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NEW GENERALIZED RESULTS ON GRÜSS-TYPE FRACTIONAL INTEGRAL INEQUALITIES INVOLVING THE CAPUTO–FABRIZIO INTEGRAL OPERATOR

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Abstract

The Grüss inequality is a well-known result in mathematical analysis that gives an upper estimate for the difference between the integral of the product of two functions and the product of their individual integrals. Its generalizations in various mathematical contexts, including fractional calculus. In this paper, we present a comprehensive generalization of certain Grüss-type integral inequalities and other related integral inequalities using the Caputo-Fabrizio fractional integral operator. Specifically, we extend and generalize classical integral inequalities to obtain new results within the framework of fractional calculus. Unlike traditional fractional operators, the Caputo-Fabrizio operator is characterized by a non-singular exponential kernel that provides a more regular and physically meaningful approach to fractional integration. By utilizing this operator into the analysis, we derive new generalized versions of the classical Grüss inequality.
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1 Introduction

Chebyshev [5] introduced the well-known Chebyshev inequality

$$\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\wp(\theta)\,d\theta \geq \left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\,d\theta\right)\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\wp(\theta)\,d\theta\right), \tag{1.1}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ which are synchronous on $[\varepsilon_1,\varepsilon_2]$, that is,

$$(\mathfrak{H}(\theta)-\mathfrak{H}(\varphi))(\wp(\theta)-\wp(\varphi))\geq 0,\quad \theta,\varphi\in[\varepsilon_1,\varepsilon_2].$$

The well-known Chebyshev functional [5] is defined by

$$T(\mathfrak{H},\wp)=\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\wp(\theta)\,d\theta-\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{H}(\theta)\,d\theta\right)\left(\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\wp(\theta)\,d\theta\right), \tag{1.2}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ which are synchronous on $[\varepsilon_1,\varepsilon_2]$ then

$$T(\mathfrak{H},\wp)\geq 0. \tag{1.3}$$

A weighted extension of the Chebyshev functional was presented by the Chebyshev in [5], generalizing the formulation in (1.1).

$$T(\mathfrak{H},\wp,u)=\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\mathfrak{H})\zeta(\theta)\wp(\theta)\,d\theta-\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\mathfrak{H}(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\wp(\theta)\,d\theta, \tag{1.4}$$

where $\mathfrak{H}$ and $\wp$ are two Lebesgue integrable functions on $[\varepsilon_1,\varepsilon_2]$ and u is positive and integrable functions on a finite interval $[\varepsilon_1,\varepsilon_2]$.

Again Chebyshev [5] considered the extended Chebyshev functional

$$T(\mathfrak{H},\wp,u,v)=\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}v(\theta)\mathfrak{H}(\theta)\wp(\theta)\,d\theta+\int_{\varepsilon_1}^{\varepsilon_2}v(\theta)\,d\theta\int_{\varepsilon_1}^{\varepsilon_2}u(\theta)\mathfrak{H}(\theta)\wp(\theta)\,d\theta$$

$$- \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\mathfrak{H}(\theta)\, d\theta \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\wp(\theta)\, d\theta - \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\wp(\theta)\, d\theta \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\mathfrak{H}(\theta)\, d\theta, \tag{1.5}$$

where $\mathfrak{H}$ and $\wp$ are two real-valued integrable functions which are synchronous on $[\varepsilon_1, \varepsilon_2]$ and u, v are two positive integrable functions on a finite interval $[\varepsilon_1, \varepsilon_2]$.

In 1935, Grüss [15] introduced the following inequality

$$\left| \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\wp(\theta)\, d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\, d\theta \right) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\, d\theta \right) \right| \leq \frac{1}{4}(\mathfrak{p} - \mathfrak{q})(\mathfrak{d} - \mathfrak{e}), \tag{1.6}$$

such that $\mathfrak{H}$ and $\wp$ are integrable functions on $[\varepsilon_1, \varepsilon_2]$ and satisfy the condition

$$\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}, \mathfrak{d} \leq \wp(\theta) \leq \mathfrak{e}; \mathfrak{p}, \mathfrak{q}, \mathfrak{d}, \mathfrak{e} \in \mathbb{R}, \theta \in [\varepsilon_1, \varepsilon_2].$$

In 1999, Matić *et al.* [21] introduced the pre-Grüss inequality

$$\begin{aligned} &\left| \frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\wp(\theta)\, d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \mathfrak{H}(\theta)\, d\theta \right) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\, d\theta \right) \right| \\ &\leq \frac{1}{2}(\mathfrak{p} - \mathfrak{q}) \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp^2(\theta)\, d\theta - \left(\frac{1}{\varepsilon_2 - \varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \wp(\theta)\, d\theta \right)^2 \right)^{\frac{1}{2}}, \end{aligned} \tag{1.7}$$

such that $\mathfrak{H}$ is integrable functions on $[\varepsilon_1, \varepsilon_2]$ and satisfy the condition

$$\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}; \mathfrak{p}, \mathfrak{q} \in \mathbb{R}, \theta \in [\varepsilon_1, \varepsilon_2].$$

In 2018, Choi [9] proved Grüss inequality for two weighted functions.
Let $\mathfrak{H}, \wp : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be integrable functions such that $\mathfrak{p} \leq \mathfrak{H}(\theta) \leq \mathfrak{q}$ and $\mathfrak{d} \leq \wp(\theta) \leq \mathfrak{e}$ for all $\theta \in [\varepsilon_1, \varepsilon_2]$, where $\mathfrak{p}, \mathfrak{q}, \mathfrak{d}, \mathfrak{e}$ are constants. Also let $u, v : [\varepsilon_1, \varepsilon_2] \to [0, \infty)$ be integrable functions on $[\varepsilon_1, \varepsilon_2]$ such that

$$\min \left\{ \int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\, d\theta, \int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\, d\theta \right\} > 0,$$

then

$$\begin{aligned} |T(\mathfrak{H}, \wp, u, v)| \leq &\left[\left(\frac{(\mathfrak{p} - \mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|_\infty^2 \right) \left(\frac{(\mathfrak{d} - \mathfrak{e})^2}{2} + 2\|\wp\|_\infty^2 \right) \right]^{\frac{1}{2}} \\ &\left(\int_{\varepsilon_1}^{\varepsilon_2} u(\theta)\, d\theta \right) \left(\int_{\varepsilon_1}^{\varepsilon_2} v(\theta)\, d\theta \right), \end{aligned} \tag{1.8}$$

$$\|\mathfrak{H}\|_\infty = \sup_{\theta \in [\varepsilon_1, \varepsilon_2]} |\mathfrak{H}(\theta)| \quad \text{and} \quad \|\wp\|_\infty = \sup_{\theta \in [\varepsilon_1, \varepsilon_2]} |\wp(\theta)|.$$

Fractional calculus generalizes classical calculus to derivatives and integrals of non-integer order. Fractional integral inequalities provide bounds for fractional operators and are important tools in the analysis of fractional differential and integral equations.

The introduction of fractional integral operators has played a significant role in extending classical integral inequalities. In particular, Grüss inequalities and their various extensions have been established for the Riemann-Liouville, Hadamard, Erdlyi-Kober, Saigo, Katugampola fractional integral operators, including those involving the Mittag-Leffler kernel and the references therein [1, 8, 12, 16, 20, 25, 26, 27, 28, 29].

In recent years, fractional integral inequalities have received considerable attention in fractional analysis. Numerous extensions of classical inequalities, including Grüss, weighted Grüss, Chebyshev, and pre-Grüss inequalities, have been established for a variety of fractional integral operators and the references therein [2, 11, 13, 14, 18, 19].

In [4] Caputo and Fabrizio introduced a novel class of fractional derivatives characterized by exponential kernels, which has been successfully applied to both time- and space-fractional models. Subsequently,[24] in 2019 Nchama et al. derived new fractional integral inequalities using the Caputo-Fabrizio operator.More recently, several authors have established Chebyshev, Grüss, weighted, Pólya-Szegö, and Minkowski-type inequalities involving the Caputo-Fabrizio fractional integral operator [6, 7, 22, 23].

Motivated by the aforementioned studies and results, the primary objective of this paper is to derive new generalized fractional integral inequalities involving the Caputo-Fabrizio integral operator. In particular, we establish novel inequalities based on Youngs inequality, the arithmetic-geometric mean inequality, as well as Grüss-type, weighted Grüss-type, and pre-Grüss-type integral inequalities.

Definition 1.1 ([3] (Cauchy–Schwarz Inequality))**.** *Let $\mathfrak{H}(\alpha,\beta)$ and $\wp(\alpha,\beta)$ be measurable functions defined on a product measure space $D=[c_1,c_2]\times[b_1,b_2]$. Then the Cauchy–Schwarz inequality states that*

$$\left(\iint_D \mathfrak{H}(x,y)\wp(x,y)\,dx\,dy\right)^2 \le \left(\iint_D \mathfrak{H}(x,y)^2\,dx\,dy\right)\left(\iint_D \wp(x,y)^2\,dx\,dy\right), \tag{1.9}$$

where c_1,c_2,b_1,b_2 are real numbers and measurable set $D\subset\mathbb{R}^2$.

Definition 1.2 ([10, 17])**.** *Let $\mathfrak{H}$ and $\wp$ be continuous, strictly increasing and mutually inverse for non-negative argument, with $\mathfrak{H}(0)=\wp(0)=0$. Then*

$$mn \le \int_0^m \mathfrak{H}(x)\,dx + \int_0^n \wp(x)\,dx, \tag{1.10}$$

equality is satisfied if and only if $n=\mathfrak{H}(m)$. In this form, if m,n are non-negative and $r,s>1$ such that $\frac{1}{r}+\frac{1}{s}=1$, then

$$\frac{1}{r}m^r+\frac{1}{s}n^s \ge mn, \qquad m,n>0, \quad \frac{1}{r}+\frac{1}{s}=1.$$

Definition 1.3 ([4])**.** *Let $\eta\in\mathbb{R}$ such that $0<\eta\le 1$. The Caputo–Fabrizio fractional integral of order η of a function $\mathfrak{f}$ is defined by*

$$\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{f}(\theta)=\frac{1}{\eta}\int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta-\xi)}\mathfrak{f}(\xi)\,d\xi. \tag{1.11}$$

For $\eta=1$, we get

$$\mathfrak{U}^{1}_{0,\theta}\mathfrak{f}(\theta)=\int_0^\theta \mathfrak{f}(\xi)\,d\xi. \tag{1.12}$$

2 On certain fractional integral inequalities

This section explores the establishment and generalization of new fractional integral inequalities and their extensions through classical integral inequalities such as Young's inequality and the arithmetic–geometric mean inequality.

Theorem 2.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\ge 0$, $0<\eta\le 1$, $0<\vartheta\le 1$ then the following inequalities hold:*

$$\begin{aligned}
(a)\quad & \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^s(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^r(\theta)\\
& \ge \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta),\\
(b)\quad & \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{1}{s}\mathfrak{U}^{\vartheta}_{0,\theta}\wp^s(\theta)\ge \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\wp^{s-1}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta),\\
(c)\quad & \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)\\
& \ge \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{\frac{2}{s}}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta),\\
(d)\quad & \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^s(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^r(\theta)\\
& \ge \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\wp^{s-1}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{\frac{2}{s}}(\theta).
\end{aligned}$$

Proof. We use the following Young's inequality given by

$$\frac{1}{r}m^r+\frac{1}{s}n^s \ge mn, \quad m,n>0, \quad \frac{1}{r}+\frac{1}{s}=1. \tag{2.1}$$

Now for $m=\mathfrak{H}(\alpha)\wp(\beta)$ and $n=\mathfrak{H}(\beta)\wp(\alpha)$, $\alpha,\beta>0$, we get

$$\frac{1}{r}(\mathfrak{H}(\alpha)\wp(\beta))^r+\frac{1}{s}(\mathfrak{H}(\beta)\wp(\alpha))^s \ge (\mathfrak{H}(\alpha)\wp(\beta))(\mathfrak{H}(\beta)\wp(\alpha)). \tag{2.2}$$

Multiplying (2.2) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{\wp^r(\beta)}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{\mathfrak{H}^s(\beta)}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\ge \mathfrak{H}(\beta)\wp(\beta)\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta)\wp(\theta)). \tag{2.3}$$

Multiplying (2.3) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^s(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta)\wp(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}(\theta)\wp(\theta)). \tag{2.4}$$

Similarly, inequalities (b), (c), (d) can be established by substituting the following identities in (2.1), respectively

$$m=\frac{\mathfrak{H}(\alpha)}{\mathfrak{H}(\beta)},\quad n=\frac{\wp(\alpha)}{\wp(\beta)},\quad \mathfrak{H}(\beta),\wp(\beta)\neq 0, \tag{2.5}$$

$$m=\mathfrak{H}(\alpha)\wp^{2r}(\beta),\quad n=\mathfrak{H}^{2s}(\beta)\wp(\alpha), \tag{2.6}$$

$$m=\mathfrak{H}^{2r}(\alpha)\mathfrak{H}(\beta),\quad n=\wp^{2s}(\alpha)\wp(\beta). \tag{2.7}$$

□

Theorem 2.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable function on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\geq 0$, $0<\eta\leq 1$ then following inequalities satisfied:*

$$(a)\ \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\geq\left[\frac{1-\eta}{1-e^{-\frac{1-\eta}{\eta}\theta}}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta),$$

$$(b)\ \frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2,$$

$$\begin{aligned}(c)\ &\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^s(\theta)\\ &\geq\left(\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp)^{r-1}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp)^{s-1}(\theta)\right),\end{aligned}$$

$$(d)\ \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^r(\theta)\geq\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\wp^{s-1}(\theta)\right).$$

Proof. (a) We use the following Youngs inequality given by

$$\frac{1}{r}m^r+\frac{1}{s}n^s\geq mn,\qquad m,n>0,\quad \frac{1}{r}+\frac{1}{s}=1. \tag{2.8}$$

Now for $m=\mathfrak{H}(\alpha)$ and $n=\wp(\beta)$, $\alpha,\beta>0$ in (2.8), we get

$$\frac{1}{r}\mathfrak{H}^r(\alpha)+\frac{1}{s}\wp^s(\beta)\geq\mathfrak{H}(\alpha)\wp(\beta). \tag{2.9}$$

Multiplying (2.9) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{\wp^s(\beta)}{s(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\geq\wp(\beta)\mathfrak{U}^{\eta}_{0,\theta}\zeta(\theta). \tag{2.10}$$

Multiplying (2.10) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ and using (1.11) we have

$$\frac{1}{r}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^r(\theta)+\frac{1}{s}\mathfrak{U}^{\eta}_{0,\theta}\wp^s(\theta)\geq\frac{1-\eta}{1-e^{-\frac{1-\eta}{\eta}\theta}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta). \tag{2.11}$$

(b) For proof of (b), put $m=\mathfrak{H}(\alpha)\wp(\beta)$ and $n=\mathfrak{H}(\beta)\wp(\alpha)$ in (2.8) we get the desired result.

(c) For proof of (c), put $m=\dfrac{\mathfrak{H}(\alpha)}{\wp(\alpha)}$ and $n=\dfrac{\mathfrak{H}(\beta)}{\wp(\beta)}$, $\wp(\alpha)\neq 0$, $\wp(\beta)\neq 0$ in (2.8) we get the desired result.

(d) For proof of (d), put $m=\dfrac{\mathfrak{H}(\alpha)}{\mathfrak{H}(\beta)}$ and $n=\dfrac{\wp(\alpha)}{\wp(\beta)}$, $\mathfrak{H}(\beta)\neq 0$, $\wp(\beta)\neq 0$ in (2.8) we get the desired result. □

Theorem 2.3. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0, \infty)$ and if $r, s > 1$ be such that $\frac{1}{r} + \frac{1}{s} = 1$. For $\theta \geq 0$, $0 < \eta \leq 1$ then following inequalities satisfied:*

$$\begin{aligned}
\text{(a)} \quad & \frac{1}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right) + \frac{1}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right) \\
& \quad \geq \left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{r}(\theta)\mathfrak{H}^{s}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right), \\
\text{(b)} \quad & \frac{1}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right) + \frac{1}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right) \\
& \quad \geq \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\delta^{s-1}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\wp^{s}(\theta)\right).
\end{aligned}$$

Proof. **(a)** Substituting $m = \mathfrak{H}(\alpha)\wp^{\frac{2}{r}}(\beta), n = \mathfrak{H}^{\frac{2}{s}}(\beta)\wp(\alpha), \alpha, \beta > 0$ in (2.8), we get

$$\frac{1}{r}\mathfrak{H}^{r}(\alpha)\wp^{2}(\beta) + \frac{1}{s}\wp^{s}(\alpha)\mathfrak{H}^{2}(\beta) \geq \mathfrak{H}(\alpha)\wp(\alpha)\mathfrak{H}^{\frac{2}{r}}(\beta)\wp^{\frac{2}{s}}(\beta). \tag{2.12}$$

Multiplying (2.12) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$\frac{\wp^{2}(\beta)}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r}(\theta)\right) + \frac{\mathfrak{H}^{2}(\beta)}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\right) \geq \mathfrak{H}^{\frac{2}{r}}(\beta)\wp^{\frac{2}{s}}(\beta)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right). \tag{2.13}$$

Now multiplying (2.13) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ we get (a).

(b) Substituting $m = \frac{\mathfrak{H}^{\frac{2}{r}}(\alpha)}{\mathfrak{H}(\beta)}, n = \frac{\wp^{\frac{2}{s}}(\alpha)}{\wp(\beta)}, \mathfrak{H}(\beta), \wp(\beta) \neq 0$ in (2.8), we get

$$\frac{\mathfrak{H}^{2}(\alpha)}{r\mathfrak{H}^{r}(\beta)} + \frac{\wp^{2}(\alpha)}{s\wp^{s}(\beta)} \geq \frac{\mathfrak{H}^{\frac{2}{r}}(\alpha)\wp^{\frac{2}{s}}(\alpha)}{\mathfrak{H}(\beta)\wp(\beta)},$$

which can be written as

$$\frac{\mathfrak{H}^{2}(\alpha)\wp^{s}(\beta)}{r} + \frac{\wp^{2}(\alpha)\mathfrak{H}^{r}(\beta)}{s} \geq \mathfrak{H}^{\frac{2}{r}}(\alpha)\wp^{\frac{2}{s}}(\alpha)\mathfrak{H}^{r-1}(\beta)\wp^{s-1}(\beta). \tag{2.14}$$

Again, multiplying (2.14) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$\frac{\wp^{s}(\beta)}{r}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right) + \frac{\mathfrak{H}^{r}(\beta)}{s}\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right) \geq \mathfrak{H}^{r-1}(\beta)\wp^{s-1}(\beta)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{\frac{2}{s}}(\theta)\right). \tag{2.15}$$

Multiplying (2.15) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ we get (b). $\square$

Theorem 2.4. *Let $\theta \geq 0$, $0 < \eta \leq 1$, $\mathfrak{H}$ and $\wp$ be two positive and integrable functions on $[0, \infty)$ and if $r, s > 1$ be such that*

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Let

$$\mathbb{S} = \min_{0 \leq \alpha \leq \theta} \frac{\mathfrak{H}(\alpha)}{\wp(\alpha)} \quad \textit{and} \quad \mathbb{P} = \max_{0 \leq \alpha \leq \theta} \frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}. \tag{2.16}$$

Then the following inequalities satisfied:

$$\begin{aligned}
\text{(a)} \quad & 0 \leq \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right) \leq \frac{(\mathbb{P}+\mathbb{S})^{2}}{4\mathbb{P}\mathbb{S}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^{2}, \\
\text{(b)} \quad & 0 \leq \sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right) - \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)} \\
& \leq \frac{(\sqrt{\mathbb{P}}-\sqrt{\mathbb{S}})^{2}}{2\sqrt{\mathbb{S}\mathbb{P}}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right), \\
\text{(c)} \quad & 0 \leq \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\right) - \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right) \\
& \leq \frac{(\mathbb{P}-\mathbb{S})^{2}}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^{2}.
\end{aligned}$$

Proof. **(a)** From equation (2.16) and inequality

$$\left(\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}-\mathbb{S}\right)\left(\mathbb{P}-\frac{\mathfrak{H}(\alpha)}{\wp(\alpha)}\right)\wp^2(\alpha)\geq 0, 0\leq\alpha\leq\theta, \tag{2.17}$$

which can be written as

$$(\mathfrak{H}(\alpha)-\mathbb{S}\wp(\alpha))(\mathbb{P}\wp(\alpha)-\xi(\alpha))\geq 0,$$

it follows that

$$(\mathbb{P}+\mathbb{S})\mathfrak{H}(\alpha)\wp(\alpha)\geq\mathfrak{H}^2(\alpha)+\mathbb{S}\mathbb{P}\,\wp^2(\alpha). \tag{2.18}$$

Multiplying (2.18) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$(\mathbb{P}+\mathbb{S})\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta). \tag{2.19}$$

Now, since $\mathbb{S}\mathbb{P}>0$,

$$\left(\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}-\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}\right)^2\geq 0, \tag{2.20}$$

so that

$$\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\geq 2\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}. \tag{2.21}$$

Using equations (2.19) and (2.21) we get

$$2\sqrt{\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)}\sqrt{\mathbb{S}\mathbb{P}\,\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)}\leq(\mathbb{P}+\mathbb{S})\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta), \tag{2.22}$$

which follows that

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)\leq\frac{(\mathbb{P}+\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

(b) By using equation (2.22)

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}\leq\frac{\mathbb{P}+\mathbb{S}}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta). \tag{2.23}$$

Subtracting $\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)$ from both sides of (2.23), we get

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\leq\frac{\mathbb{P}+\mathbb{S}}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta).$$

It follows that

$$\sqrt{\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)}-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\leq\frac{(\sqrt{\mathbb{P}}-\sqrt{\mathbb{S}})^2}{2\sqrt{\mathbb{S}\mathbb{P}}}\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta).$$

(c) By using equation (2.22)

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)\leq\frac{(\mathbb{P}+\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2. \tag{2.24}$$

Subtracting $\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2$ from both sides of (2.24), we get

$$\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\right)-\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

$$(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta))(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta))-(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta))^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

It follows that

$$(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta))(\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta))-(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta))^2\leq\frac{(\mathbb{P}-\mathbb{S})^2}{4\mathbb{S}\mathbb{P}}\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\wp(\theta)\right)^2.$$

$\square$

Theorem 2.5. *Suppose that $\mathfrak{H}$ and $\wp$ are two positive and integrable functions on $[0,\infty)$ and if $r,s>1$ be such that $\frac{1}{r}+\frac{1}{s}=1$. For $\theta\geq 0$, $0<\eta\leq 1$, $0<\vartheta\leq 1$ then following inequalities satisfied.*

$$\begin{aligned}
(a)\quad & r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\\
&\geq \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\,,\\
(b)\quad & r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{r-1}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}(\mathfrak{H}(\theta)\wp^{s}(\theta))+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp^{s-1}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}(\wp^{s}(\theta)\mathfrak{H}(\theta))\\
&\geq \mathfrak{U}^{\eta}_{0,\theta}\wp^{s}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^{r}(\theta),\\
(c)\quad & r\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp^{\frac{2}{r}}(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\,\mathfrak{U}-^{\vartheta}_{0,\theta}\,\mathfrak{H}^{\frac{2}{s}}(\theta)\\
&\geq \mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}^{r}(\theta)\wp^{2}(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(\wp^{r}(\theta)\mathfrak{H}^{2}(\theta)\big)\,,\\
(d)\quad & r\,\mathfrak{U}^{\eta}_{0,\theta}\Big(\mathfrak{H}^{\frac{2}{r}}(\theta)\wp^{s}(\theta)\Big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\wp^{s-1}(\theta)+s\,\mathfrak{U}^{\eta}_{0,\theta}\wp^{s-1}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\Big(\wp^{\frac{2}{s}}(\theta)\mathfrak{H}^{r}(\theta)\Big)\\
&\geq \mathfrak{U}^{\eta}_{0,\theta}\wp^{2}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^{2}(\theta).
\end{aligned}$$

Proof. (a) From the arithmetic mean and geometric mean inequality (AM-GM) given by

$$rm+sn\geq m^{r}n^{s},\quad m,n\geq 0,\quad r+s=1. \tag{2.25}$$

Now putting $m=\mathfrak{H}(\alpha)\wp(\beta)$ and $n=\mathfrak{H}(\beta)\wp(\alpha)$, $\alpha,\beta>0$ in (2.25), we get

$$r\mathfrak{H}(\alpha)\wp(\beta)+s\mathfrak{H}(\beta)\wp(\alpha)\geq(\mathfrak{H}(\alpha)\wp(\beta))^{r}(\mathfrak{H}(\beta)\wp(\alpha))^{s}. \tag{2.26}$$

Multiplying (2.26) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ and using (1.11) we have

$$r\wp(\beta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)+s\mathfrak{H}(\beta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\geq\mathfrak{H}^{r}(\beta)\wp^{s}(\beta)\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta)). \tag{2.27}$$

Multiplying (2.27) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ, and using (1.11) we have

$$r\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)+s\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\geq\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}^{r}(\theta)\wp^{s}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(\wp^{r}(\theta)\mathfrak{H}^{s}(\theta)),$$

which is the required inequality.

Other inequalities can be arrived by substituting identities in (2.25), respectively

$$\text{(b)}\quad m=\frac{\mathfrak{H}(\beta)}{\mathfrak{H}(\alpha)}\quad\text{and}\quad n=\frac{\wp(\alpha)}{\wp(\beta)},\quad \mathfrak{H}(\alpha),\wp(\beta)\neq 0, \tag{2.28}$$

$$\text{(c)}\quad m=\mathfrak{H}(\alpha)\wp^{\frac{2}{5}}(\beta)\quad\text{and}\quad n=\mathfrak{H}^{\frac{2}{5}}(\beta)\wp(\alpha), \tag{2.29}$$

$$\text{(d)}\quad m=\frac{\mathfrak{H}^{\frac{2}{5}}(\alpha)}{\wp(\beta)}\quad\text{and}\quad n=\frac{\mathfrak{H}^{\frac{2}{5}}(\beta)}{\wp(\alpha)},\quad \wp(\alpha),\wp(\beta)\neq 0. \tag{2.30}$$

□

3 Grüss-type fractional integral inequalities

This section presents new generalized forms of Grüss-type fractional integral inequalities formulated via the Caputo-Fabrizio fractional integral operator.

($\mathcal{H}$)Assume that $\mathfrak{H},\wp:[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$ are two integrable functions on $[\varepsilon_1,\varepsilon_2]$ for which there exist constants $\mathfrak{p},\mathfrak{q},\mathfrak{d},\mathfrak{e}\in\mathbb{R}$ such that

$$\mathfrak{p}\leq\mathfrak{H}(\theta)\leq\mathfrak{q},\qquad \mathfrak{d}\leq\wp(\theta)\leq\mathfrak{e},\qquad \theta\in[\varepsilon_1,\varepsilon_2]. \tag{3.1}$$

Lemma 3.1. *Suppose $\mathfrak{H}$ is integrable function on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\leq 1$, we have:*

$$\begin{aligned}
&\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^{2}(\theta)-\Big(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\Big)^{2}\\
&=\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\xi(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\\
&\quad-\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big).
\end{aligned} \tag{3.2}$$

Proof. Suppose $\mathfrak{H}$ is integrable function on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. For any $\alpha,\beta\in[0,\infty)$, we have

$$\begin{aligned}&(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\alpha)-\mathfrak{q})+(\mathfrak{p}-\mathfrak{H}(\alpha))(\mathfrak{H}(\beta)-\mathfrak{q})-(\mathfrak{p}-\mathfrak{H}(\alpha))(\mathfrak{H}(\alpha)-\mathfrak{q})\\&-(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\beta)-\mathfrak{q})=\mathfrak{H}^2(\alpha)+\mathfrak{H}^2(\beta)-2\mathfrak{H}(\alpha)\mathfrak{H}(\beta).\end{aligned}\tag{3.3}$$

Multiplying (3.3) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}$ and integrating with respect to α from 0 to θ we get

$$\begin{aligned}&(\mathfrak{p}-\mathfrak{H}(\beta))\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\\&+\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)(\mathfrak{H}(\beta)-\mathfrak{q})\\&-\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)-(\mathfrak{p}-\mathfrak{H}(\beta))(\mathfrak{H}(\beta)-\mathfrak{q})\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\\&=\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathfrak{H}^2(\beta)\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{H}(\beta)\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta).\end{aligned}\tag{3.4}$$

Now, multiplying (3.4) by$\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$and integrating with respect to β from 0 to θ which gives (3.2) and proves the lemma. □

Theorem 3.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$ and $0<\eta\le 1$, we have the inequality*

$$\begin{aligned}&\left|\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\right|\\&\le\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)^2(\mathfrak{p}-\mathfrak{q})(\mathfrak{d}-\mathfrak{e}).\end{aligned}\tag{3.5}$$

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Define

$$J(\alpha,\beta)=\big(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta)\big)\big(\wp(\alpha)-\wp(\beta)\big).\tag{3.6}$$

Multiplying (3.6) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to α and β over $(0,\theta)^2$, we get

$$\begin{aligned}&\frac{1}{\eta^2}\int_0^{\theta}\int_0^{\theta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}\big(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta)\big)\big(\wp(\alpha)-\wp(\beta)\big)\,d\alpha\,d\beta\\&=2\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-2\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big).\end{aligned}\tag{3.7}$$

Applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\wp(\theta)\big)-\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\right)^2\\&\le\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\left[\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}^2(\theta)\big)-\left(\mathfrak{U}^{\eta}_{0,\theta}\big(\mathfrak{H}(\theta)\big)\right)^2\right]\\&\quad\times\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right)\left[\mathfrak{U}^{\eta}_{0,\theta}\big(\wp^2(\theta)\big)-\left(\mathfrak{U}^{\eta}_{0,\theta}\big(\wp(\theta)\big)\right)^2\right].\end{aligned}\tag{3.8}$$

Since $(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\ge 0$ and $(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\ge 0$, we have

$$\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\ge 0,\tag{3.9}$$

and

$$\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)\ge 0.\tag{3.10}$$

Using (3.2), (3.9) and (3.10), we have

$$\frac{1}{(1-\eta)}\left[1-e^{-(\frac{1-\eta}{\eta})\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\,\mathfrak{H}^2(\theta)-\left(\mathfrak{U}^{\eta}_{0,\theta}\,\mathfrak{H}(\theta)\right)^2$$

$$\leq \left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.11}$$

and

$$\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)-\left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2$$
$$\leq \left(\wp\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)-e\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.12}$$

By using (3.2) and inequalities (3.8), (3.11) and (3.12), we have

$$\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp(\theta))-\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta))\right)^2$$
$$\leq \left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\cdot\left(\wp\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)-e\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \tag{3.13}$$

Now using the elementary identities $4ab \leq (a+b)^2$, $a, b \in \mathbb{R}$, we can state that

$$4\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\leq \left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right](\mathfrak{p}-\mathfrak{q})\right)^2, \tag{3.14}$$

$$4\left(\mathfrak{d}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{e}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)$$
$$\leq \left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right](\mathfrak{d}-\mathfrak{e})\right)^2. \tag{3.15}$$

Again using (3.13), (3.14) and (3.15), we get the desired result. □

Lemma 3.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\leq 1$, $0<\vartheta\leq 1$, we have the inequality*

$$\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}\wp(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\zeta\wp(\theta)-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\right.$$
$$\left.-\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2$$
$$\leq\left(\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^2(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)-2\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)$$
$$\left(\frac{1}{1-\eta}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)+\frac{1}{1-\vartheta}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)-2\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\right). \tag{3.16}$$

Proof. Multiplying (3.7) by $\frac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to α and β over $(0,\theta)^2$ and applying Cauchy-Schwarz inequality for double integrals, we get the desired result. □

Lemma 3.3. *Suppose that $\mathfrak{H}$ be an integrable function on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\leq 1$, $0<\vartheta\leq 1$, we have:*

$$\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^2(\theta)-2\,\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\,\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)$$
$$=\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)$$

$$+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)$$
$$-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)$$
$$-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big). \tag{3.17}$$

Proof. Multiplying (3.5) by $\frac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}$ and integrating with respect to β from 0 to θ which gives (3.17) and proves the lemma. □

Theorem 3.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, $0<\vartheta\le 1$ we have the inequality*

$$\Bigg(\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}\wp(\theta)+\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}\wp(\theta)$$
$$-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)-\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\Bigg)^2$$
$$\le\Bigg[\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)$$
$$+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\Bigg]$$
$$\times\Bigg[\left(\mathfrak{d}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)$$
$$+\left(\mathfrak{d}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)-e\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\Bigg]. \tag{3.18}$$

Proof. Since $(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\ge 0$ and $(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\ge 0$,

Therefore we write

$$-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)$$
$$-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\le 0. \tag{3.19}$$

and

$$-\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}_{0,\theta}^{\vartheta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)$$
$$-\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\big((\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\big)\le 0. \tag{3.20}$$

Now applying (3.17) to $\mathfrak{H}$ and $\wp$ we get

$$\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}^2(\theta)+\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}^2(\theta)\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)$$
$$\le\left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\right)$$
$$+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\vartheta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]\right). \tag{3.21}$$

and

$$\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\mathfrak{U}_{0,\theta}^{\eta}\wp^2(\theta)+\mathfrak{U}_{0,\theta}^{\vartheta}\wp^2(\theta)\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-2\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)$$
$$\le\left(\mathfrak{d}\frac{1}{(1-\eta)}\left[1-e^{-\frac{1-\eta}{\eta}\theta}\right]-\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\right]\right)$$

$$+\left(\mathfrak{d}\frac{1}{(1-\vartheta)}\Big[1-e^{-\frac{1-\vartheta}{\vartheta}\theta}\Big]-\mathfrak{U}_{0,\theta}^{\vartheta}\wp(\theta)\right)\left(\mathfrak{U}_{0,\theta}^{\eta}\wp(\theta)-\mathfrak{e}\frac{1}{(1-\eta)}\Big[1-e^{-\frac{1-\eta}{\eta}\theta}\Big]\right). \tag{3.22}$$

Using (3.21) and (3.22) in (3.16) we get the desired result. □

4 Grüss-type fractional integral inequalities for two weighted functions

This section is devoted to the study of generalized versions of Grüss-type fractional integral inequalities for two weighted functions involving the Caputo–Fabrizio fractional integral operator.

Lemma 4.1. *Assume Ω is an integrable function on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathfrak{p}\le\Omega(\theta)\le\mathfrak{q}$ on $[\varepsilon_1,\varepsilon_2]$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, we have:*

$$\begin{aligned}&\left(\mathfrak{U}_{0,\theta}^{\eta}[(\mathfrak{p}-\Omega(\theta))v(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[(\Omega(\alpha)-q)u(\theta)]\right)\\&+\left(\mathfrak{U}_{0,\theta}^{\eta}[(\mathfrak{p}-\Omega(\theta))u(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[(\Omega(\theta)-q)v(\theta)]\right)\\&-\left(\mathfrak{U}_{0,\theta}^{\eta}[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})u(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[v(\theta)]\right)\\&-\left(\mathfrak{U}_{0,\theta}^{\eta}[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})v(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[u(\theta)]\right)\\&=\left(\mathfrak{U}_{0,\theta}^{\eta}[u(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[v\Omega^2(\theta)]\right)+\left(\mathfrak{U}_{0,\theta}^{\eta}[v(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[u\Omega^2(\theta)]\right)\\&\quad-2\left(\mathfrak{U}_{0,\theta}^{\eta}[v\Omega(\theta)]\right)\left(\mathfrak{U}_{0,\theta}^{\eta}[u\Omega(\theta)]\right).\end{aligned}\tag{4.1}$$

Proof. Let Ω be an integrable function on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathfrak{p}\le\Omega(\theta)\le\mathfrak{q}$ on $[\varepsilon_1,\varepsilon_2]$. For any $\alpha,\beta\in[\varepsilon_1,\varepsilon_2]$, we have

$$\begin{aligned}&(\mathfrak{p}-\Omega(\beta))(\Omega(\alpha)-\mathfrak{q})+(\mathfrak{U}-\Omega(\alpha))(\Omega(\beta)-\mathfrak{q})\\&\quad-(\mathfrak{p}-\Omega(\alpha))(\Omega(\alpha)-\mathfrak{q})-(\mathfrak{p}-\Omega(\beta))(\Omega(\beta)-\mathfrak{q})\\&=\Omega^2(\alpha)+\Omega^2(\beta)-2\Omega(\alpha)\Omega(\beta).\end{aligned}\tag{4.2}$$

Multiplying (4.2) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$, we get the required lemma. □

Theorem 4.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le 1$, we have the inequality:*

$$\begin{aligned}&\Big|\mathfrak{U}_{0,\theta}^{\eta}(v(\theta))\,\mathfrak{U}_{0,\theta}^{\eta}(u\zeta\delta(\theta))+\mathfrak{U}_{0,\theta}^{\eta}(u(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v\mathfrak{H}\wp(\theta))\\&\quad-\mathfrak{U}_{0,\theta}^{\eta}(u\mathfrak{H}(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v\wp(\theta))-\mathfrak{U}_{0,\theta}^{\eta}(v\mathfrak{H}(\theta))\mathfrak{U}_{0,\theta}^{\eta}(u\wp(\theta))\Big|\\&\le\left[\left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2}+2\|\mathfrak{H}\|_\infty^2\right)\left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2}+2\|\wp\|_\infty^2\right)\right]^{\frac{1}{2}}\mathfrak{U}_{0,\theta}^{\eta}(u(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v(\theta)),\end{aligned}\tag{4.3}$$

where $\mathfrak{H},\wp\in L_\infty[\varepsilon_1,\varepsilon_2]$.

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathcal{H}$. Define the functional

$$J(\alpha,\beta)=(\mathfrak{H}(\alpha)-\mathfrak{H}(\beta))(\wp(\alpha)-\wp(\beta)).\tag{4.4}$$

Multiplying (4.4) by $\frac{1}{\eta^2}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$ we get

$$\begin{aligned}&\frac{1}{\eta^2}\int_0^\theta\int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\eta}{\eta}(\theta-\beta)}u(\alpha)v(\beta)(\zeta(\alpha)-\zeta(\beta))(\delta(\alpha)-\delta(\beta))\,d\alpha d\beta\\&=\mathfrak{U}_{0,\theta}^{\eta}(v(\theta))\mathfrak{U}_{0,\theta}^{\eta}(u\mathfrak{H}\wp(\theta))+\mathfrak{U}_{0,\theta}^{\eta}(u(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v\mathfrak{H}\wp(\theta))-\mathfrak{U}_{0,\theta}^{\eta}(u\mathfrak{H}(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v\wp(\theta))\\&\quad-\mathfrak{U}_{0,\theta}^{\eta}(v\mathfrak{H}(\theta))\mathfrak{U}_{0,\theta}^{\eta}(u\wp(\theta)).\end{aligned}\tag{4.5}$$

Applying Cauchy–Schwarz inequality for double integral and using Lemma 4.1 and using the result $4ab\le(a+b)^2;\ a,b\in\mathbb{R}$ and utilizing the two inequalities

$$\left[\mathfrak{U}_{0,\theta}^{\eta}(u(\theta))u_{0,\theta}^{\eta}(v\mathfrak{H}(\theta))\right]^2\le\|\mathfrak{H}\|_\infty^2\left[\mathfrak{U}_{0,\theta}^{\eta}(u(\theta))\mathfrak{U}_{0,\theta}^{\eta}(v(\theta))\right]^2,$$

$$\left[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right]^2 \leq \|\mathfrak{H}\|^2_\infty \left[u^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2, \tag{4.6}$$

we get

$$\begin{aligned}
&(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)]) + (\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)])\\
&\qquad - 2(\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}(\theta)])\\
&\leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v(\theta)).
\end{aligned} \tag{4.7}$$

Similarly, we obtain

$$\begin{aligned}
&(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[v\wp^2(\theta)]) + (\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)])\\
&\qquad - 2(\mathfrak{U}^{\eta}_{0,\theta}[v\wp(\theta)])(\mathfrak{U}^{\eta}_{0,\theta}[u\wp(\theta)])\\
&\leq \left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2} + 2\|\wp\|^2_\infty\right)\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v(\theta)).
\end{aligned} \tag{4.8}$$

Using (4.5), (4.7) and (4.8) we get the desired result. □

Corollary 4.1. *Suppose that $\mathfrak{H}$ is integrable function on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u is positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, we have the inequality:*

$$\left|\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}^2(\theta)) - \left(\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right)^2\right| \leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{4} + \|\mathfrak{H}\|^2_\infty\right)\left(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\right)^2. \tag{4.9}$$

where $\mathfrak{H} \in L_\infty[\varepsilon_1, \varepsilon_2]$.

Proof. Applying Theorem 4.1 for $\mathfrak{H}(\theta) = \wp(\theta)$ and $u(\theta) = v(\theta)$, we get (4.9). □

Lemma 4.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$, we have the inequality:*

$$\begin{aligned}
&\Big[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}\wp(\theta))\\
&\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\
&\quad - \mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big]^2\\
&\leq \Big(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}^2(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}^2(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}^2(\theta))\\
&\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}^2(\theta)) - 2\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - 2\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big)\\
&\quad \times \Big(\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,{}^{\vartheta}_{0,\theta}(v\wp^2(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp^2(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\wp^2(\theta))\\
&\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\wp^2(\theta)) - 2\mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - 2\mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\Big).
\end{aligned} \tag{4.10}$$

Proof. Let us consider the functional

$$J(\alpha, \beta) = (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta)). \tag{4.11}$$

Multiplying (4.17) by $\frac{1}{\eta\vartheta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$, we get

$$\begin{aligned}
&\frac{1}{\eta\vartheta}\int_0^\theta\int_0^\theta (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta))e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)\,d\alpha d\beta\\
&= \mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta)) + \mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}\wp(\theta))\\
&\quad + \mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\
&\quad - \mathfrak{U}^{\eta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta)) - \mathfrak{U}^{\vartheta}_{0,\theta}(u\wp(\theta))\,\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta)).
\end{aligned} \tag{4.12}$$

Applying Cauchy–Schwarz inequality for double integral, we get the lemma. □

Lemma 4.3. *Assume Ω is an integrable function on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathfrak{p}\le\Omega(\theta)\le\mathfrak{q}$ on $[\varepsilon_1,\varepsilon_2]$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le1$, $0<\vartheta\le1$, we have:*

$$\begin{aligned}
&\mathfrak{U}^{\eta}_{0,\theta}\big[(\Omega(\theta)-\mathfrak{q})u(\theta)\big]\,\mathfrak{U}^{\vartheta}_{0,\theta}\big[(\mathfrak{p}-\Omega(\theta))v(\theta)\big]\\
&+\mathfrak{U}^{\vartheta}_{0,\theta}\big[(\Omega(\theta)-\mathfrak{q})u(\theta)\big]\,\mathfrak{U}^{\eta}_{0,\theta}\big[(\mathfrak{p}-\Omega(\theta))v(\theta)\big]\\
&+\mathfrak{U}^{\eta}_{0,\theta}\big[(\mathfrak{p}-\Omega(\theta))u(\theta)\big]\,\mathfrak{U}^{\vartheta}_{0,\theta}\big[(\Omega(\theta)-\mathfrak{q})v(\theta)\big]\\
&+\mathfrak{U}^{\vartheta}_{0,\theta}\big[(\mathfrak{p}-\Omega(\theta))u(\theta)\big]\,\mathfrak{U}^{\eta}_{0,\theta}\big[(\Omega(\theta)-\mathfrak{q})v(\theta)\big]\\
&-\Big(\mathfrak{U}^{\eta}_{0,\theta}\left[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})u(\theta)\right]\Big)\left(\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\right)\\
&-\left(\mathfrak{U}^{\vartheta}_{0,\theta}\left[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})u(\theta)\right]\right)\Big(\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\Big)\\
&-\Big(\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\Big)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\left[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})v(\theta)\right]\right)\\
&-\left(\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\right)\Big(\mathfrak{U}^{\eta}_{0,\theta}\left[(\mathfrak{p}-\Omega(\theta))(\Omega(\theta)-\mathfrak{q})v(\theta)\right]\Big)\\
&=\mathfrak{U}^{\eta}_{0,\theta}[u\Omega^2(\theta)]\,\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]+\mathfrak{U}^{\vartheta}_{0,\theta}[u\Omega^2(\theta)]\,\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\\
&+\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\,\mathfrak{U}^{\vartheta}_{0,\theta}[v\Omega^2(\theta)]+\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\,\mathfrak{U}^{\eta}_{0,\theta}[v\Omega^2(\theta)]\\
&-2\,\mathfrak{U}^{\eta}_{0,\theta}[u\Omega(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\Omega(\theta)]-2\mathfrak{U}^{\vartheta}_{0,\theta}[u\Omega(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\Omega(\theta)]
\end{aligned}\tag{4.13}$$

Proof. Multiplying (4.2) by $\dfrac{1}{\eta}e^{-\frac{1-\eta}{\eta}(\theta-\alpha)}\dfrac{1}{\vartheta}e^{-\frac{1-\vartheta}{\vartheta}(\theta-\beta)}u(\alpha)v(\beta)$ and integrating with respect to α and β over $(0,\theta)^2$ we get the required lemma. □

Theorem 4.2. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[\varepsilon_1,\varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1,\varepsilon_2]$. Then for $0<\theta<\varepsilon_2$, $0<\eta\le1$, $0<\vartheta\le1$, we have the inequality:*

$$\begin{aligned}
&\Big|\mathfrak{U}^{\vartheta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}\wp(\theta))+\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}\wp(\theta))+\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{U}\wp(\theta))\\
&+\mathfrak{U}^{\vartheta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}\wp(\theta))-\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\wp(\theta))-\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\wp(\theta))\\
&-\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\vartheta}_{0,\theta}(v\mathfrak{H}(\theta))-\mathfrak{U}^{\vartheta}_{0,\theta}(u\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\Big|\\
&\le\left[\left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2}+2\|\mathfrak{H}\|^2_\infty\right)\left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2}+2\|\wp\|^2_\infty\right)\right]^{\frac12}\Big[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]+\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\Big],
\end{aligned}\tag{4.14}$$

where $\mathfrak{H},\wp\in L_\infty[\varepsilon_1,\varepsilon_2]$.

Proof. Since

$$(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\ge0\quad\text{and}\quad(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})\ge0,$$

therefore we write

$$\begin{aligned}
&-\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(p-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})u(\theta)]\\
&-\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})u(\theta)]\\
&-\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})v(\theta)]\\
&-\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})v(\theta)]\le0,
\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}
&-\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(b-\wp(\theta))(\wp(\theta)-e)u(\theta)]\\
&-\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{d}-\wp(\theta))(\wp(\theta)-e)u(\theta)]\\
&-\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})v(\theta)]\\
&-\mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[(\mathfrak{d}-\wp(\theta))(\wp(\theta)-\mathfrak{e})v(\theta)]\le0,
\end{aligned}\tag{4.16}$$

Now applying (4.13) to $\mathfrak{H}$ and $\wp$ and using the result $4ab \leq (a+b)^2$, $a, b \in \mathbb{R}$ and utilize the two inequalities

$$\begin{aligned}
\left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v\mathfrak{H}(\theta))\right]^2 &\leq \|\mathfrak{H}\|^2_\infty \left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2, \\
\left[\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\mathfrak{U}^{\eta}_{0,\theta}(u\mathfrak{H}(\theta))\right]^2 &\leq \|\mathfrak{H}\|^2_\infty \left[\mathfrak{U}^{\eta}_{0,\theta}(u(\theta))\mathfrak{U}^{\eta}_{0,\theta}(v(\theta))\right]^2,
\end{aligned} \tag{4.17}$$

we get

$$\begin{aligned}
&\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)] \\
&+ \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}^2(\theta)] - 2\mathfrak{U}^{\eta}_{0,\theta}[u\mathfrak{H}(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)] \\
&- 2\mathfrak{U}^{\vartheta}_{0,\theta}[u\mathfrak{H}(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\mathfrak{H}(\theta)] \\
&\leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\left[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right].
\end{aligned} \tag{4.18}$$

Similarly, we obtain

$$\begin{aligned}
&\mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u\wp^2(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\wp^2(\theta)] \\
&+ \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\wp^2(\theta)] - 2\mathfrak{U}^{\eta}_{0,\theta}[u\wp(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v\wp(\theta)] - 2\mathfrak{U}^{\vartheta}_{0,\theta}[u\wp(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v\wp(\theta)] \\
&\leq \left(\frac{(\mathfrak{d}-\mathfrak{e})^2}{2} + 2\|\wp\|^2_\infty\right)\left[\mathfrak{U}^{\eta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\vartheta}_{0,\theta}[v(\theta)] + \mathfrak{U}^{\vartheta}_{0,\theta}[u(\theta)]\mathfrak{U}^{\eta}_{0,\theta}[v(\theta)]\right].
\end{aligned} \tag{4.19}$$

Using (4.10), (4.18) and (4.19) we get the desired result. □

Corollary 4.2. *Suppose that $\mathfrak{H}$ is an integrable function on $[\varepsilon_1, \varepsilon_2]$ satisfying the condition $\mathcal{H}$ and suppose u and v are two positive functions on $[\varepsilon_1, \varepsilon_2]$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$, we have the inequality*

$$\begin{aligned}
&\Big|\mathfrak{U}^{\eta}_{0,\theta}\big(v(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(u\mathfrak{H}^2(\theta)\big) + \mathfrak{U}^{\eta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v\mathfrak{H}^2(\theta)\big) + \mathfrak{U}^{\vartheta}_{0,\theta}\big(v(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(u\mathfrak{H}^2(\theta)\big) \\
&\quad + \mathfrak{U}^{\vartheta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v\mathfrak{H}^2(\theta)\big) - 2\,\mathfrak{U}^{\eta}_{0,\theta}\big(u\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v\mathfrak{H}(\theta)\big) - 2\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(u\mathfrak{H}(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v\mathfrak{H}(\theta)\big)\Big| \\
&\leq \left(\frac{(\mathfrak{p}-\mathfrak{q})^2}{2} + 2\|\mathfrak{H}\|^2_\infty\right)\Big[\mathfrak{U}^{\eta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\vartheta}_{0,\theta}\big(v(\theta)\big) + \mathfrak{U}^{\vartheta}_{0,\theta}\big(u(\theta)\big)\,\mathfrak{U}^{\eta}_{0,\theta}\big(v(\theta)\big)\Big].
\end{aligned} \tag{4.20}$$

where $\mathfrak{H} \in L_\infty[\varepsilon_1, \varepsilon_2]$.

Proof. Applying Theorem 4.2 for $\mathfrak{H}(\theta) = \wp(\theta)$ and $u(\theta) = v(\theta)$, we get (4.20). □

5 η-order pre-Grüss fractional integral inequality

This section investigates generalized forms of the η-order pre-Grüss fractional integral inequality using the Caputo–Fabrizio fractional integral operator.

Theorem 5.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0,\infty)$ satisfying the condition $\mathcal{H}$ on $[0,\infty)$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, we have the inequality*

$$\begin{aligned}
&\left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta))\right)^2 \\
&\leq \left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right)^2 (\mathfrak{p}-\mathfrak{q})^2 \\
&\quad \times \left(\frac{1}{1-\eta}\left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta) - \left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2\right).
\end{aligned} \tag{5.1}$$

Proof. Let $\mathfrak{H}$ and $\wp$ be two integrable functions on $[0, \infty)$ satisfying the condition $\mathcal{H}$ on $[0, \infty)$. Define

$$J(\alpha, \beta) = (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta)). \tag{5.2}$$

Multiplying (5.2) by $\frac{1}{\eta^2} e^{-\frac{1-\eta}{\eta}(\theta-\alpha)} e^{-\frac{1-\eta}{\eta}(\theta-\beta)}$ and integrating with respect to α and β over $(0, \theta)^2$ we get

$$\begin{aligned} &\frac{1}{\eta^2} \int_0^\theta \int_0^\theta e^{-\frac{1-\eta}{\eta}(\theta-\alpha)} e^{-\frac{1-\eta}{\eta}(\theta-\beta)} (\mathfrak{H}(\alpha) - \mathfrak{H}(\beta))(\wp(\alpha) - \wp(\beta))\, d\alpha d\beta \\ &= 2\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\varrho(\theta)) - 2\mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta)). \end{aligned} \tag{5.3}$$

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left(\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta))\right)^2 \\ &\leq \left(\frac{1}{1-\eta}\right) \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta) - \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)^2 \\ &\times \left(\frac{1}{1-\eta}\right) \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta) - \left(\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2. \end{aligned} \tag{5.4}$$

Since $(\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q}) \geq 0$, we have

$$\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p} - \mathfrak{H}(\theta))(\mathfrak{H}(\theta) - \mathfrak{q})\big) \geq 0. \tag{5.5}$$

Using (3.2) and (5.5), we have

$$\begin{aligned} &\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta) - \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)^2 \\ &\leq \left(\mathfrak{p}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right) \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{q}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \end{aligned} \tag{5.6}$$

By using (3.2) and inequalities (5.4) and (5.6), we have

$$\begin{aligned} &\left(\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}\wp(\theta)) - \mathfrak{U}^{\eta}_{0,\theta}(\mathfrak{H}(\theta))\mathfrak{U}^{\eta}_{0,\theta}(\wp(\theta))\right)^2 \\ &\leq \left(\mathfrak{p}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right) \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{U}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right) \\ &\times \left(\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta) - \left(u^{\eta}_{0,\theta}\wp(\theta)\right)^2\right). \end{aligned} \tag{5.7}$$

Now using the elementary identity $4ab \leq (a+b)^2$, $a, b \in \mathbb{R}$, we can state that

$$\begin{aligned} &4\left(\mathfrak{p}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] - \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right) \left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta) - \mathfrak{q}\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right) \\ &\leq \left(\frac{1}{1-\eta} \left[1 - e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right] (\mathfrak{p} - \mathfrak{q})\right)^2. \end{aligned} \tag{5.8}$$

Further using (5.7) and (5.8) we get the desired result. □

6 (η, ϑ)-order pre-Grüss fractional integral inequality

In this section, we present and analyze the generalized version of the (η, ϑ)-order pre-Grüss fractional integral inequality using the Caputo–Fabrizio fractional integral operator.

Theorem 6.1. *Suppose that $\mathfrak{H}$ and $\wp$ are two integrable functions on $[0, \infty)$ satisfying the condition $\mathcal{H}$ on $[0, \infty)$. Then for $0 < \theta < \varepsilon_2$, $0 < \eta \leq 1$, $0 < \vartheta \leq 1$ we have the inequality*

$$\begin{aligned} &\left(\frac{1}{(1-\eta)} \left[1 - e^{-\frac{(1-\eta)}{\eta}\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}\wp(\theta) + \frac{1}{(1-\vartheta)} \left[1 - e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right] \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}\wp(\theta) \right. \\ &\left. \quad - \mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta) - \mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathfrak{p}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\right)\\
&\quad+\Bigg(\frac{1}{(1-\eta)}\left[1-e^{-\frac{(1-\eta)}{\eta}\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\wp^2(\theta)+\frac{1}{(1-\vartheta)}\left[1-e^{-\frac{(1-\vartheta)}{\vartheta}\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\wp^2(\theta)\\
&\qquad-2\mathfrak{U}^{\eta}_{0,\theta}\wp(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\wp(\theta)\Bigg). \qquad (6.1)
\end{aligned}
$$

Proof. Since

$$(\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\geq 0.$$

Therefore we write

$$
\begin{aligned}
&-\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\mathfrak{U}^{\vartheta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\\
&-\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\big((\mathfrak{p}-\mathfrak{H}(\theta))(\mathfrak{H}(\theta)-\mathfrak{q})\big)\leq 0. \qquad (6.2)
\end{aligned}
$$

Now applying (3.17) to $\mathfrak{H}$ we get

$$
\begin{aligned}
&\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}^2(\theta)+\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}^2(\theta)\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-2\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\\
&\leq\left(\mathfrak{p}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]-\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]\right)\\
&\quad+\left(\mathfrak{p}\frac{1}{1-\vartheta}\left[1-e^{-\left(\frac{1-\vartheta}{\vartheta}\right)\theta}\right]-\mathfrak{U}^{\vartheta}_{0,\theta}\mathfrak{H}(\theta)\right)\left(\mathfrak{U}^{\eta}_{0,\theta}\mathfrak{H}(\theta)-\mathfrak{q}\frac{1}{1-\eta}\left[1-e^{-\left(\frac{1-\eta}{\eta}\right)\theta}\right]\right). \qquad (6.3)
\end{aligned}
$$

Using (6.3) in (3.16), we get the desired result. □

7 Conclusion

In the present study, we employ the Caputo–Fabrizio fractional integral operator to investigate several fractional integral inequalities, utilizing Youngs inequality and the arithmetic–geometric mean inequality to establish generalized and newly extended results. Specifically, we have derived generalized forms of Grüss-type inequalities, weighted Grüss inequalities, and pre-Grüss inequalities.The nonsingular kernel of the Caputo–Fabrizio operator provides a useful framework for developing new bounds and has potential applications in fractional differential equations, norm estimates, error analysis, and the qualitative analysis of fractional models.

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