [25, 1977]; Patil *et al.* [28, 1974]; Galambos and Kotz [12, 1978]; Kakosyan *et al.* [17, 1984]; and Ahsanullah [1,2, 2017, 2019].

Several characterizations of normal distribution appear in literature investigated by various authors. As pointed out by Kotz [20, 1974] "while the normal distribution with a more complex distribution function is well characterized, the characteristic property of the logistic distribution has not been thoroughly developed". This comment remains valid nowadays, although some characterization results are recently available in the literature (Lin and Hu [23, 2008]). The present paper reviews some well-known characterization results for the normal distribution and proposes new characterization results for the logistic distribution, thereby extending existing results in the theory of probability distributions.

The organization of paper is as follows: In Section 2 we provide some well-known characterization results for the normal distribution. Section 3 contains some basic properties of logistic distribution. In Section 4, we present main results on some new characterization results and inferences of logistic distribution. Some conclusions are drawn in Section 5.

2 Review of Some Well-known Characterization Results for the Normal Distribution

In what follows, we first present some well-known characterization results for the normal distribution, as provided below:

2.1 Theorem of Cramer or Cramér's Decomposition Theorem

Theorem 2.1. *One of the most important results on the characterizations of normal distribution was proved by Cramér* [10, 1936] *as stated below:*

If X_1 and X_2 are two independent random variables, $Z = X_1 + X_2$, and if Z is normally distributed, the X_1 and X_2 are normally distributed.

Proof. See Cramér [10, 1936]. □

2.2 Theorem of Darmois, Skitovich and Basu

Theorem 2.2. *Darmois* [11, 1953]*, Skitovitch* [34, 1953]*, and Basu* [9,1953] *independently obtained the following general result related to the characterization of the normal distribution using the property of* *stochastic independence of linear functions of independent random variables:*

Let $X_1, X_2, \ldots, X_n$ be independent random variables. Suppose

$$L_1 = a_1X_1 + a_2X_2 + \ldots\ldots + a_nX_n,$$

and

$$L_2 = b_1X_1 + b_2X_2 + \ldots\ldots + b_nX_n.$$

If L_1 and L_2 are independent, then, for each index $i (i = 1, 2, \ldots, n), a_i$ and $b_i \neq 0, X_i$ is normal.

Proof. See Darmois [11], Skitovitch [34], and Basu [9]. □

2.3 Theorem of Heyde

Theorem 2.3. *The Heyde theorem (Heyde [14, 1970]) characterizes the normal distribution by the symmetry of the conditional distribution of one linear form of independent random variables given another. Specifically, let $X_1, X_2, \ldots, X_n$ be independent random variables and $a_i's$ and $b_i's, i = 1, 2, \ldots, n$, be non-zero constants such that $a_ib_i^{-1} + a_jb_j^{-1} \neq 0$ for all $i \neq j$. Suppose*

$$L_1 = a_1X_1 + a_2X_2 + \ldots + a_nX_n$$

and

$$L_2 = b_1X_1 + b_2X_2 + \ldots + b_nX_n.$$

Then, if the conditional distribution of L_1 given L_2 is symmetric, the distributions of the $X_i's, (i = 1, 2, \ldots, n)$, are normal.

Proof. See Heyde [14]. □

2.4 Theorem of Kagan-Linnik-Rao

Theorem 2.4 (Kagan *et al.* [16, 1973])**.** *This theorem establishes that the admissibility of sample means as an estimator (specifically, the linearity of regression of the sample mean on residuals) characterizes the normal distribution. It states that if $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $E(X_i) = 0, i = 1, 2, \ldots, n$, and $n \geq 3$, and if*

$$E\left(\bar{X} \mid X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X}\right) = 0,$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the distributions of the $X_i's, i = 1, 2, \ldots, n$, are normal.

Proof. See Kagan *et al.* [16]. □

2.5 Theorem of Polya (or Pólya's characterization of the normal distribution)

Theorem 2.5 (Pólya [29, 1923])**.** *This theorem identifies the normal distribution as the only distribution (with finite variance) that is "stable" under the specific averaging transformation described. Specifically, suppose that X_1 and X_2 are two independent and identically distributed random variables with finite variance σ^2, then $(X_1 + X_2)/\sqrt{2}$ and X_1 are identically distributed if and only if X_1 is normally distributed with mean zero (or a shifted normal distribution if the mean is non-zero).*

Proof. See Pólya [29]. □

2.6 Theorem of Rao

Theorem 2.6 (Rao [30, 1967])**.** *If $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $E(X_i) = 0$ and $E\left(X_i^2\right) < \infty, i = 1, 2, \ldots, n$, for $n \geq 3$. Then, if $E\left(\bar{X} \mid X_i - \bar{X}\right) = 0$ for any $i, i = 1, 2, \ldots, n$, the distributions of the $X_i's, i = 1, 2, \ldots, n$, are normal.*

Proof. See Rao [30]. □

Remark 2.1. *It is also noted that, for $n = 2$, the above result (in Theorem of Rao) does not hold, as seen in the following example.*

Example 2.1. Suppose that the random variables X_1 and X_2 are independent and identically distributed Uniform [-1, 1], and have the joint pdf $f(x_1, x_2)$ as

$$f(x_1, x_2) = \frac{1}{4}, -1 \le x_1, x_2 \le 1,$$
$$= 0, \text{ otherwise .}$$

Let $2Y_1 = X_1 + X_2$ and $2Y_2 = X_1 - X_2$, where $Y_1 = \bar{X} = \frac{X_1+X_2}{2}$ (the sample mean). Then, the conditional pdf of $Y_1 \mid Y_2 = y_2$ is

$$f(y_1 \mid Y_2 = y_2) = \frac{1}{2(1-|y_2|)}, -(1-|y_2|) \le y_1 \le (1-|y_2|).$$

Thus

$$E(Y_1 \mid Y_2 = y_2) = \int_{-(1-|y_2|)}^{(1-|y_2|)} (y_1) \frac{1}{2(1-|y_2|)} dy_1 = 0$$

Since $Y_1 = \bar{X}$ and Y_2 is a linear function of $X_1 - X_2$, the condition $Y_2 = y_2$ is equivalent to specifying a value for $X_1 - X_2$.

Hence

$$E\left(\bar{X} \mid X_1 - X_2\right) = 0.$$

This completes the proof of Example 2.6.

2.7 Theorem of Zinger and Linnik

Theorem 2.7 (Zinger and Linnik [36, 1964]). *Suppose that $a_1, a_2, \ldots, a_n$ are positive numbers and $\varphi_i(t), i = 1, 2, .., n$, are characteristic functions of independent random variables X_i 's repectively. Further, if*

$$(\varphi_1(t))^{a_1} \left(\varphi_2(t)^{a_2} \ldots \left(\varphi_n(t)^{a_n} = e^{-\frac{t^2}{2}} \text{ for } |t| < t_0, t_0 > 0,\right.\right.$$

then $\varphi_i(t), i = 1, 2, .., n$, are characteristic functions of normal distribution. Thus $Xi's, i = 1, 2, \ldots, n$, are normally distributed.

Proof. See Zinger and Linnik [36]. □

2.8 Theorem of Ahsanullah and Hamedani

Theorem 2.8 (Ahsanullah and Hamedani [7, 1988]). *Suppose X_1 and X_2 are independent and identically symmetric (about zero) random variables and $Z = \min(X_1, X_2)$. If Z^2 is distributed as chi-square one degree of freedom, then X_1 and X_2 are normally distributed.*

Proof. See Ahsanullah and Hamedani [7]. □

Remark 2.2. *The above theorem is also true if minimum is replaced by maximum for Z.*

3 Some Basic Properties of Logistic Distribution

This paper aims to contribute to the understanding of logistic distribution by presenting its basic properties and characterizations. We will discuss some of its characterization techniques that enhance its applicability in statistical analysis, thus enhancing its use in model building and its connection to other distributions. In what follows, we present some basic properties of logistic distribution.

3.1 Logistic Distribution

The logistic distribution is a symmetric distribution, unimodal and similar in shape to the normal distribution. The cumulative distribution function of the logistic distribution is also a scaled version of the of the hyperbolic tangent, as given below:

$$F(x, \mu, \sigma) = \frac{1}{1 + e^{-\frac{x-\mu}{\sigma}}} = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x-\mu}{2\sigma}\right), \tag{3.1}$$

where $-\infty < x < \infty, -\infty < \mu < x < \infty, \sigma > 0, \mu$ and σ being location and scale parameters respectively. The corresponding probability density function (*pdf*) is given by

$$f(x, \mu, \sigma) = \frac{1}{4\sigma}\operatorname{sech}^2\left(\frac{x-\mu}{2\sigma}\right). \tag{3.2}$$

To illustrate the behavior of the logistic distribution, the graphs of its *pdf* (2.2) for some selected values of the location, μ, and scale, σ, parameters, are given in Figure 3.1. It is observed that, though the shapes of the logistic distribution and the normal distribution resemble each other, the logistic distribution has heavier tails in view of the fact its kurtosis is high which is approximately 1.2 .

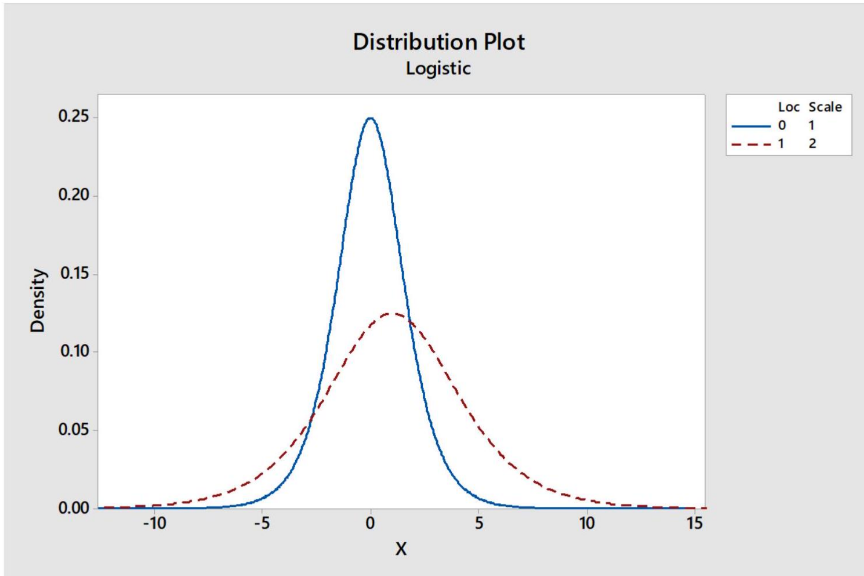

***Figure* 3.1:** *pdfs of the logistic distribution for $\mu = 0, \sigma = 1$; and $\mu = 1, \sigma = 2$, respectively.*

Logistic distribution is widely used in statistical modeling, often preferred over the normal distribution due to its simpler cumulative distribution function (CDF). This simplicity, alongside its symmetric and unimodal shape, makes the logistic distribution particularly useful in applications such as logistic regression and modeling of logistic growth processes. The CDF of the logistic distribution is a scaled version of the hyperbolic tangent, which further enhances its applicability.

Notably, both the United States Chess Federation *(USCF)* and the World Chess Federation (commonly referred to by its French acronym *FIDE*) employ the logistic distribution to assess the skill levels of chess players, replacing the traditional use of the normal distribution for this purpose. This shift underscores the utility of the logistic distribution in various practical settings where normal distribution models might fall short.

3.2 Standard Logistic Distribution

We say a random variable X has a standard logistic distribution if the probability density function (pdf) $f(x)$ is as follows:

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, -\infty < x < \infty. \tag{3.3}$$

The corresponding cumulative distribution function (*cdf*)$F(x)$ is given by

$$F(x) = \frac{1}{1 + e^{-x}}. \tag{3.4}$$

The *pdf* and *cdf* of the standard logistic distribution satisfy the following relations:

1. $\frac{f'(x)}{f(x)} = \frac{1-e^{x}}{1+e^{-x}}$,
2. $f(x) = F(x)(1 - F(x))$,
3. The odd moments are zero, that is,

$$E\left(X^{2n+1}\right) = 0, \quad n = 1, 2, \ldots.$$

4. Even moments $E\left(X^{2n}\right), n = 1, 2, \ldots$, are given as follows:

$$E\left(X^{2n}\right) = \int_{-\infty}^{\infty} x^{2n} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx, n = 1, 2, \ldots$$
$$= \int_0^1 \left(\ln \frac{p}{1-p}\right)^n dp$$
$$= \pi^n \left(2^n - 2\right) \left|B_n\right|,$$

where B_n is the nth Bernoulli number.

4 Some New Characterization Results and Inferences of Logistic Distribution

In what follows, we present some new characterization results and inferences of logistic distribution.

4.1 Characterizations Based on Characteristic Function

Let $\varphi(t)$ be the characteristic function of the standard logistic distribution, given by

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx$$
$$= \int_0^1 u^{it}(1-u)^{-it} du$$
$$= \Gamma(1-it)E(1+it)$$
$$= \frac{\pi t}{\sinh(\pi t)}.$$

The following theorems give characterizations of the standard logistic distribution based on characteristic function.

Theorem 4.1. *If $\varphi(t)$ is the characteristic function of an absolutely continuous random variable X with finite expectation. Then X has the standard logistic random variable if and only if*

$$\varphi'(t) = \varphi(t)\left(\frac{1}{t} - \pi \frac{\cosh(\pi t)}{\sinh(\pi t)}\right).$$

Proof. Suppose the random variable has the pdf $f(x)$ as

$$f(x) = \frac{e^{-x}}{\left(1+e^{-x}\right)^2}.$$

We have

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \frac{e^{-x}}{\left(1+e^{-x}\right)^2} dx.$$

Let $u = \left(1+e^{-x}\right)^{-1}$, then

$$\varphi(t) = \int_0^1 e^{it \ln\left(\frac{u}{1-u}\right)} du$$
$$= \int_0^1 u^{it}(1-u)^{-it} du$$
$$= B(1+it, 1-it)$$
$$= \Gamma(1-it)\Gamma(1+it)$$
$$= -it\Gamma(-it)\Gamma(1+it)$$
$$= -it\pi \csc(\pi it)$$
$$= \pi t \operatorname{csech}(\pi t),$$

from which by taking natural log of both sides, we obtain

$$\ln \varphi(t) = \ln(i\pi t) + \ln \operatorname{csech}(\pi t).$$

Now differentiating the above expression with respect to t, we obtain

$$\varphi'(t) = \varphi(t)\left(\frac{1}{t} - \pi \frac{\cosh(\pi t)}{\sinh^2(\pi t)}\right).$$

Dividing both sides of the above equation by $\varphi(t)$, we obtain

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{1}{t} - \pi\frac{\cosh(\pi t)}{\sinh(\pi t)}.$$

Now integrating the above equation with respect to t, we obtain

$$\ln\varphi(t) = c + \ln t - \ln\sinh(\pi t), t > 0,$$

from which by taking the limit of both sides of the above equation, when $t \to 0$, we obtain

$$\lim_{t\to 0}\ln\varphi(t) = c + \lim_{t\to 0}\ln\frac{t}{\sinh(\pi t)} = c + \ln\frac{1}{\pi}.$$

Now, using the characteristic function property in the above equation, that is,

$$\varphi(0) = 1 \Rightarrow \ln\varphi(0) = 0,$$

we obtain

$$0 = c - \ln\pi \Rightarrow c = \ln\pi.$$

Hence, substituting back $c = \ln\pi$ in the above equation, we obtain

$$\begin{aligned}\operatorname{Ln}\varphi(t) &= \ln\pi + \ln t - \ln\sinh(\pi t)\\ &= \ln\left(\frac{\pi t}{\sinh(\pi t)}\right).\end{aligned}$$

Thus, exponentiating both sides of the above equation, we obtain

$$\varphi(t) = \frac{\pi t}{\sinh(\pi t)},$$

or

$$\varphi(t) = \pi t\,\operatorname{csch}(\pi t),$$

which is the characteristic function of standard logistic distribution. This completes the proof of Theorem 4.1. □

Theorem 4.2. *Suppose $X_1, X_2, \ldots, X_n$ are i.i.d random variables from standard logistic distribution and $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics. Let $W_{n-1,n-1} = \max(W_1, W_2, \ldots, W_{n-1})$, where $W_1, W_2, \ldots, W_n$ are i. i.d exponential random variables with* pdf $f(w) = e^{-w}, 0 < w < \infty$. *For an absolutely continuous random variable X, if X and $X_{1,n} + W_{n-1,n-1}$ are identically distributed, then X has standard logistic distribution.*

Proof. Let $\Phi_1(t)$ and $\Phi_2(t)$ be the characteristic functions of $X_{1,n}$ and $W_{n,n}$ respectively. We have

$$\Phi_1(t) = n\int_{-\infty}^{\infty} e^{itx}\left(\frac{e^{-x}}{1+e^{-x}}\right)^{n-1}\frac{e^{-x}}{(1+e^{-x})^2}dx.$$

Let $u = \frac{1}{1+e^{-x}}$, then (Ahsanullah *et al.* [4, 2012])

$$\begin{aligned}\Phi_1(t) &= n\int_0^1 u^{n-1}\left(\frac{u}{1-u}\right)^{it}\left(\frac{1-u}{u}\right)^{n-1}du\\ &= n\int_0^1 u^{it}(1-u)^{n-it-1}du\\ &= nB(1+it, n-it)\\ &= \frac{\Gamma(1+it)\Gamma(n-it)}{\Gamma(n)}, \text{ using the definition of the Beta function, } B(x,y).\end{aligned}$$

Also

$$\Phi_2(t) = (n-1)\int_0^{\infty} e^{itx}\left(1-e^{-x}\right)^{n-2}e^{-x}dx$$

Let $u = 1 - e^{-x}$. Then

$$
\begin{aligned}
\Phi_2(t) &= (n-1)\int_0^1 u^{n-2}(1-u)^{-it}du \\
&= (n-1)B(1-it, n-1) \\
&= \frac{\Gamma(1-it)\Gamma(n)}{\Gamma(n-it)}.
\end{aligned}
$$

The characteristic function of $X_{1,n} + W_{n-1,n-1}$ is

$$
\begin{aligned}
\Phi_1(t)\Phi_2(t) &= \frac{\Gamma(1+it)\Gamma(n-it)}{\Gamma(n)}\frac{\Gamma(1-it)\Gamma(n)}{\Gamma(n-it)} \\
&= \Gamma(1+it)\Gamma(1-it),
\end{aligned}
$$

which is the characteristic function of the standard logistic distribution. This completes the proof of Theorem 4.2. ☐

Theorem 4.3. *If an absolutely continuous random variable X has the characteristic function $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$ if and only if the pdf of the random variable has the standard logistic distribution, where $f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, -\infty < x < \infty$.*

Proof. We have already seen that the characteristic function $\varphi(t)$ of the standard logistic distribution is $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.

We will prove here the only if case.
Suppose $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.
Using the inverse formulae of the characteristic function, we have

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\Gamma(1-it)\Gamma(1+it)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\frac{\pi t}{\cosh(\pi t)}dt \\
&= \int_{-\infty}^{\infty} e^{-itx}\frac{t}{e^{\pi t} - e^{-\pi t}}dt \\
&= \int_0^{\infty} e^{-itx}\frac{t}{e^{\pi t} - e^{-\pi t}}dt + \int_{-\infty}^{0} e^{-itx}\frac{t}{e^{\pi t} - e^{-\pi t}}dt \\
&= \int_0^{\infty} e^{-itx}\frac{t}{e^{\pi t} - e^{-\pi t}}dt + \int_0^{\infty} e^{itx}\frac{t}{e^{\pi t} - e^{-\pi t}}dt \\
&= \int_0^{\infty} e^{-itx}\frac{te^{-\pi t}}{1 - e^{-2\pi t}}dt + \int_0^{\infty} e^{itx}\frac{te^{-\pi t}}{1 - e^{-2\pi t}}dt \\
&= \sum_{n=0}^{\infty}\int_0^{\infty} te^{-t(ix+(2n+1)\pi)}dt + \sum_{n=0}^{\infty}\int_0^{\infty} te^{t(ix-(2n+1)\pi)}dt \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(ix+(2n+1)\pi)^2} + \frac{1}{(ix-(2n+1)\pi)^2}\right) \\
&= \sum_0^{\infty}\frac{2(2n+1)^2\pi^2 - 2x^2}{\left((2n+1)^2\pi^2 + x^2\right)^2}.
\end{aligned}
\tag{4.1}
$$

Now, it is well-known that the identity $\frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right) = \sum_0^{\infty}\frac{2(2n+1)^2\pi^2 - 2x^2}{((2n+1)^2\pi^2+x^2)^2}$ is a valid mathematical identity derived from the partial fraction expansion of the hyperbolic secant function. Hence, (4.1) represents the series representation for $\frac{1}{4}\operatorname{sech}^2\left(\frac{x}{2}\right)$, and thus (4.1) reduces to

$$
f(x) = \frac{1}{4}\operatorname{sech}^2(x/2). \tag{4.2}
$$

The hyperbolic secant function given in (4.2) can easily be expressed in its exponential form and simplifying, as follows:

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, -\infty < x < \infty,$$

which is the required pdf of the logistic distribution. This completes the proof of Theorem 4.3. □

4.2 Characterizations Based on Domain of Attraction

In this section on characterization by domain of attraction, the following definition and examples are relevant:

Definition of Domain of Attraction: A distribution $F(x)$ is said to lie in the domain of attraction of a limit distribution $G(x)$ if, after appropriate normalization (shifting and scaling), the distribution of the sum, maximum, or minimum of n independent and identically distributed random variables converges in distribution to $G(x)$ as n tends to infinity.

Examples:

- **Normal Distribution:** By the central limit theorem, for many distributions with finite variance, the normalized sum is in the domain of attraction of the normal distribution.
- **Extreme Value Distributions:** The normalized maximum or minimum of many distributions is in the domain of attraction of the Gumbel, Fréchet, or Weibull distributions.
  In what follows, we present some characterization results of logistic distribution based on domain of attraction, as provided in Theorems 4.4, 4.5 and 4.6.

Theorem 4.4. *Suppose $X_1, X_2, \ldots, X_n$ are $n, (n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $S_n = X_1 + X_2 + \ldots, +X_n$. Then, the normalized sum $\left(= \frac{\sqrt{\left(\frac{3}{n}\right)} s_n}{\pi}\right)$ belongs to the domain of the standard normal distribution.*

Proof. If the random variable X has the standard logistic distribution, then its characteristic function $\varphi(t) = \Gamma(1-it)\Gamma(1+it)$.

The characteristic function $\Psi_n(t)$ of $\frac{\sqrt{\left(\frac{3}{n}\right)} S_n}{\pi}$ is

$$\Psi n(t) = \left(\Gamma\left(1 - \sqrt{\left(\frac{3}{n}\right)}\frac{it}{\pi}\right)\Gamma\left(1 + \sqrt{\left(\frac{3}{n}\right)}\frac{it}{\pi}\right)\right)^n.$$

Then, using the identity $\Gamma(1+z)\Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}$ and for small z :

$$\Psi_n(t) = \left(\frac{\frac{\sqrt{3}}{\sqrt{n}}t}{\sinh\left(\frac{\sqrt{3}}{\sqrt{n}}t\right)}\right)^n.$$

Substitute $x = \frac{\sqrt{3}}{\sqrt{n}}t$ in the above equation. Then, we have

$$\Psi_n(t) = \left(\frac{x}{\sinh(x)}\right)^n.$$

Expanding $\sinh(x)$ for small x, we obtain

$$\frac{x}{\sinh(x)} = 1 - \frac{x^2}{6} + o\left(x^2\right).$$

Substitute $x = \frac{\sqrt{3}}{\sqrt{n}}t$ in the above equation. Then, we have

$$\Psi_n(t) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n.$$

Thus, taking the limit of both sides of the above equation as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \Psi_n(t) = \lim_{n\to\infty}\left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n = e^{-t^2/2},$$

which is the characteristic function of the standard normal distribution. Thus $\frac{\sqrt{\left(\frac{3}{n}\right)} S_n}{\pi}$ belongs to the domain of the standard normal distribution. This completes the proof of Theorem 4.4. □

Now, we prove the following two Theorems 4.5 and 4.6 for the domain of attraction of the maximum and the minimum of the logistic random variables. To prove these two theorems, we need the following lemmas.

Lemma 4.1 (Von Mises [26, 1964])**.** *Let $X_1, X_2, \ldots, X_n$ be $n, (n > 1)$, independent and identically distributed continuous random variables with cdf $F(x)$, pdf $f\ (x)$, $\bar{F}(x) = 1$ - $F(x)$ and $\beta(f) = \sup\{x \mid F(x) < 1$. If $F(x)$ has derivatives on $[c_0, \beta(f)]$, for some $c_0 > 0$, then, if $\lim_{x\uparrow\beta(F)} \frac{f(x)}{\bar{F}(x)} = c, c > 0$, then F belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. See Von Mises [26]. □

Lemma 4.2 (Ahsanullah and Nevzorov [8, 2001, p 87])**.** *Let F^{-1} denote the inverse of cdf F. For a continuous random variable, the necessary and sufficient condition that $X_{n,n}$ belongs to the domain of attraction of type 1 extreme value distribution of the maximum is*

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1,$$

and the normalizing constants are $a_n = F^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = F^{-1}\left(1 - \frac{1}{ne}\right) - F^{-1}\left(1 - \frac{1}{n}\right)$.

Proof. See Ahsanullah and Nevzorov [8, 2001, p 87]. □

Lemma 4.3 (Ahsanullah and Nevzorov [8, 2001, p 87])**.** *For a continuous random variable, the necessary and sufficient condition that $X_{1,n}$ belong to the domain of attraction of type 1 extreme value distribution of the maximum is*

$$\lim_{c\to 0} \frac{F^{-1}(c) - F^{-1}(2c)}{F^{-1}(2c) - F^{-1}(4c)} = 1.$$

The normalizing constants are $a_n = F^{-1}\left(1 - \frac{1}{n}\right)$ and $b_n = F^{-1}\left(1 - \frac{1}{ne}\right) - F^{-1}\left(1 - \frac{1}{n}\right)$.

Proof. See Ahsanullah and Nevzorov [8, 2001, p 87]. □

Theorem 4.5. *Theorem 4.5. Suppose $X_1, X_2, \ldots, X_n$ are $n, (n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $X_{n,n} = \max(X_1, X_2, \ldots, X_n)$. Then the normalized $X_{n,n}$ belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. We shall prove the sufficient condition for the domain of attraction of logistic distribution for $X_{n,n}$. It can be shown that, for standard logistic distribution,

$$\lim_{c\to 0} \frac{F^{-1}(1-c) - F^{-1}(1-2c)}{F^{-1}(1-2c) - F^{-1}(1-4c)} = 1,$$

and $a_n = \ln(n)$ and $b_n = 1$.

Thus

$$P\left(X_{n,n} \le (x + \ln n)\right.$$
$$= \lim_{n\to\infty} \left(\frac{1}{1 + e^{-(x+\ln n)}}\right)^n = e^{-e^{-x}}.$$

Hence $X_{n,n}$ belongs to the domain of attraction of Gumbel (type 1 extreme value) distribution of maximum. This completes the proof of the sufficient condition for the domain of attraction of logistic distribution for $X_{n,n}$. □

Theorem 4.6. *Suppose $X_1, X_2, \ldots, X_n$ are n, $(n > 1)$, independent and identically distributed random variables from the standard logistic distribution, and $X_{1,n} = \min(X_1, X_2, \ldots, X_n)$. Then the normalized $X_{1,n}$ belongs to the domain of Gumbel (type 1 extreme value) distribution.*

Proof. We shall prove the sufficient condition for the domain of attraction of logistic distribution for $X_{1,n}$. It can be shown that for standard logistic distribution

$$\lim_{c\to 0} \frac{F^{-1}(c) - F^{-1}(2c)}{F^{-1}(2c) - F^{-1}(4c)} = 1,$$

and $a_n = -\ln(n)$ and $b_n = 1$.

Thus

$$
\begin{aligned}
& P\left(X_{1,n} \geq x-\ln n\right) \\
& =1-P\left(X_{n.n}>x-\ln n\right) \\
& =\lim_{n \rightarrow \infty}\left(1+e^{x-\ln n}\right)^{-n} \\
& =e^{-e^{x}} .
\end{aligned}
$$

Hence

$F_{1,n}(x)=1-e^{-e^{x}}$, which is the type 1 extreme value distribution of the minimum.

The normalizing constants are the same as that of $X_{n,n}$. Thus

$$
\begin{aligned}
& \lim_{n \rightarrow \infty} P\left(X_{1,n} \leq x+\ln n\right) \\
& =\lim_{n \rightarrow \infty}\left(1+e^{(x+\ln n)}\right)^{-n} \\
& =\lim_{n \rightarrow \infty}\left(1+\frac{1}{n} e^{-x}\right)^{-n} \\
& =e^{-e^{-x}} .
\end{aligned}
$$

Hence $X_{1,n}$ belongs to the domain of attraction of Gumbel (type 1 extreme value) distribution of minimum. This completes the proof of the sufficient condition for the domain of attraction of logistic distribution for $X_{1,n}$. □

4.3 Characterizations by Truncated Moments

In what follows, we will establish our proposed characterization results of logistic distribution by right truncated moment, that is, by taking a relation between the right truncated moment and reversed failure rate function. To prove this, we need the following Lemmas.

Lemma 4.4. *Suppose the random variable X is absolutely continuous with* cdf $F(x)$ *and pdf $f(x)$. Let $a=\inf\{x \mid F(x)>0\}$, and $b=\sup\{x \mid F(x)<1\}$. We also assume that $E(X)$ and $f'(x)$ exists for all $x, x \in(a, b)$. If $E(X \geq x)=h(x) \tau(x)$, where $\tau(x)=\frac{f(x)}{1-F(x)}, h'(x)$ exists and $\int_{a}^{b} \frac{u+h'(x)}{h(u)} du$ is finite for all $x, x \in(a, b)$, then $f(x)=c e^{-\int_{a}^{x} \frac{u+h/(x)}{h(u)} du}$, where c is determined by the condition $\int_{a}^{b} f(x) dx=1$.*

Proof. See Ahsanullah [1, 2017, p.50]. □

Lemma 4.5. *Suppose that X is a standard logistic random variable, with the pdf given by*

$$
f(x)=\frac{e^{-x}}{\left(1+e^{-x}\right)^{2}}=\frac{e^{x}}{\left(1+e^{x}\right)^{2}}, x \in(-\infty, \infty).
$$

Then,

$$
\int_{x}^{\infty} u f(u) du=\ln \left(1+e^{x}\right)-\frac{x e^{x}}{\left(1+e^{x}\right)} .
$$

Proof. Using the integration by parts formula, and, on simplification, it can easily be seen that the integral

$$
\int_{x}^{\infty} u \frac{e^{u}}{\left(1+e^{u}\right)^{2}} du=\ln \left(1+e^{x}\right)-\frac{x e^{x}}{\left(1+e^{x}\right)} .
$$

□

Theorem 4.7. *Suppose that the random variable X is absolutely continuous (with respect to Lebesgue measure) with cdf $F(x)$ and pdf $f(x)$ for $x \in(-\infty, \infty)$. Suppose that the random variable X has finite expectation and $f'(x)$ exists for all $x \in(-\infty, \infty)$. Then X is a standard logistic random variable if and only if $E(X \geq x)=h(x) \tau(x)$, where $\tau(x)=\frac{f(x)}{1-F(x)}$ and*

$$
h(x)=\frac{\left(1+e^{x}\right)^{2}}{e^{x}} \ln \left(1+e^{x}\right)-x\left(1+e^{x}\right) .
$$

Proof. The proof of "if" part easily follows from Lemma 4.6. We will prove here the "only if" case.

Suppose that $E(X \geq x) = h(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{1-F(x)}$.

Then, we have

$$\frac{E(X \geq x)}{\tau(x)} = h(x),$$

or

$$\frac{\int_x^\infty uf(u)du}{f(x)} = h(x). \tag{4.3}$$

In the above equation (4.3), using Lemma 4.5, after simplification, we have

$$h(x) = \frac{(1+e^x)^2}{e^x} \ln(1+e^x) - x(1+e^x). \tag{4.4}$$

Differentiating both sides of the above equation with respect to x, we obtain

$$h'(x) = \left(e^x - e^{-x}\right) \ln(1+e^x) - xe^x. \tag{4.5}$$

Now, expressing (4.3) as

$$\int_x^\infty uf(u)du = h(x)f(x),$$

and differentiating both sides of the above equation with respect to x, we obtain

$$-xf(x) = h'(x)f(x) + h(x)f'(x),$$

or

$$\frac{f'(x)}{f(x)} = -\frac{x + h'(x)}{h(x)}. \tag{4.6}$$

In the above equation, using (4.5) and (4.6), after simplification, we obtain

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{(1-e^x)\left[x - \frac{e^x+1}{e^x}\ln(1+e^x)\right]}{(1+e^x)\left[x - \frac{e^x+1}{e^x}\ln(1+e^x)\right]} \\ &= \frac{1-e^x}{1+e^x}. \end{aligned} \tag{4.7}$$

Now, it is easily seen that the expression on the right hand side of the above equation (4.7) is given by

$$\frac{d\ln\left(\frac{e^x}{(1+e^x)^2}\right)}{dx} = \frac{1-e^x}{1+e^x}.$$

Hence, we have

$$\frac{d\ln f(x)}{dx} = \frac{d\ln\left(\frac{e^x}{(1+e^x)^2}\right)}{dx}.$$

On integrating the above equation with respect to x, we obtain

$$\ln f(x) = c + \ln\frac{e^x}{(1+e^x)^2}, \tag{4.8}$$

where c is a constant to be determined.

Exponentiating both sides of the equation (4.8), we obtain

$$f(x) = c\frac{e^x}{(1+e^x)^2}, \tag{4.9}$$

where c is the normalizing constant.

On integrating the equation (4.8) with respect to x from $x = -\infty$ to $x = +\infty$, we easily obtain $\int_{-\infty}^\infty \frac{e^x}{(1+e^x)^2}dx = 1$, and hence using the condition $\int_{-\infty}^\infty f(x)dx = 1$, we obtain $c = 1$.

Thus, we have

$$f(x) = \frac{e^x}{(1+e^x)^2} = \frac{e^{-x}}{(1+e^{-x})^2},$$

which is the probability density function of the standard logistic distribution. This completes the proof of Theorem 4.7. $\square$

Remark 4.1. *For characterization of logistic distribution based on the left truncated moment, $E(X \mid X \leq x)$, that is, by taking a relation between left truncated moment and failure rate function, see Ahsanullah and Shakil* [3, 2015].

5 Conclusion

Logistic distribution offers several advantages over the normal distribution in certain statistical modeling contexts. Its simple *CDF*, symmetric shape, and applicability in logistic regression make it a powerful tool in statistical science. The characterization of this distribution, as well as methods for determining its suitability for real-world data, is an ongoing area of research. As demonstrated by the works of Khatri, Rao, Shanbhag, Glänzel, and others, the characterization of probability distributions through moments and truncated moments provides valuable insights that aid in understanding and applying the logistic distribution effectively.

In this paper, we have presented an overview of the key properties and characterizations of the logistic distribution, underscoring its significance in model building and statistical inference. Further research into advanced characterization methods will continue to expand the utility of this distribution in various domains of statistical analysis.

Acknowledgement. The authors are grateful to the Editor for his kind encouragement, support, and valuable and helpful comments and suggestions for modifying and updating the manuscript, which greatly and considerably helped in improving the quality and presentation of the manuscript. Finally, the authors are thankful to the anonymous referees for their helpful comments.

Declarations

Authors' Contributions: All the authors have equally made their contributions and have read and approved the final manuscript.

Funding: The authors state that they have no funding source for this paper.

Availability of Data and Materials: Not applicable.

Declarations Conflict of Interest: The authors declare that they have no competing interests. The authors state that no funding source or sponsor has participated in the realization of this work.

References

[1] M. Ahsanullah, *Characterizations of Univariate Distributions,* Atlantis Press, Paris, France, 2017.

[2] M. Ahsanullah, *Characterizations of Exponential Distribution by Ordered Random Variables.* Nova Science Publishers, New York, USA, 2019.

[3] M. Ahsanullah, and M. Shakil, Characterizations of continuous probability distributions occurring in physics and applied sciences by truncated moments, *International Journal of Advances in Statistics and Probability,* **3**(1) (2015), 100-114.

[4] M. Ahsanullah, G. P. Yanev, and C. Onica, Characterizations of logistic distribution through order statistics with independent exponential shifts, *Economic Quality Control,* **27** (2012), 85-96.

[5] M. Ahsanullah, M. Shakil and Ahmed M. T. Abd El-Bar, Some Inferences on Offset Normal Distribution: Properties and Characterizations, *Jñānābha*, **51**(2) (2021), 143-151.

[6] M. Ahsanullah, and M. Shakil, Characterizations of Continuous Uniform Distribution by Some Functional Relations of Characteristic Functions, *J. Stat. Appl. Pro. Lett.* **11**(1) (2024), 33-40.

[7] M. Ahsanullah, and G. G. Hamedani, Some characterizations of normal distribution, *Calcutta Statistical Association Bulletin,* **37** (1988), 95-99.

[8] M. Ahsanullah, and V. B. Nevzorov, *Ordered Random Variables,* Nova Science Publishers, New York, USA, 2001.

[9] D. Basu, On the independence of linear functions of chance variables, *Bulletin of the International Statistical Institute,* **33** (1953), 83-96.

[10] H. Cramer, Uber eine Eigenschaft der normal en Verteilungsfunktion, *Mathematische Zeitschrift,* **41** (1936), 405-414.

[11] G. Darmois, Analyse Générale Des Liaisons Stochastiques: Etude Particuliére de l'analyse Factorielle Linéaire, R*evue de l'Institut International de Statistique / Review of the International Statistical Institute,* **21**(1/2) (1953), 2-8.

[12] J. Galambos, and S. Kotz, *Characterizations of Probability Distributions, Lecture Notes in Mathematics,* **675**, Springer-Verlag, New York, USA, 1978.

[13] W. Glänzel, A. Telcs, and A. Schubert, Characterization by truncated moments and its application to Pearson-type distributions, *Zeitschrift fr Wahrscheinlichkeitstheorie und Verwandte Gebiete,* **66** (1984), 173-183.

[14] C. C. Heyde, Characterization of the Normal Law by the Symmetry of a Certain Conditional Distribution, *Sankhya: The Indian Journal of Statistics,* **Series A, 32** (1) (1970), 115-118.

[15] A. M. Kagan, Yu. V. Linnik, and C. R. Rao, On a characterization of the normal law based on a property of the sample average, *Sankhya Ser. A,* **27** (1965), 405-406

[16] A. M. Kagan, Yu. V. Linnik, and C. R. Rao, *Characterization Problems in Mathematical Statistics,* John Wiley & Sons, New York, USA, 1973.

[17] A. V. Kakosyan, L. B. Klebanov, and J. A. Melamed, *Characterizations of Distributions by the Method of Intensively Monotone Operators,* Springer-Verlag, New York, USA, 1984.

[18] C. G. Khatri, and C. R. Rao, Functional equations and characterization of probability laws through linear functions of random variables, *Journal of Multivariate Analysis,* **2**(2) (1972), 162-173.

[19] A. E. Koudou, and C. Ley, Characterizations of GIG laws: A survey, *Probability surveys,* **11** (2014), 161-176.

[20] S. Kotz, Characterizations of statistical distributions: a supplement to recent surveys, *Internet. Statist. Rev.,* **42** (1974), 39-65.

[21] S. Kotz, and D. N. Shanbhag, Some new approaches to probability distributions, *Advances in Applied Probability,* **12** (1980), 903-921.

[22] K.-S. Lau, and C. R. Rao, Integrated Cauchy functional equation and characterizations of the exponential law, *Sankhya: The Indian Journal of Statistics,* Series A, **44**(1) (1982), 72-90.

[23] G. D. Lin, and C.-Y. Hu, On characterizations of the logistic distribution, *J. Statist. Plann. Inference,* **138** (2008), 1147-1156.

[24] Yu. V. Linnik, *Decomposition of Probability Distributions,* Dover Publications, Inc., New York, USA, 1964.

[25] A. M. Mathai, and G. Pederzoli, *Characterizations of Normal Probability Law,* John Wiley & Sons, New York, USA, 1977.

[26] R. Von Mises, La distribution de la plus grande de n valeurs, *Revue Math. Union Interbalcanique,* **1** (1936), 141-160. Reprinted in Selected Papers of Richard von Mises, *Amer. Math. Soc.,* **2** (1964), 271-294.

[27] H. Nagaraja, *Characterizations of Probability Distributions,* In Springer Handbook of Engineering Statistics. Springer, London, UK., (2006), 79-95.

[28] G. P. Patil, S. Kotz, and J. K. Ord, *Statistical Distributions in Scientific Work: Characterizations and Applications,* Vol. **3**, D. Reidel Publishing Company, Boston, USA, 1974.

[29] G. Pólya, Herleitung des Gaußschen Fehlergesetzes auseiner Funktionalgleichung, *Mathematische Zeitschrift,* **18**(1) (1923), 96-108.

[30] C. R. Rao, On some characterizations of the normal law, *Sankhya: The Indian Journal of Statistics, Series A,* **29** (1967), 1-14.

[31] C. R. Rao, and D. N. Shanbhag, Recent results on characterization of probability distributions: A unified approach through extensions of Deny's theorem, *Advances in Applied Probability,* **18**(3) (1986), 660-678.

[32] M. Shakil, M. Ahsanullah, and B. M. G. Kibria, On the characterizations of Chen's twoparameter exponential power life-testing distribution, *Journal of Statistical Theory and Applications,* **17**(3) (2018), 393-407.

[33] M. Shakil, and M. Ahsanullah, Von Mises Distribution: Properties and Characterizations, *Jñānābha*, **45**(2) (2015), 13-30.

[34] V. P. Skitovitch, On a property of the normal distribution, *Doklady Akademii Nauk SSSR,* **89** (1953), 217-219.

[35] J. C. Su, and W. J. Huang, Characterizations based on conditional expectations, *Statistical Papers,* **41**(4) (2000), 423-435.

[36] A. A. Zinger, and Yu. V. Linnik, Characterizations of normal distribution, *Theory of Probability and Its Applications,* **2**(5) (1964), 432-456.