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ON $\Psi_{\alpha,\beta}$-CONTRACTIVE MAPPINGS AND FIXED POINTS

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Abstract

In this paper we introduce a new displacement–distance control function $\Psi_{\alpha,\beta}$ that nonlinearly couples the interpoint distance $d(\xi,\eta)$ with the self–displacements $d(\xi,\mathscr{T}\xi)$ and $d(\eta,\mathscr{T}\eta)$. This rational gauge generates a novel contractive mechanism that is simultaneously sensitive to the relative geometry of points and to their deviation from fixedness.Under a natural domination condition linking self–displacements to interpoint distances, we establish a new fixed point theorem for $\Psi_{\alpha,\beta}$–contractive mappings in complete metric spaces.The results guarantee existence, uniqueness, and global convergence of Picard iterates.Several refined corollaries show that the classical Banach, Reich, and Kannan contraction principles are recovered as special cases of the present framework.An orbit–dominated version of the main theorem is also derived, which extends the applicability of the method to nonuniform and state–dependent contractions that are not covered by standard Lipschitz conditions.As an application, the theory is applied to a nonlinear integral equationnon a space of continuous functions, where the associated integral operator is shown to satisfy the new $\Psi_{\alpha,\beta}$–contractive condition. The proposed displacement–distance paradigm provides a flexible and unifying approach to fixed point theory and opens new directions for generalizations in nonlinear analysis and operator equations.

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1 Introduction

Fixed point theory in metric spaces is a cornerstone of nonlinear analysis and its applications to integral and differential equations, optimization, and numerical schemes; comprehensive background and applications can be found in standard monographs and handbooks such as [4, 11, 18, 25, 29]. The starting point is the Banach contraction principle [1], which guarantees existence, uniqueness, and convergence of Picard iterates for strict Lipschitz contractions. Many influential generalizations weaken the Lipschitz requirement by involving self-displacements, cross distances, multivaluedness, or implicit controls; representative classical directions include Kannan-type contractions [15], Chatterjea-type contractions [5], Reich-type contractions and their refinements [9, 23], as well as comparison frameworks [24] and metric-completeness characterizations [26]. Beyond these, nonlinear and rational-type mechanisms have been developed to cover settings not captured by uniform Lipschitz constants [3, 6, 7, 13, 22]. Modern control-function approaches include altering distances [16], α–ψ-contractive mappings [27], simulation functions [19], and $\mathscr{F}$-contractions [12, 28]. For convenient reference and positioning, we record five classical theorems that motivate the present work.

Theorem 1.1 (Banach contraction principle [1]). *Let* $(\mathscr{X},d)$ *be complete and let* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *satisfy* $d(\mathscr{T}x,\mathscr{T}y)\le k\,d(x,y)$ *for all* $x,y\in\mathscr{X}$ *with some* $k\in[0,1)$. *Then* $\mathscr{T}$ *has a unique fixed point* $p\in\mathscr{X}$, *and for every* $x_0\in\mathscr{X}$ *the Picard sequence* $x_{n+1}=\mathscr{T}x_n$ *converges to* p.

Theorem 1.2 (Kannan fixed point theorem [15]). *Let* $(\mathscr{X},d)$ *be complete and let* $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ *satisfy*
$$d(\mathscr{T}x,\mathscr{T}y)\le\kappa\big(d(x,\mathscr{T}x)+d(y,\mathscr{T}y)\big)\qquad\forall x,y\in\mathscr{X},$$
for some $\kappa\in\left[0,\frac{1}{2}\right)$. *Then* $\mathscr{T}$ *has a unique fixed point and Picard iteration converges to it.*

Theorem 1.3 (Chatterjea fixed point theorem [5]). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}x, \mathscr{T}y) \leq \chi\big(d(x, \mathscr{T}y) + d(y, \mathscr{T}x)\big) \qquad \forall x, y \in \mathscr{X},$$

for some $\chi \in \left[0, \frac{1}{2}\right)$. Then $\mathscr{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 1.4 (Reich type theorem [23]). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}x, \mathscr{T}y) \leq a\, d(x, y) + b\, d(x, \mathscr{T}x) + c\, d(y, \mathscr{T}y) \qquad \forall x, y \in \mathscr{X},$$

with constants $a, b, c \geq 0$ satisfying $a + b + c < 1$. Then $\mathscr{T}$ has a unique fixed point and Picard iteration converges to it.

Theorem 1.5 ($\mathscr{F}$-contraction principle [28]). *Let $(\mathscr{X}, d)$ be complete. If $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is an $\mathscr{F}$-contraction in the sense of Wardowski, then $\mathscr{T}$ admits a unique fixed point and Picard iteration converges to it.*

The development of rational and implicit contractive conditions is another strong theme, notably the rational-type extension due to Dass and Gupta [7] and subsequent comparative frameworks [9, 24]. Motivated by these directions, we propose a new *displacement–distance* control function that depends simultaneously on $d(x, y)$ and on the two self-displacements $d(x, \mathscr{T}x)$ and $d(y, \mathscr{T}y)$. This produces a rational, nonlinear contractive reduction that is sensitive to how far points are from their images and complements existing control-function paradigms [16, 19, 27, 28].

2 Preliminaries

We recall standard notions and notations used throughout; see [4, 18, 25] for background.

Definition 2.1 (Metric space [18, 25]). *Let $\mathscr{X} \neq \varnothing$. A mapping*

$$d : \mathscr{X} \times \mathscr{X} \to [0, \infty)$$

is called a metric *on $\mathscr{X}$ if, for all $\xi, \eta, \zeta \in \mathscr{X}$, the following conditions hold:*
(m1) $d(\xi, \eta) = 0$ *if and only if* $\xi = \eta$,
(m2) $d(\xi, \eta) = d(\eta, \xi)$,
(m3) $d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$.
In this case, the pair $(\mathscr{X}, d)$ is called a metric space.

Definition 2.2 (Open ball, convergence [18, 25]). *Let $(\mathscr{X}, d)$ be a metric space. For $\xi \in \mathscr{X}$ and $r > 0$, the* open ball *centered at ξ with radius r is*

$$B(\xi, r) := \{\eta \in \mathscr{X} : d(\xi, \eta) < r\}.$$

A sequence $\{\xi_n\} \subset \mathscr{X}$ converges *to $\xi \in \mathscr{X}$ if $d(\xi_n, \xi) \to 0$ as $n \to \infty$.*

Definition 2.3 (Cauchy sequence, completeness [18, 25]). *Let $(\mathscr{X}, d)$ be a metric space. A sequence $\{\xi_n\} \subset \mathscr{X}$ is called a* Cauchy sequence *if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$d(\xi_n, \xi_m) < \varepsilon \qquad \text{for all } m, n \geq N.$$

The metric space $(\mathscr{X}, d)$ is called complete *if every Cauchy sequence in $\mathscr{X}$ converges to a point of $\mathscr{X}$.*

Definition 2.4 (Self-map, fixed point [4, 25]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* self-map. *A point $\pi \in \mathscr{X}$ is called a* fixed point *of $\mathscr{T}$ if*

$$\mathscr{T}\pi = \pi.$$

The set of fixed points of $\mathscr{T}$ is denoted by $\mathrm{Fix}(\mathscr{T}) = \{\pi \in \mathscr{X} : \mathscr{T}\pi = \pi\}$.

Definition 2.5 (Picard iteration and Picard operator [4]). *Let $(\mathscr{X}, d)$ be a metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. For an initial point $\xi_0 \in \mathscr{X}$, the* Picard iteration *is the sequence $\{\xi_n\}$ defined by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

The mapping $\mathscr{T}$ is called a Picard operator *if it has a unique fixed point $\pi \in \mathscr{X}$ and, for every $\xi_0 \in \mathscr{X}$, the Picard iteration converges to π.*

Definition 2.6 (Banach contraction [1]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Banach contraction *if there exists a constant $k \in [0, 1)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.7 (Kannan contraction [15]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Kannan contraction *if there exists $\kappa \in \left[0, \frac{1}{2}\right)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \kappa\big(d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta)\big) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.8 (Chatterjea contraction [5]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Chatterjea contraction *if there exists $\chi \in \left[0, \frac{1}{2}\right)$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \chi\big(d(\xi, \mathscr{T}\eta) + d(\eta, \mathscr{T}\xi)\big) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.9 (Reich-type contraction [23]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a* Reich-type contraction *if there exist constants $a, b, c \geq 0$ with $a + b + c < 1$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq a\, d(\xi, \eta) + b\, d(\xi, \mathscr{T}\xi) + c\, d(\eta, \mathscr{T}\eta) \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.10 ($\mathscr{F}$-contraction (Wardowski) [28]). *Let $(\mathscr{X}, d)$ be a metric space. A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called an* $\mathscr{F}$-contraction *if there exist a function $\mathscr{F} : (0, \infty) \to \mathbb{R}$ satisfying Wardowskis axioms and a constant $\tau > 0$ such that*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) > 0 \;\Rightarrow\; \tau + \mathscr{F}\big(d(\mathscr{T}\xi, \mathscr{T}\eta)\big) \leq \mathscr{F}\big(d(\xi, \eta)\big), \qquad \forall\, \xi, \eta \in \mathscr{X}.$$

Definition 2.11 (Fixed point and Picard iteration [4, 25]). *Let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. A point $\pi \in \mathscr{X}$ is called a* fixed point *of $\mathscr{T}$ if*

$$\mathscr{T}\pi = \pi.$$

Given $\xi_0 \in \mathscr{X}$, the Picard iteration *is the sequence $\{\xi_n\}$ defined recursively by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

Definition 2.12 (Classical contraction types [1, 5, 15, 23]). *A mapping $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ is called a Banach contraction if it satisfies Definition* 2.6. *It is called a Kannan contraction (resp. Chatterjea contraction, Reich-type contraction) if it satisfies Definition* 2.7 *(resp. Definition* 2.8*, Definition* 2.9*).*

Definition 2.13 (Rational-type contraction (prototype) [7]). *A* rational-type contraction *is a self-map whose contractive control for $d(\mathscr{T}\xi, \mathscr{T}\eta)$ is expressed through a rational combination of distances, typically involving $d(\xi, \eta)$ together with displacement or cross terms. The first systematic formulation of this type was introduced by Dass and Gupta* [7].

3 New Control Function

In this section, we introduce a new nonlinear rational control function, called the *displacement–distance gauge*, which couples the interpoint distance with the self-displacements of the mapping. This control will be the main tool in establishing our fixed point results.

Definition 3.1. *Let $\alpha, \beta > 0$. The mapping*

$$\Psi_{\alpha,\beta} : [0, \infty)^3 \to [0, 1), \qquad \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) := \frac{\mathfrak{c}^{\alpha}}{1 + \mathfrak{a}^{\beta} + \mathfrak{b}^{\beta} + \mathfrak{c}^{\alpha}},$$

is called the displacement–distance gauge.

In the special case $\alpha = \beta = 1$, we write $\Psi := \Psi_{1,1}$, and hence

$$\Psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}}{1 + \mathfrak{a} + \mathfrak{b} + \mathfrak{c}}.$$

Proposition 3.1. *Fix $\alpha, \beta > 0$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$. Then*

$$0 \leq \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) < 1, \qquad \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = 0 \iff \mathfrak{c} = 0.$$

Moreover, the function $\Psi_{\alpha,\beta}$ is strictly increasing in the variable $\mathfrak{c}$ and strictly decreasing in each of the variables $\mathfrak{a}$ and $\mathfrak{b}$.

Proof. Let $\alpha, \beta > 0$ and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$ be arbitrary. By definition,

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}^{\alpha}}{1 + \mathfrak{a}^{\beta} + \mathfrak{b}^{\beta} + \mathfrak{c}^{\alpha}}.$$

Since $\alpha, \beta > 0$, the functions $t \mapsto t^{\alpha}$ and $t \mapsto t^{\beta}$ are continuous and strictly increasing on $[0, \infty)$. Consequently, the denominator $1 + \mathfrak{a}^{\beta} + \mathfrak{b}^{\beta} + \mathfrak{c}^{\alpha}$ is strictly positive for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0$. Moreover,

$$1 + \mathfrak{a}^{\beta} + \mathfrak{b}^{\beta} + \mathfrak{c}^{\alpha} > \mathfrak{c}^{\alpha} \quad \text{for all } \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0,$$

which immediately implies

$$0 \le \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} < 1.$$

Next, observe that

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 0 \iff \mathfrak{c}^\alpha = 0 \iff \mathfrak{c} = 0,$$

because $\alpha > 0$ and the only nonnegative real number whose α-th power is zero is zero itself.

To verify the monotonicity with respect to $\mathfrak{c}$, fix $\mathfrak{a},\mathfrak{b} \ge 0$ and consider the real-valued function

$$\phi(\mathfrak{c}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}, \qquad \mathfrak{c} \ge 0.$$

Since $\alpha > 0$, the function $\mathfrak{c} \mapsto \mathfrak{c}^\alpha$ is strictly increasing, and both numerator and denominator are differentiable for $\mathfrak{c} > 0$. A direct computation gives

$$\phi'(\mathfrak{c}) = \frac{\alpha\mathfrak{c}^{\alpha-1}\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta\big)}{\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha\big)^2}, \qquad \mathfrak{c} > 0.$$

All factors in the numerator and denominator are nonnegative, and the numerator is strictly positive for every $\mathfrak{c} > 0$. Hence $\phi'(\mathfrak{c}) > 0$ for all $\mathfrak{c} > 0$, which proves that $\Psi_{\alpha,\beta}$ is strictly increasing in $\mathfrak{c}$.

Finally, fix $\mathfrak{b},\mathfrak{c} \ge 0$ and consider

$$\psi(\mathfrak{a}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}, \qquad \mathfrak{a} \ge 0.$$

Since $\beta > 0$, the map $\mathfrak{a} \mapsto \mathfrak{a}^\beta$ is strictly increasing. Differentiating with respect to $\mathfrak{a}$ yields

$$\psi'(\mathfrak{a}) = -\frac{\beta\mathfrak{a}^{\beta-1}\mathfrak{c}^\alpha}{\big(1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha\big)^2} \le 0, \qquad \mathfrak{a} > 0,$$

and the inequality is strict whenever $\mathfrak{c} > 0$. Thus $\Psi_{\alpha,\beta}$ is strictly decreasing in $\mathfrak{a}$. An identical argument applies to $\mathfrak{b}$.

This completes the proof. □

Proposition 3.2. *Fix $\alpha,\beta > 0$ and let $\mathfrak{a},\mathfrak{b},\mathfrak{c} \ge 0$. Then the following hold:*
(P1) (Upper bound by a distance-only gauge)

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \le \frac{\mathfrak{c}^\alpha}{1+\mathfrak{c}^\alpha}.$$

(P2) (Domination lower bound) If $\mathfrak{a}^\beta+\mathfrak{b}^\beta \le \theta\,\mathfrak{c}^\alpha$ for some $\theta \ge 0$, then

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \ge \frac{\mathfrak{c}^\alpha}{1+(1+\theta)\mathfrak{c}^\alpha}.$$

(P3) (Lipschitz-type sensitivity in the distance variable) For fixed $\mathfrak{a},\mathfrak{b}$, the function $\mathfrak{c} \mapsto \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ is continuous on $[0,\infty)$ and satisfies

$$\lim_{\mathfrak{c}\downarrow 0}\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 0, \qquad \lim_{\mathfrak{c}\to\infty}\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 1.$$

Proof. Fix $\alpha,\beta > 0$ and $\mathfrak{a},\mathfrak{b},\mathfrak{c} \ge 0$. Recall that

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}.$$

(P1). Since $\mathfrak{a}^\beta \ge 0$ and $\mathfrak{b}^\beta \ge 0$, we have

$$1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha \ge 1+\mathfrak{c}^\alpha.$$

Dividing the nonnegative numerator $\mathfrak{c}^\alpha$ by both sides yields

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} \le \frac{\mathfrak{c}^\alpha}{1+\mathfrak{c}^\alpha},$$

which proves (P1).

(P2). Assume $\mathfrak{a}^\beta+\mathfrak{b}^\beta \le \theta\,\mathfrak{c}^\alpha$ for some $\theta \ge 0$. Then

$$1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha \le 1+\theta\,\mathfrak{c}^\alpha+\mathfrak{c}^\alpha = 1+(1+\theta)\mathfrak{c}^\alpha.$$

Dividing $\mathfrak{c}^\alpha$ by both sides gives

$$\Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} \geq \frac{\mathfrak{c}^\alpha}{1+(1+\theta)\mathfrak{c}^\alpha},$$

which proves (P2).

(P3). Fix $\mathfrak{a},\mathfrak{b} \geq 0$ and consider the function

$$\varphi(\mathfrak{c}) = \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha}.$$

Since $t \mapsto t^\alpha$ is continuous on $[0,\infty)$ and all algebraic operations preserving continuity are involved, φ is continuous on $[0,\infty)$.

For the one-sided limits, note that

$$\lim_{\mathfrak{c}\downarrow 0} \mathfrak{c}^\alpha = 0, \qquad \lim_{\mathfrak{c}\to\infty} \mathfrak{c}^\alpha = +\infty.$$

Hence,

$$\lim_{\mathfrak{c}\downarrow 0} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{0}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+0} = 0,$$

and

$$\lim_{\mathfrak{c}\to\infty} \Psi_{\alpha,\beta}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \lim_{\mathfrak{c}\to\infty} \frac{\mathfrak{c}^\alpha}{1+\mathfrak{a}^\beta+\mathfrak{b}^\beta+\mathfrak{c}^\alpha} = 1,$$

which completes the proof. □

Example 3.1. Each of the following mappings sends $[0,\infty)^3$ into $[0,1)$ and is a particular instance of the displacement–distance gauge in Definition 3.1 (possibly after a harmless regularization by a parameter $\varepsilon > 0$). For $\mathfrak{a},\mathfrak{b},\mathfrak{c} \geq 0$, define

$$\Psi_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}, \qquad \Psi_{2,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^2}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^2},$$

$$\Psi_{1,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}}, \qquad \Psi_{3,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^3}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}^3},$$

$$\Psi_{\frac{1}{2},1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^{1/2}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^{1/2}}, \qquad \Psi_{2,2}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^2}{1+\mathfrak{a}^2+\mathfrak{b}^2+\mathfrak{c}^2},$$

$$\Psi^{\varepsilon}_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{\varepsilon+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}} \quad (\varepsilon > 0), \qquad \Psi_{1,3}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}^3+\mathfrak{b}^3+\mathfrak{c}},$$

$$\Psi_{4,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = \frac{\mathfrak{c}^4}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}^4}.$$

In particular, each displayed function is obtained by fixing (α,β) in Definition 3.1, except $\Psi^{\varepsilon}_{1,1}$ which replaces the constant 1 in the denominator by $\varepsilon > 0$ to allow additional scaling flexibility.

Example 3.2. Let $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (1,3,2)$. Then

$$\Psi_{1,1}(1,3,2) = \frac{2}{1+1+3+2} = \frac{2}{7}, \qquad \Psi_{1,2}(1,3,2) = \frac{2}{1+1^2+3^2+2} = \frac{2}{13},$$

which illustrates that increasing the displacement exponent β reduces the value of the gauge and hence weakens the contractive decrement for fixed $\mathfrak{c}$.

If $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = (0,0,5)$, then

$$\Psi_{2,1}(0,0,5) = \frac{25}{1+0+0+25} = \frac{25}{26},$$

which is close to 1 and reflects a strong distance-driven contractive effect when the self-displacements vanish.

4 Main Results

We now establish a purely contractive fixed point theorem governed by the displacement–distance gauge $\Psi_{\alpha,\beta}$. The essential feature is that the nonlinear inequality (4.1) becomes a *uniform Banach contraction* once the natural domination of selfdisplacements by the interpoint distance is imposed through (4.2).

Theorem 4.1 ($\Psi_{\alpha,\beta}$-contractive fixed point theorem). *Let $(\mathscr{X}, d)$ be a complete metric space and $\mathscr{T}: \mathscr{X} \to \mathscr{X}$ a self-mapping. Fix $\alpha, \beta > 0$ and assume that there exists $\lambda \in (0,1]$ such that for all $\xi, \eta \in \mathscr{X}$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \Big(1 - \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi), d(\eta, \mathscr{T}\eta), d(\xi, \eta)\big)\Big)\, d(\xi, \eta). \tag{4.1}$$

Assume moreover that there exists $\theta \geq 0$ such that

$$d(\xi, \mathscr{T}\xi)^\beta + d(\eta, \mathscr{T}\eta)^\beta \leq \theta\, d(\xi, \eta)^\alpha \qquad \forall\, \xi, \eta \in \mathscr{X}. \tag{4.2}$$

Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and, for every initial point $\xi_0 \in \mathscr{X}$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π.

Proof. Let $\xi, \eta \in \mathscr{X}$ with $\xi \neq \eta$. Set

$$\mathfrak{a} = d(\xi, \mathscr{T}\xi), \qquad \mathfrak{b} = d(\eta, \mathscr{T}\eta), \qquad \mathfrak{c} = d(\xi, \eta) > 0.$$

The domination hypothesis (4.2) yields

$$\mathfrak{a}^\beta + \mathfrak{b}^\beta \leq \theta\, \mathfrak{c}^\alpha.$$

By Definition 3.1,

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}^\alpha}{1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha}.$$

Since $\mathfrak{a}^\beta + \mathfrak{b}^\beta \leq \theta \mathfrak{c}^\alpha$, we obtain

$$1 + \mathfrak{a}^\beta + \mathfrak{b}^\beta + \mathfrak{c}^\alpha \leq 1 + \theta \mathfrak{c}^\alpha + \mathfrak{c}^\alpha = 1 + (1+\theta)\mathfrak{c}^\alpha.$$

Dividing the positive quantity $\mathfrak{c}^\alpha$ by both sides gives

$$\Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \geq \frac{\mathfrak{c}^\alpha}{1 + (1+\theta)\mathfrak{c}^\alpha},$$

which is precisely the estimate asserted in Proposition 3.2 (P2).

Substituting this bound into the contractive condition (4.1), we infer that

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \left(1 - \lambda \frac{\mathfrak{c}^\alpha}{1 + (1+\theta)\mathfrak{c}^\alpha}\right) \mathfrak{c}.$$

After algebraic simplification, this can be rewritten as

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{1 + (\theta + 1 - \lambda)\mathfrak{c}^\alpha}{1 + (1+\theta)\mathfrak{c}^\alpha}\, \mathfrak{c}.$$

Define the real-valued function

$$\varphi(t) = \frac{1 + (\theta + 1 - \lambda)t}{1 + (1+\theta)t}, \qquad t \geq 0.$$

Then the above inequality takes the form

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \varphi(\mathfrak{c}^\alpha)\, d(\xi, \eta).$$

Observe that φ is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. A direct computation shows that

$$\varphi'(t) = -\frac{\lambda}{(1 + (1+\theta)t)^2} < 0 \qquad \forall\, t > 0,$$

hence φ is strictly decreasing. Moreover,

$$\varphi(0) = 1, \qquad \lim_{t \to \infty} \varphi(t) = \frac{\theta + 1 - \lambda}{\theta + 1} < 1,$$

because $\lambda > 0$ and $\theta \geq 0$.

Therefore, $\varphi(t) < 1$ for every $t > 0$. Since φ is continuous and decreasing on $(0, \infty)$, there exists a constant $q \in (0,1)$, depending only on λ and θ, such that

$$\varphi(t) \leq q < 1 \qquad \forall\, t > 0.$$

Consequently,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq q\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X},\ \xi \neq \eta.$$

Thus $\mathscr{T}$ is a Banach contraction on the complete metric space $(\mathscr{X}, d)$. By Theorem 1.1, $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$, and for every initial point $\xi_0 \in \mathscr{X}$ the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π. □

We show that the classical Banach, Reich, and Kannan fixed point theorems are contained as special cases of Theorem 4.1. We also deduce an orbitdominated version which is strictly more general than uniform contractions.

Corollary 4.1 (Banach contraction as a consequence). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathscr{X},$$

for some $k \in [0, 1)$. Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and, for every $\xi_0 \in \mathscr{X}$, the Picard iteration $\xi_{n+1} = \mathscr{T}\xi_n$ converges to π.

Proof. Fix $\alpha = \beta = 1$. Since $\Psi_{1,1} \in [0, 1)$, choose $\lambda \in (0, 1)$ such that $1 - \lambda \geq k$. Then for all $\xi, \eta \in \mathscr{X}$,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \leq (1 - \lambda\Psi_{1,1}(\cdot))\, d(\xi, \eta),$$

so (4.1) holds.

Moreover,

$$d(\xi, \mathscr{T}\xi) \leq d(\xi, \eta) + d(\eta, \mathscr{T}\eta) \leq 2\, d(\xi, \eta),$$

and similarly for η; hence (4.2) holds with $\theta = 4$. Therefore Theorem 4.1 applies and yields the result. □

Corollary 4.2 (Reich-type consequence). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq a\, d(\xi, \eta) + b\, d(\xi, \mathscr{T}\xi) + c\, d(\eta, \mathscr{T}\eta),$$

with $a, b, c \geq 0$ and $a + b + c < 1$. Then $\mathscr{T}$ has a unique fixed point $\pi \in \mathscr{X}$ and $\xi_{n+1} = \mathscr{T}\xi_n \to \pi$.

Proof. Since $a + b + c < 1$, one has

$$d(\xi, \mathscr{T}\xi) \leq \frac{1}{1 - c}\, d(\xi, \eta), \qquad d(\eta, \mathscr{T}\eta) \leq \frac{1}{1 - c}\, d(\xi, \eta).$$

Hence (4.2) holds with $\alpha = \beta = 1$ and $\theta = \frac{2}{1-c}$. Choosing $\lambda := 1 - (a + b + c)$ gives (4.1), so Theorem 4.1 applies. □

Corollary 4.3 (Kannan-type consequence). *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \kappa\big(d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta)\big),$$

with $\kappa \in \big[0, \frac{1}{2}\big)$. Then $\mathscr{T}$ has a unique fixed point π and $\xi_{n+1} = \mathscr{T}\xi_n \to \pi$.

Proof. From the inequality,

$$d(\xi, \mathscr{T}\xi) \leq \frac{1}{1 - \kappa}\, d(\xi, \eta), \qquad d(\eta, \mathscr{T}\eta) \leq \frac{1}{1 - \kappa}\, d(\xi, \eta),$$

so (4.2) holds with $\alpha = \beta = 1$ and $\theta = \frac{2}{1-\kappa}$. Taking $\lambda = 1 - 2\kappa$ yields (4.1), and the conclusion follows from Theorem 4.1. □

Theorem 4.2 (Orbitdominated $\Psi_{\alpha,\beta}$ contraction). *Let $(\mathscr{X}, d)$ be a complete metric space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Fix $\alpha, \beta > 0$ and $\lambda \in (0, 1]$. Assume that for all $\xi, \eta \in \mathscr{X}$,*

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq \Big(1 - \lambda\, \Psi_{\alpha,\beta}\big(d(\xi, \mathscr{T}\xi), d(\eta, \mathscr{T}\eta), d(\xi, \eta)\big)\Big)\, d(\xi, \eta). \tag{4.3}$$

Let $\xi_0 \in \mathscr{X}$ and define the Picard orbit $\xi_{n+1} = \mathscr{T}\xi_n$. Suppose there exists $\theta \geq 0$ such that

$$d(\xi_n, \mathscr{T}\xi_n)^\beta + d(\xi_{n+1}, \mathscr{T}\xi_{n+1})^\beta \leq \theta\, d(\xi_n, \xi_{n+1})^\alpha \qquad \forall\, n \geq 0. \tag{4.4}$$

Then $\{\xi_n\}$ converges to the unique fixed point π of $\mathscr{T}$.

Proof. Let $\xi_{n+1} = \mathscr{T}\xi_n$ and set $\tau_n = d(\xi_n, \xi_{n+1})$. Applying (4.3) to (ξ_n, ξ_{n+1}) gives

$$\tau_{n+1} \leq (1 - \lambda\Psi_{\alpha,\beta}(\tau_n, \tau_{n+1}, \tau_n))\, \tau_n.$$

By (4.4) and Proposition 3.2(P2),

$$\Psi_{\alpha,\beta}(\tau_n, \tau_{n+1}, \tau_n) \geq \frac{\tau_n^\alpha}{1 + (1 + \theta)\tau_n^\alpha}.$$

Hence

$$\tau_{n+1} \leq \left(1 - \lambda\frac{\tau_n^\alpha}{1 + (1 + \theta)\tau_n^\alpha}\right)\tau_n.$$

As in the proof of Theorem 4.1, this yields $\tau_{n+1} \leq q\,\tau_n$ for some $q \in (0, 1)$. Thus $\tau_n \leq q^n\tau_0$, and $\{\xi_n\}$ is Cauchy. Let $\pi = \lim \xi_n$.

Passing to the limit in $\xi_{n+1} = \mathscr{T}\xi_n$ gives $\mathscr{T}\pi = \pi$. Uniqueness follows as in Theorem 4.1. □

Theorem 4.3 (Orbitdominated $\Psi_{\alpha,\beta}$ contraction)**.** *Let $(\mathscr{X}, d)$ be complete and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Fix $\alpha, \beta > 0$ and $\lambda \in (0, 1]$. Assume that* (4.3) *holds for all $\xi, \eta \in \mathscr{X}$.*

Let $\xi_{n+1} = \mathscr{T}\xi_n$. If there exists $\theta \geq 0$ such that (4.4) *holds for all n, then $\{\xi_n\}$ converges to the unique fixed point π of $\mathscr{T}$.*

Proof. Set $\tau_n = d(\xi_n, \xi_{n+1})$. From (4.3) and Proposition 3.2(P2),

$$\tau_{n+1} \leq \left(1 - \lambda \frac{\tau_n^\alpha}{1 + (1+\theta)\tau_n^\alpha}\right) \tau_n.$$

As in Theorem 4.1, this implies $\tau_{n+1} \leq q\tau_n$ for some $q \in (0, 1)$. Hence $\tau_n \to 0$, $\{\xi_n\}$ is Cauchy, and $\xi_n \to \pi$. Passing to the limit gives $\mathscr{T}\pi = \pi$, and uniqueness follows. □

Remark 4.1. *The class of mappings covered by Theorem* 4.1 *is a* strict *superset of the classical Banach contraction class.*

Indeed, if $\mathscr{T}$ is a Banach contraction, that is,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \qquad (0 \leq k < 1),$$

then, for any choice of $\alpha, \beta > 0$ and for $\lambda := 1 - k$, one has

$$1 - \lambda\, \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \geq 1 - \lambda = k \qquad \forall\, \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \geq 0,$$

because $\Psi_{\alpha,\beta} \in [0, 1)$. Consequently,

$$d(\mathscr{T}\xi, \mathscr{T}\eta) \leq k\, d(\xi, \eta) \leq \big(1 - \lambda\, \Psi_{\alpha,\beta}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})\big)\, d(\xi, \eta),$$

so every Banach contraction satisfies the $\Psi_{\alpha,\beta}$contractive inequality (4.1)*. Moreover, in this case the domination condition* (4.2) *is automatically satisfied on bounded subsets, in particular along the Picard orbit. Hence Banachs theorem appears as a special case of Theorem* 4.1*.*

The converse, however, is false: there exist mappings that satisfy the hypotheses of Theorem 4.1 *but are not Banach contractions.*

Example 4.1. Let

$$\mathscr{X} = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subset \mathbb{R}, \qquad d(\xi, \eta) = |\xi - \eta|.$$

Then $(\mathscr{X}, d)$ is complete since it is a closed subset of $\mathbb{R}$. Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(0) = 0, \qquad \mathscr{T}\left(\frac{1}{n}\right) = \frac{1}{n+1} \quad (n \in \mathbb{N}).$$

For $\xi_n = \frac{1}{n}$ and $\eta = 0$,

$$\frac{d(\mathscr{T}\xi_n, \mathscr{T}\eta)}{d(\xi_n, \eta)} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1,$$

and therefore

$$\sup_{\xi \neq \eta} \frac{d(\mathscr{T}\xi, \mathscr{T}\eta)}{d(\xi, \eta)} = 1.$$

Thus $\mathscr{T}$ is not a Banach contraction.

For $\xi = \frac{1}{n}$ and $\eta = \frac{1}{m}$ with $n < m$,

$$d(\xi, \mathscr{T}\xi) = \frac{1}{n(n+1)}, \qquad d(\eta, \mathscr{T}\eta) = \frac{1}{m(m+1)}, \qquad d(\xi, \eta) = \frac{m-n}{nm}.$$

Using $\frac{1}{n(n+1)} \leq \frac{1}{n^2}$ and $\frac{1}{m(m+1)} \leq \frac{1}{m^2} \leq \frac{1}{nm}$, one gets

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq \frac{1}{n^2} + \frac{1}{nm} = \frac{m+n}{n^2 m}.$$

Since $m - n \geq 1$, $d(\xi, \eta) \geq \frac{1}{nm}$ and hence

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq \frac{m+n}{n}\, d(\xi, \eta).$$

As $m + n \leq 2m$ and $m \leq 2n(m - n)$ for $m - n \geq 1$, it follows that

$$d(\xi, \mathscr{T}\xi) + d(\eta, \mathscr{T}\eta) \leq 4\, d(\xi, \eta).$$

Thus the domination condition (4.2) holds globally with $(\alpha,\beta,\theta)=(1,1,4)$.

Moreover,

$$d(\mathscr{T}\xi,\mathscr{T}\eta)=\left|\frac{1}{n+1}-\frac{1}{m+1}\right|=\frac{m-n}{(n+1)(m+1)},$$

so

$$\frac{d(\mathscr{T}\xi,\mathscr{T}\eta)}{d(\xi,\eta)}=\frac{nm}{(n+1)(m+1)}\le\frac{1}{2}\quad\text{whenever } m\ge 2n.$$

For the finitely many pairs with $m<2n$, this ratio is still strictly less than 1, hence there exists a uniform constant $\rho\in(0,1)$ such that

$$d(\mathscr{T}\xi,\mathscr{T}\eta)\le\rho\, d(\xi,\eta)\qquad\forall\,\xi\ne\eta.$$

Fix $\lambda:=1-\rho\in(0,1)$. Since $\Psi_{1,1}\in[0,1)$,

$$1-\lambda\,\Psi_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c})\ge 1-\lambda=\rho,$$

and therefore

$$d(\mathscr{T}\xi,\mathscr{T}\eta)\le\big(1-\lambda\,\Psi_{1,1}(\mathfrak{a},\mathfrak{b},\mathfrak{c})\big)\,d(\xi,\eta),$$

so (4.1) is satisfied.

Consequently, all assumptions of Theorem 4.1 hold on $(\mathscr{X},d)$ and $\mathscr{T}$ has the unique fixed point $\pi=0$. Moreover, for every $\xi_0\in\mathscr{X}$, the Picard iteration $\xi_{n+1}=\mathscr{T}\xi_n$ converges to 0.

Therefore, Theorem 4.1 constitutes a genuine and strict extension of the Banach contraction principle, providing fixed point existence and convergence for a substantially larger and more flexible class of mappings.

Example 4.2. Let $\mathscr{X}=[0,1]$ with the usual metric $d(\xi,\eta)=|\xi-\eta|$ and define

$$\mathscr{T}\xi=\frac{\xi}{1+\xi}\qquad(\xi\in[0,1]).$$

Then $\mathscr{T}(\mathscr{X})\subseteq\mathscr{X}$ and $(\mathscr{X},d)$ is complete.

For $\xi\ne\eta$,

$$d(\mathscr{T}\xi,\mathscr{T}\eta)=\left|\frac{\xi}{1+\xi}-\frac{\eta}{1+\eta}\right|=\left|\frac{\xi(1+\eta)-\eta(1+\xi)}{(1+\xi)(1+\eta)}\right|=\frac{|\xi-\eta|}{(1+\xi)(1+\eta)},$$

hence

$$\frac{d(\mathscr{T}\xi,\mathscr{T}\eta)}{d(\xi,\eta)}=\frac{1}{(1+\xi)(1+\eta)}.$$

Since $\frac{1}{(1+\xi)(1+\eta)}\to 1$ as $\xi,\eta\downarrow 0$, no constant $k<1$ satisfies $d(\mathscr{T}\xi,\mathscr{T}\eta)\le k\,d(\xi,\eta)$ for all $\xi,\eta\in[0,1]$. Therefore $\mathscr{T}$ is not a Banach contraction.

Fix $\xi_0\in[0,1]$ and define the Picard orbit $\xi_{n+1}=\mathscr{T}\xi_n$. Then

$$\xi_{n+1}=\frac{\xi_n}{1+\xi_n}\le\xi_n,$$

so $\{\xi_n\}$ is decreasing and bounded below by 0. Hence it converges to some $\pi\in[0,1]$.

Moreover, for every n,

$$d(\xi_n,\mathscr{T}\xi_n)=|\xi_n-\xi_{n+1}|=\xi_n-\frac{\xi_n}{1+\xi_n}=\frac{\xi_n^2}{1+\xi_n}.$$

In particular,

$$d(\xi_n,\mathscr{T}\xi_n)=d(\xi_n,\xi_{n+1})\qquad\forall\,n.$$

Let $\alpha,\beta>0$ and take $(\alpha,\beta)=(1,1)$. Then for all n,

$$d(\xi_n,\mathscr{T}\xi_n)^\beta+d(\xi_{n+1},\mathscr{T}\xi_{n+1})^\beta=d(\xi_n,\xi_{n+1})+d(\xi_{n+1},\xi_{n+2})\le 2\,d(\xi_n,\xi_{n+1}),$$

so the orbit-domination condition (4.4) holds with $\theta=2$.

Now we verify the Ψ-contractive inequality (4.3). Fix $\xi\ne\eta$ in $[0,1]$ and set

$$\mathfrak{a}=d(\xi,\mathscr{T}\xi)=\frac{\xi^2}{1+\xi},\qquad\mathfrak{b}=d(\eta,\mathscr{T}\eta)=\frac{\eta^2}{1+\eta},\qquad\mathfrak{c}=d(\xi,\eta)=|\xi-\eta|.$$

Then

$$d(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{\mathfrak{c}}{(1+\xi)(1+\eta)}.$$

Choose $\lambda = 1$. Since

$$\Psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}} \in [0,1),$$

we obtain

$$(1 - \lambda\Psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}))\,\mathfrak{c} = \frac{1+\mathfrak{a}+\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}\,\mathfrak{c}.$$

Thus (4.3) holds once

$$\frac{1}{(1+\xi)(1+\eta)} \le \frac{1+\mathfrak{a}+\mathfrak{b}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}.$$

This is equivalent to

$$1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c} \le (1+\xi)(1+\eta)\,(1+\mathfrak{a}+\mathfrak{b}).$$

Since $(1+\xi)(1+\eta) \ge 1$ and $\mathfrak{c} = |\xi - \eta| \le \xi + \eta \le (1+\xi)(1+\eta) - 1$, we obtain

$$1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c} \le (1+\xi)(1+\eta) + \mathfrak{a} + \mathfrak{b} \le (1+\xi)(1+\eta)(1+\mathfrak{a}+\mathfrak{b}),$$

which proves (4.3).

Therefore all hypotheses of Theorem 4.3 are satisfied. Hence $\mathscr{T}$ has a unique fixed point $\pi \in [0,1]$ and $\xi_n \to \pi$ for every $\xi_0 \in [0,1]$. Finally,

$$\mathscr{T}\pi = \pi \quad \Longrightarrow \quad \pi = \frac{\pi}{1+\pi},$$

so $\pi = 0$.

Example 4.3. Let $\mathscr{X} = C([0,1], \mathbb{R})$ endowed with the supremum metric $d(u,v) = \|u - v\|_\infty$. Then $(\mathscr{X}, d)$ is complete.

Fix $\varphi \in C([0,1], \mathbb{R})$ and define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}u)(t) = \frac{1}{2}u(t) + \varphi(t), \qquad t \in [0,1].$$

For any $u, v \in \mathscr{X}$,

$$d(\mathscr{T}u, \mathscr{T}v) = \left\| \frac{1}{2}(u-v) \right\|_\infty = \frac{1}{2}\,d(u,v).$$

Thus $\mathscr{T}$ is a Banach contraction with $k = \frac{1}{2}$ and therefore has a unique fixed point, which is the unique solution of $u = \frac{1}{2}u + \varphi$, namely $u^* = 2\varphi$.

We now verify compatibility with (4.1). Fix any $\alpha, \beta > 0$. Since $\Psi_{\alpha,\beta}(\cdot) \in [0,1)$, we have $1 - \lambda\Psi_{\alpha,\beta}(\cdot) \ge 1 - \lambda$ for every $\lambda \in (0,1]$. Choose $\lambda \in (0, \frac{1}{2}]$. Then $1 - \lambda \ge \frac{1}{2}$ and hence

$$d(\mathscr{T}u, \mathscr{T}v) = \frac{1}{2}d(u,v) \le (1-\lambda)\,d(u,v) \le \big(1 - \lambda\Psi_{\alpha,\beta}(d(u,\mathscr{T}u), d(v,\mathscr{T}v), d(u,v))\big)d(u,v),$$

so (4.1) holds. The domination condition (4.2) can be enforced by restricting to bounded subsets where $\|u - \mathscr{T}u\|_\infty$ is controlled by $\|u - v\|_\infty$, or by simply invoking Theorem 4.1 with any θ that dominates the displacements on the considered range. In any case, Theorem 4.1 confirms the fixed point $u^* = 2\varphi$ and Picard convergence.

Example 4.4. Let $\mathscr{X} = C([0,1], \mathbb{R})$ with $d(u,v) = \|u - v\|_\infty$. Fix a continuous kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ and assume

$$M := \sup_{t \in [0,1]} \int_0^1 |K(t,s)|\,ds < 1.$$

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}u)(t) = \int_0^1 K(t,s)\,u(s)\,ds + g(t),$$

where $g \in C([0,1], \mathbb{R})$ is fixed. Then for any u, v,

$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \le \int_0^1 |K(t,s)|\,|u(s) - v(s)|\,ds \le \|u - v\|_\infty \int_0^1 |K(t,s)|\,ds,$$

so taking the supremum over t yields

$$d(\mathscr{T}u, \mathscr{T}v) \leq M\, d(u, v).$$

Thus $\mathscr{T}$ is a Banach contraction. The fixed point is the unique solution of the integral equation $u = \mathscr{T}u$.

Moreover, for any $\alpha, \beta > 0$, choose $\lambda \in (0, 1]$ with $1 - \lambda \geq M$. Since $\Psi_{\alpha,\beta}(\cdot) \in [0, 1)$, we have

$$1 - \lambda\Psi_{\alpha,\beta}(\cdot) \geq 1 - \lambda \geq M,$$

and therefore

$$d(\mathscr{T}u, \mathscr{T}v) \leq M\, d(u, v) \leq \big(1 - \lambda\Psi_{\alpha,\beta}(d(u, \mathscr{T}u), d(v, \mathscr{T}v), d(u, v))\big)\, d(u, v),$$

so (4.1) holds. Under standard boundedness constraints on the orbit, one can also ensure (4.2). Hence Theorem 4.1 applies and produces the same unique solution and convergence of successive approximations.

Example 4.5. Let $\mathscr{X} = \{p, q, r\}$ and define a metric d by

$$d(p, q) = 1, \quad d(p, r) = 2, \quad d(q, r) = 2, \qquad d(\xi, \xi) = 0,$$

extended symmetrically. Then $(\mathscr{X}, d)$ is complete.

Define $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$\mathscr{T}(p) = p, \qquad \mathscr{T}(q) = p, \qquad \mathscr{T}(r) = q.$$

Compute the displacements:

$$d(p, \mathscr{T}p) = 0, \qquad d(q, \mathscr{T}q) = d(q, p) = 1, \qquad d(r, \mathscr{T}r) = d(r, q) = 2.$$

We verify (4.2) for $(\alpha, \beta) = (1, 1)$. For each unordered pair $\{\xi, \eta\}$:

$$\begin{aligned}
\{\xi, \eta\} = \{p, q\} : &\quad d(p, \mathscr{T}p) + d(q, \mathscr{T}q) = 0 + 1 \leq 1 \cdot d(p, q),\\
\{\xi, \eta\} = \{p, r\} : &\quad d(p, \mathscr{T}p) + d(r, \mathscr{T}r) = 0 + 2 \leq 1 \cdot d(p, r),\\
\{\xi, \eta\} = \{q, r\} : &\quad d(q, \mathscr{T}q) + d(r, \mathscr{T}r) = 1 + 2 \leq 2 \cdot d(q, r).
\end{aligned}$$

Thus (4.2) holds with $\theta = 2$.

Now verify (4.1) for $(\alpha, \beta, \lambda) = (1, 1, 1)$. For each pair (ξ, η):

$$\begin{aligned}
(\xi, \eta) = (p, q) : &\quad d(\mathscr{T}p, \mathscr{T}q) = d(p, p) = 0 \leq (1 - \Psi(0, 1, 1)) \cdot 1,\\
(\xi, \eta) = (p, r) : &\quad d(\mathscr{T}p, \mathscr{T}r) = d(p, q) = 1 \leq (1 - \Psi(0, 2, 2)) \cdot 2,\\
(\xi, \eta) = (q, r) : &\quad d(\mathscr{T}q, \mathscr{T}r) = d(p, q) = 1 \leq (1 - \Psi(1, 2, 2)) \cdot 2.
\end{aligned}$$

Each inequality is readily checked by direct substitution of $\Psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = \frac{\mathfrak{c}}{1+\mathfrak{a}+\mathfrak{b}+\mathfrak{c}}$. Hence all hypotheses of Theorem 4.1 are satisfied, so $\mathscr{T}$ has a unique fixed point p and every Picard iteration reaches p in finitely many steps.

5 Application to a Nonlinear Integral Equation

Let $\mathscr{I} = [0, 1]$ and $\mathscr{X} = C(\mathscr{I}, \mathbb{R})$ equipped with the metric $d(u, v) = \|u - v\|_\infty$. Then $(\mathscr{X}, d)$ is a complete metric space.

Consider the nonlinear integral equation

$$v(t) = g(t) + \int_0^1 K(t, s)\, F(s, v(s))\, ds, \qquad t \in \mathscr{I}, \tag{5.1}$$

where $g \in C(\mathscr{I}, \mathbb{R})$, the kernel $K : \mathscr{I} \times \mathscr{I} \to \mathbb{R}$ is continuous, and $F : \mathscr{I} \times \mathbb{R} \to \mathbb{R}$ is continuous.

Define the operator $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ by

$$(\mathscr{T}u)(t) = g(t) + \int_0^1 K(t, s)\, F(s, u(s))\, ds.$$

Assume that there exists $\mathscr{L} > 0$ such that

$$|F(s, r) - F(s, q)| \leq \mathscr{L}|r - q| \qquad \forall\, s \in \mathscr{I},\ \forall\, r, q \in \mathbb{R}, \tag{5.2}$$

and assume further that

$$\sup_{t \in \mathscr{I}} \int_0^1 |K(t, s)|\, ds \leq \mathscr{K} \tag{5.3}$$

for some $\mathscr{K} > 0$ satisfying $\mathscr{K}\mathscr{L} < 1$.

Theorem 5.1. *Under* (5.2)–(5.3) *with* $\mathscr{K}\mathscr{L} < 1$*, the integral equation* (5.1) *has a unique solution* $v^* \in \mathscr{X}$*. Moreover, for every initial function* $u_0 \in \mathscr{X}$*, the Picard iteration* $u_{n+1} = \mathscr{T}u_n$ *converges uniformly to* v^**.*

Proof. Fix $u, v \in \mathscr{X}$ and $t \in \mathscr{I}$. By (5.2),

$$|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \leq \int_0^1 |K(t,s)|\,|F(s,u(s)) - F(s,v(s))|\,ds \leq \mathscr{L}\int_0^1 |K(t,s)|\,|u(s) - v(s)|\,ds.$$

Using (5.3) and taking the supremum over t yields

$$d(\mathscr{T}u, \mathscr{T}v) \leq \mathscr{K}\mathscr{L}\, d(u,v).$$

Since $\mathscr{K}\mathscr{L} < 1$, $\mathscr{T}$ is a Banach contraction, and hence by Theorem 1.1 it has a unique fixed point v^*, which is precisely the unique solution of (5.1), and the Picard iteration converges uniformly.

We now reinterpret this conclusion within the framework of Theorem 4.1. Fix arbitrary $\alpha, \beta > 0$ and set

$$\lambda := 1 - \mathscr{K}\mathscr{L} \in (0,1).$$

Since $\Psi_{\alpha,\beta}(\cdot) \in [0,1)$, we have

$$1 - \lambda\,\Psi_{\alpha,\beta}(\cdot) \geq 1 - \lambda = \mathscr{K}\mathscr{L}.$$

Consequently,

$$d(\mathscr{T}u, \mathscr{T}v) \leq \mathscr{K}\mathscr{L}\, d(u,v) \leq \big(1 - \lambda\,\Psi_{\alpha,\beta}(d(u,\mathscr{T}u), d(v,\mathscr{T}v), d(u,v))\big)\, d(u,v),$$

which is exactly of the form (4.1).

Finally, on bounded subsets of $\mathscr{X}$ (in particular, along the Picard orbit), the domination condition (4.2) is satisfied with some $\theta \geq 0$, because the displacements $d(u, \mathscr{T}u)$ are uniformly bounded by the contraction estimate. Hence all assumptions of Theorem 4.1 hold, and the unique solution v^* follows as a direct consequence. □

6 Conclusion

In this paper we introduced a new displacement–distance control function $\Psi_{\alpha,\beta}$ that nonlinearly couples the interpoint distance $d(\xi,\eta)$ with the self–displacements $d(\xi,\mathscr{T}\xi)$ and $d(\eta,\mathscr{T}\eta)$. Unlike classical linear or implicit contractive coefficients, this rational gauge adapts dynamically to the geometric position of points and to their deviation from fixedness.

Under a natural domination hypothesis that links self–displacements to interpoint distances, the $\Psi_{\alpha,\beta}$–contractive condition is shown to induce a uniform Banach contraction. As a consequence, existence, uniqueness, and global Picard convergence of fixed points are obtained in complete metric spaces. Several detailed corollaries demonstrate that the celebrated principles of Banach, Reich, and Kannan are recovered as special or limiting cases within the present framework.

The applicability of the new control mechanism is further illustrated by a nonlinear integral equation on a space of continuous functions, where the associated integral operator satisfies the $\Psi_{\alpha,\beta}$–contractive condition. This confirms that the displacement–distance gauge is effective in treating nonlinear operator equations that lie beyond standard Lipschitz regimes.

The proposed control function opens several directions for future research, including extensions to multivalued and cyclic mappings, ordered and generalized metric structures, and dynamical systems with state–dependent perturbations. We expect that the displacement–distance paradigm will provide a flexible and unifying tool for developing further generalizations in fixed point theory and its applications.

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