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RATIONAL-TYPE CONTRACTIONS IN DOUBLE CONTROLLED METRIC TYPE SPACES WITH APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

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Abstract

In this paper, we introduce several classes of rational-type contractive mappings in double controlled metric type spaces, namely rational, quadratic rational, maximum rational, interpolative rational, and mixed rational contractions. Sufficient conditions are established for the existence of fixed points and the convergence of Picard iterative sequences. Existence and uniqueness results are obtained for the rational, quadratic rational, maximum rational, and mixed rational contractions, while existence together with Picard convergence is proved for the interpolative rational contraction. Several examples are presented to illustrate the applicability of the proposed results. Furthermore, an application to a nonlinear Fredholm integral equation is included to demonstrate the effectiveness of the developed theory.
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1 Introduction

Fixed point techniques constitute a central tool in nonlinear analysis, appearing in integral equations, equilibrium problems, and dynamical systems. Classical results such as the Banach contraction principle rely on metric structures, where the geometry is governed by a uniform triangle inequality. However, many nonlinear models evolve in settings where such rigid assumptions fail, motivating the development of generalized distance frameworks.

Among the most flexible of these are *double controlled metric type spaces*, where the standard triangle inequality is replaced by a two-parameter inequality. In this setting, for any $\xi, \eta, \zeta \in \mathscr{X}$,

$$\mathscr{D}(\xi, \zeta) \leq \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

The functions α and β adapt the geometry of $\mathscr{X}$ to the local behaviour of sequences, thereby unifying a broad class of generalized metric constructions [1, 19].

Another major advance in fixed point theory is the use of *rational contractive conditions*. Unlike classical linear contractions, rational inequalities incorporate nonlinear terms involving the defect quantities $\mathscr{D}(\xi, \mathscr{T}\xi)$ and $\mathscr{D}(\eta, \mathscr{T}\eta)$. This approach, initiated by Dass and Gupta and later refined by Jaggi and Arshad, yields convergence even in cases where standard Lipschitz conditions are not available [4, 10, 13].

The objective of this article is to combine these two directions by establishing a new rational contraction principle in the framework of double controlled metric type spaces. Our contractive scheme simultaneously reduces the mutual distance $\mathscr{D}(\xi, \eta)$ and controls the defect terms through a rational correction. Under natural orbit-dependent bounds on the control functions α and β, we prove the existence, uniqueness, and convergence of fixed points. We also derive a common fixed point theorem, a stability result for perturbed operators, and an application to a nonlinear integral equation.

For comparison, let $(\mathcal{M}, d)$ be a complete metric space and let $T : \mathcal{M} \to \mathcal{M}$.

Theorem 1.1 (Banach [6]). *If there exists $\kappa \in (0,1)$ such that*

$$d(T\xi, T\eta) \leq \kappa\, d(\xi, \eta), \qquad \forall\, \xi, \eta \in \mathcal{M},$$

then T admits a unique fixed point ξ^*, and the Picard sequence $\xi_{n+1} = T\xi_n$ converges to ξ^* for every initial point $\xi_0 \in \mathcal{M}$.

Theorem 1.2 (Kannan [16]). *If there exists $\kappa \in \left(0, \frac{1}{2}\right)$ such that*

$$d(T\xi, T\eta) \leq \kappa \left[d(\xi, T\xi) + d(\eta, T\eta) \right], \qquad \forall\, \xi, \eta \in \mathcal{M},$$

then T has a unique fixed point.

Theorem 1.3 (Chatterjea [8]). *If there exists $\kappa \in \left(0, \frac{1}{2}\right)$ such that*

$$d(T\xi, T\eta) \leq \kappa \left[d(\xi, T\eta) + d(\eta, T\xi) \right], \qquad \forall\, \xi, \eta \in \mathcal{M},$$

then T has a unique fixed point.

Theorem 1.4 (Rational Type [10, 13]). *If there exist $\lambda, \mu \geq 0$ with $\lambda + \mu < 1$ such that*

$$d(T\xi, T\eta) \leq \lambda d(\xi, \eta) + \mu \frac{d(\xi, T\xi)\, d(\eta, T\eta)}{1 + d(\xi, \eta)},$$

for all $\xi, \eta \in \mathcal{M}$, then T has a unique fixed point.

Theorem 1.5 (Double Controlled Metric Type [1]). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete double controlled metric type space and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. If there exists $\kappa \in (0,1)$ such that*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \kappa\, \mathscr{D}(\xi, \eta), \qquad \forall\, \xi, \eta \in \mathscr{X},$$

and the control functions α and β are bounded along the Picard orbit, then $\mathscr{T}$ has a unique fixed point in $\mathscr{X}$.

2 Preliminaries

This section recalls the basic concepts and results on double controlled metric type spaces *(DCMTS)* [1,2,18,23] used throughout the paper.

Definition 2.1 (Double controlled metric type space [1]). *Let $\mathscr{X} \neq \emptyset$ and let*

$$\mathscr{D} : \mathscr{X} \times \mathscr{X} \to [0, \infty), \qquad \alpha, \beta : \mathscr{X} \times \mathscr{X} \to [1, \infty).$$

The quadruple $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is called a double controlled metric type space *(DCMTS) if, for all $\xi, \eta, \zeta \in \mathscr{X}$,*

(DC1) $\mathscr{D}(\xi, \eta) = 0$ *if and only if* $\xi = \eta$;
(DC2) $\mathscr{D}(\xi, \eta) = \mathscr{D}(\eta, \xi)$;
(DC3)

$$\mathscr{D}(\xi, \zeta) \leq \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

Remark 2.1. *If $\alpha(\xi, \eta) = \beta(\xi, \eta) = 1$ for all $\xi, \eta \in \mathscr{X}$, then $(\mathscr{X}, \mathscr{D})$ is a metric space. If $\alpha = \beta \equiv b \geq 1$, then $(\mathscr{X}, \mathscr{D})$ is a b-metric space* [9]. *Thus, DCMTS generalizes metric, b-metric, and controlled metric spaces.*

Remark 2.2. *The control functions need not be globally bounded. In many fixed point results, boundedness along the Picard orbit is sufficient* [1, 2].

Definition 2.2 (Convergence and completeness). *A sequence $\{\xi_n\} \subseteq \mathscr{X}$ converges to $\xi \in \mathscr{X}$ if*

$$\mathscr{D}(\xi_n, \xi) \to 0 \qquad (n \to \infty).$$

It is called Cauchy if

$$\mathscr{D}(\xi_n, \xi_m) \to 0 \qquad (n, m \to \infty).$$

A DCMTS is complete if every Cauchy sequence converges in $\mathscr{X}$.

Lemma 2.1 (Orbit telescoping estimate). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\{\xi_n\} \subseteq \mathscr{X}$. Then, for every $m > n$,*

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1}) \mathscr{D}(\xi_k, \xi_{k+1}),$$

where the empty product is interpreted as 1.

Proof. For $r = n, \ldots, m$, let

$$u_r = \mathscr{D}(\xi_r, \xi_m),$$

so that $u_m = 0$. Applying (DC3) to the triple $(\xi_r, \xi_{r+1}, \xi_m)$ yields

$$u_r \leq \beta(\xi_r, \xi_{r+1})\mathscr{D}(\xi_r, \xi_{r+1}) + \alpha(\xi_r, \xi_{r+1})u_{r+1}.$$

Repeated substitution, together with $u_m = 0$, gives

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1})\mathscr{D}(\xi_k, \xi_{k+1}),$$

where the empty product equals 1. This completes the proof. □

Comparison of Generalized Metric Structures

Structure	**Generalized Triangle Inequality**	**Control**	**Source**
Metric space	$d(\xi, \zeta) \leq d(\xi, \eta) + d(\eta, \zeta)$	None	[6]
b-metric space	$d(\xi, \zeta) \leq b[d(\xi, \eta) + d(\eta, \zeta)]$	Constant $b \geq 1$	[9]
Controlled metric type	$d(\xi, \zeta) \leq \alpha(\xi, \eta)d(\xi, \eta) + d(\eta, \zeta)$	One function α	[19]
Double controlled metric type	$\mathscr{D}(\xi, \zeta) \leq \alpha(\xi, \eta)\mathscr{D}(\xi, \eta)+\beta(\eta, \zeta)\mathscr{D}(\eta, \zeta)$	Two functions α, β	[1]

Example 2.1. Let $\mathscr{X} = [0, \infty)$ and define

$$\mathscr{D}(\xi, \eta) = (\sqrt{\xi} - \sqrt{\eta})^2,$$

with

$$\alpha(\xi, \eta) = 1 + \sqrt{\xi} + \sqrt{\eta}, \qquad \beta(\eta, \zeta) = 1 + \sqrt{\eta} + \sqrt{\zeta}.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are immediate. Let $u = \sqrt{\xi}$, $v = \sqrt{\eta}$, and $w = \sqrt{\zeta}$. Then

$$(u - w)^2 \leq 2(u - v)^2 + 2(v - w)^2.$$

Since

$$\alpha(\xi, \eta) \geq 2, \qquad \beta(\eta, \zeta) \geq 2,$$

we obtain

$$\mathscr{D}(\xi, \zeta) \leq \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

Hence, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

Example 2.2. Let $\mathscr{X} = (0, \infty)$ and define

$$\mathscr{D}(\xi, \eta) = (\ln \xi - \ln \eta)^2,$$

with

$$\alpha(\xi, \eta) = 1 + |\ln \xi| + |\ln \eta|, \qquad \beta(\eta, \zeta) = 1 + |\ln \eta| + |\ln \zeta|.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are obvious. Setting $a = \ln \xi$, $b = \ln \eta$, and $c = \ln \zeta$, we have

$$(a - c)^2 \leq 2(a - b)^2 + 2(b - c)^2.$$

Since

$$\alpha(\xi, \eta) \geq 2, \qquad \beta(\eta, \zeta) \geq 2,$$

it follows that

$$\mathscr{D}(\xi, \zeta) \leq \alpha(\xi, \eta)\mathscr{D}(\xi, \eta) + \beta(\eta, \zeta)\mathscr{D}(\eta, \zeta).$$

Hence, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

Example 2.3. Let $\mathscr{X} = \mathbb{N}$ and define

$$L(n) = \lfloor \log_2 n \rfloor, \qquad \mathscr{D}(m,n) = |L(m) - L(n)| + \mathbf{1}_{\{m \neq n\}},$$

with

$$\alpha(m,n) = 2 + |L(m) - L(n)|, \qquad \beta(n,k) = 2 + |L(n) - L(k)|.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*.

Proof. Conditions (DC1) and (DC2) are immediate. Moreover,

$$|L(m) - L(k)| \leq |L(m) - L(n)| + |L(n) - L(k)|,$$

and

$$\mathbf{1}_{\{m \neq k\}} \leq \mathbf{1}_{\{m \neq n\}} + \mathbf{1}_{\{n \neq k\}}.$$

Hence,

$$\mathscr{D}(m,k) \leq \mathscr{D}(m,n) + \mathscr{D}(n,k).$$

Since

$$\alpha(m,n) \geq 2, \qquad \beta(n,k) \geq 2,$$

we obtain

$$\mathscr{D}(m,k) \leq \alpha(m,n)\mathscr{D}(m,n) + \beta(n,k)\mathscr{D}(n,k),$$

which proves (DC3). Therefore, $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a *DCMTS*. □

3 Rational Contractions in *DCMTS*

In this section, we introduce several rational-type contractive conditions in the framework of double controlled metric type spaces. These definitions extend the classical Dass–Gupta scheme and provide flexible tools for treating nonlinear operators whose behaviour is not captured by standard Lipschitz inequalities.

Definition 3.1 (*DC*–rational contraction). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is a (λ, μ)* double-controlled rational contraction *if there exist $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)\mathscr{D}(\eta, \mathscr{T}\eta)}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.1}$$

Definition 3.2 (*DC*–quadratic rational contraction). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is called a (λ, μ) DC*–quadratic rational contraction *if there exist $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)^2 + \mathscr{D}(\eta, \mathscr{T}\eta)^2}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.2}$$

Definition 3.3 (*DC*–max rational contraction). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. We say that $\mathscr{T}$ is called a (λ, μ) DC*–max rational contraction *if there exist $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta) + \mu \frac{\max\{\mathscr{D}(\xi, \mathscr{T}\xi), \mathscr{D}(\eta, \mathscr{T}\eta)\}}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.3}$$

Definition 3.4 (*DC*–interpolative rational contraction). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Then $\mathscr{T}$ is called a (λ, μ, θ) DC*–interpolative rational contraction *if there exist $\lambda, \mu \in (0,1)$ and $\theta \in (0,1)$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \mathscr{D}(\xi, \eta)^{\theta} + \mu \frac{\mathscr{D}(\xi, \mathscr{T}\xi)^{1-\theta}\mathscr{D}(\eta, \mathscr{T}\eta)^{1-\theta}}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.4}$$

Definition 3.5 (*DC*–mixed rational contraction). *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$. Then $\mathscr{T}$ is called a (λ, μ) DC*–mixed rational contraction *if there exist $\lambda, \mu \in [0,1)$ with $\lambda + \mu < 1$ such that, for all $\xi, \eta \in \mathscr{X}$,*

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \leq \lambda \max\{\mathscr{D}(\xi, \eta), \mathscr{D}(\xi, \mathscr{T}\xi)\} + \mu \frac{\mathscr{D}(\eta, \mathscr{T}\eta)}{1 + \mathscr{D}(\xi, \eta)}. \tag{3.5}$$

Remark 3.1. *Inequalities (3.2)–(3.5) generalize the basic rational contraction model (3.1) by modifying the defect terms through quadratic, maximum, interpolative, and mixed rational structures. Each of these contractive conditions reduces to the Banach contraction whenever the rational term vanishes, and all are compatible with the main fixed point theorem under suitable parameter bounds.*

4 Main Results

Throughout this section, let $\xi_0 \in \mathscr{X}$ be an arbitrary initial point, and let $\{\xi_n\}_{n\geq 0}$ denote the corresponding Picard sequence defined by

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0.$$

For convenience, we write

$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}), \qquad n \geq 0.$$

Throughout this section, we assume the following orbit-control boundedness condition:

$$\alpha pha^* := \sup_{n\geq 0} \alpha pha(\xi_n, \xi_{n+1}) < \infty, \qquad \beta^* := \sup_{n\geq 0} \beta(\xi_n, \xi_{n+1}) < \infty. \tag{4.1}$$

We also assume that, for every $\omega \in \mathscr{X}$,

$$\sup_{n\geq 0} \alpha pha(\omega, \xi_n) < \infty, \qquad \sup_{n\geq 0} \beta(\xi_n, \omega) < \infty. \tag{4.2}$$

Theorem 4.1. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy the DC–rational contractive condition* (3.1)*. Fix $\xi_0 \in \mathscr{X}$, and define the corresponding Picard sequence $\{\xi_n\}_{n\geq 0}$ by*

$$\xi_{n+1} = \mathscr{T}\xi_n, n \geq 0.$$

Assume that the conditions (4.1) *and* (4.2) *hold. Suppose, in addition, that*

$$q_{\mathrm{r}} := \frac{\lambda}{1-\mu} \in [0,1), \qquad \beta^* q_{\mathrm{r}} < 1. \tag{4.3}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and the Picard sequence $\{\xi_n\}$ converges to $\xi^\star$.

Proof. For each $n \geq 0$, set

$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}).$$

Applying the DC–rational contractive condition (3.1) to the pair (ξ_n, ξ_{n-1}), where $n \geq 1$, and using

$$\mathscr{T}\xi_n = \xi_{n+1}, \qquad \mathscr{T}\xi_{n-1} = \xi_n,$$

we obtain

$$\begin{aligned}
\delta_n &= \mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_{n-1}) \\
&\leq \lambda \mathscr{D}(\xi_n, \xi_{n-1}) + \mu \frac{\mathscr{D}(\xi_n, \mathscr{T}\xi_n)\mathscr{D}(\xi_{n-1}, \mathscr{T}\xi_{n-1})}{1 + \mathscr{D}(\xi_n, \xi_{n-1})} \\
&= \lambda \delta_{n-1} + \mu \frac{\delta_n \delta_{n-1}}{1 + \delta_{n-1}}.
\end{aligned} \tag{4.4}$$

Since

$$\frac{\delta_{n-1}}{1 + \delta_{n-1}} \leq 1,$$

inequality (4.4) yields

$$\delta_n \leq \lambda \delta_{n-1} + \mu \delta_n.$$

Consequently,

$$(1-\mu)\delta_n \leq \lambda \delta_{n-1},$$

and hence, by (4.3),

$$\delta_n \leq q_{\mathrm{r}} \delta_{n-1} \leq q_{\mathrm{r}}^n \delta_0, \qquad n \geq 1. \tag{4.5}$$

Since $q_{\mathrm{r}} \in [0,1)$, it follows that

$$\delta_n \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Let $m > n \geq 0$. By Lemma 2.1,

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1}) \delta_k, \tag{4.6}$$

where an empty product is understood to be equal to 1.

Using (4.1) and (4.5), we obtain

$$\begin{aligned}\mathscr{D}(\xi_n,\xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\le \alpha^*\delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{r}}^k \\ &= \alpha^*\delta_0 q_{\mathrm{r}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{r}})^r \\ &\le \frac{\alpha^*\delta_0 q_{\mathrm{r}}^n}{1-\beta^* q_{\mathrm{r}}}. \end{aligned} \tag{4.7}$$

Since $\beta^* q_{\mathrm{r}} < 1$, the right-hand side of (4.7) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Thus,

$$\mathscr{D}(\xi_n,\xi_m) \longrightarrow 0 \qquad \text{as } m,n \to \infty.$$

Therefore, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$. Since $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is complete, there exists $\xi^\star \in \mathscr{X}$ such that

$$\xi_n \longrightarrow \xi^\star \qquad \text{as } n \to \infty.$$

It remains to prove that $\xi^\star$ is a fixed point of $\mathscr{T}$. By (DC3), with ξ_{n+1} as the intermediate point, we have

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) \le{}& \alpha(\mathscr{T}\xi^\star,\xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \\ &+ \beta(\xi_{n+1},\xi^\star)\mathscr{D}(\xi_{n+1},\xi^\star).\end{aligned} \tag{4.8}$$

Since $\xi_{n+1} = \mathscr{T}\xi_n$, we have

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star,\mathscr{T}\xi_n).$$

Applying (3.1) to the pair $(\xi^\star,\xi_n)$ gives

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) &\le \lambda\mathscr{D}(\xi^\star,\xi_n) \\ &\quad + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)\mathscr{D}(\xi_n,\mathscr{T}\xi_n)}{1+\mathscr{D}(\xi^\star,\xi_n)} \\ &= \lambda\mathscr{D}(\xi^\star,\xi_n) + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)\delta_n}{1+\mathscr{D}(\xi^\star,\xi_n)}.\end{aligned} \tag{4.9}$$

Because

$$\mathscr{D}(\xi^\star,\xi_n) \longrightarrow 0 \qquad \text{and} \qquad \delta_n \longrightarrow 0,$$

inequality (4.9) implies that

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

By (4.2), applied with $\omega = \mathscr{T}\xi^\star$ and $\omega = \xi^\star$, respectively, the sequences

$$\big\{\alpha(\mathscr{T}\xi^\star,\xi_{n+1})\big\} \qquad \text{and} \qquad \big\{\beta(\xi_{n+1},\xi^\star)\big\}$$

are bounded. Letting $n \to \infty$ in (4.8), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) = 0.$$

By (DC1),

$$\mathscr{T}\xi^\star = \xi^\star.$$

Hence, $\xi^\star$ is a fixed point of $\mathscr{T}$.

To establish uniqueness, suppose that $\upsilon,\varpi \in \mathscr{X}$ are two fixed points of $\mathscr{T}$. Applying (3.1) to the pair (υ,ϖ) yields

$$\begin{aligned}\mathscr{D}(\upsilon,\varpi) &= \mathscr{D}(\mathscr{T}\upsilon,\mathscr{T}\varpi) \\ &\le \lambda\mathscr{D}(\upsilon,\varpi) + \mu\frac{\mathscr{D}(\upsilon,\mathscr{T}\upsilon)\mathscr{D}(\varpi,\mathscr{T}\varpi)}{1+\mathscr{D}(\upsilon,\varpi)} \\ &= \lambda\mathscr{D}(\upsilon,\varpi).\end{aligned}$$

Therefore,

$$(1-\lambda)\mathscr{D}(\upsilon,\varpi) \le 0.$$

Since $q_{\mathrm{r}} < 1$, we have

$$\frac{\lambda}{1-\mu} < 1,$$

and hence

$$\lambda < 1 - \mu \leq 1.$$

Thus, $\lambda < 1$. Since $\mathscr{D}(\upsilon, \varpi) \geq 0$, it follows that

$$\mathscr{D}(\upsilon, \varpi) = 0.$$

By (DC1), we obtain $\upsilon = \varpi$.

Therefore, $\mathscr{T}$ admits a unique fixed point $\xi^{\star} \in \mathscr{X}$, and the Picard sequence $\{\xi_n\}$ converges to $\xi^{\star}$. $\square$

Theorem 4.2. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be sequentially continuous and satisfy the DC–quadratic rational contractive condition (3.2). For $\xi_0 \in \mathscr{X}$, define the Picard sequence by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \geq 0,$$

and set

$$\delta_n := \mathscr{D}(\xi_n, \xi_{n+1}), \qquad n \geq 0.$$

Assume that (4.1) and (4.2) hold, and that

$$\Delta^* := \sup_{n \geq 0} \delta_n < \infty. \tag{4.10}$$

Suppose, in addition, that

$$\mu\Delta^* < 1, \qquad q_{\mathrm{q}} := \frac{\lambda + \mu\Delta^*}{1 - \mu\Delta^*} \in [0, 1), \qquad \beta^* q_{\mathrm{q}} < 1. \tag{4.11}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^{\star} \in \mathscr{X}$, and $\xi_n \to \xi^{\star}$ as $n \to \infty$.

Proof. For $n \geq 1$, applying (3.2) to (ξ_n, ξ_{n-1}) gives

$$\begin{aligned}\delta_n &= \mathscr{D}(\mathscr{T}\xi_n, \mathscr{T}\xi_{n-1}) \\ &\leq \lambda\delta_{n-1} + \mu\frac{\delta_n^2 + \delta_{n-1}^2}{1 + \delta_{n-1}}.\end{aligned} \tag{4.12}$$

By (4.10),

$$\delta_n^2 \leq \Delta^*\delta_n, \qquad \delta_{n-1}^2 \leq \Delta^*\delta_{n-1}.$$

Since $1 + \delta_{n-1} \geq 1$, inequality (4.12) yields

$$\delta_n \leq \lambda\delta_{n-1} + \mu\Delta^*\big(\delta_n + \delta_{n-1}\big).$$

Hence,

$$(1 - \mu\Delta^*)\delta_n \leq (\lambda + \mu\Delta^*)\delta_{n-1}.$$

Therefore, by (4.11),

$$\delta_n \leq q_{\mathrm{q}}\delta_{n-1} \leq q_{\mathrm{q}}^n\delta_0, \qquad n \geq 1. \tag{4.13}$$

In particular, $\delta_n \to 0$ as $n \to \infty$.

For $m > n \geq 0$, Lemma 2.1, together with (4.1) and (4.13), gives

$$\begin{aligned}\mathscr{D}(\xi_n, \xi_m) &\leq \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\leq \alpha^*\delta_0 q_{\mathrm{q}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{q}})^r \\ &\leq \frac{\alpha^*\delta_0 q_{\mathrm{q}}^n}{1 - \beta^* q_{\mathrm{q}}}.\end{aligned} \tag{4.14}$$

Since $\beta^* q_{\mathrm{q}} < 1$, the right-hand side tends to zero as $n \to \infty$, uniformly for $m > n$. Thus, $\{\xi_n\}$ is a Cauchy sequence. By completeness, there exists $\xi^{\star} \in \mathscr{X}$ such that

$$\xi_n \longrightarrow \xi^{\star}.$$

Sequential continuity of $\mathscr{T}$ gives

$$\mathscr{T}\xi_n \longrightarrow \mathscr{T}\xi^\star.$$

Moreover, $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$. By (DC3),

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) \le \alpha(\mathscr{T}\xi^\star, \xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1})\\ + \beta(\xi_{n+1}, \xi^\star)\mathscr{D}(\xi_{n+1}, \xi^\star).\end{aligned} \tag{4.15}$$

The control sequences in (4.15) are bounded by (4.2), while

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star, \mathscr{T}\xi_n) \longrightarrow 0$$

and

$$\mathscr{D}(\xi_{n+1}, \xi^\star) \longrightarrow 0.$$

Letting $n \to \infty$ in (4.15), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) = 0.$$

Thus, (DC1) implies

$$\mathscr{T}\xi^\star = \xi^\star.$$

For uniqueness, let υ and ϖ be fixed points of $\mathscr{T}$. Then (3.2) gives

$$\mathscr{D}(\upsilon, \varpi) \le \lambda\mathscr{D}(\upsilon, \varpi),$$

since

$$\mathscr{D}(\upsilon, \mathscr{T}\upsilon) = \mathscr{D}(\varpi, \mathscr{T}\varpi) = 0.$$

Furthermore, $q_{\mathrm{q}} < 1$ implies

$$\lambda + \mu\Delta^* < 1 - \mu\Delta^*,$$

and hence $\lambda < 1$. Therefore,

$$\mathscr{D}(\upsilon, \varpi) = 0,$$

so $\upsilon = \varpi$ by (DC1). This completes the proof. $\square$

Theorem 4.3. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS, and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ be sequentially continuous and satisfy the DC–max rational contractive condition (3.3). For $\xi_0 \in \mathscr{X}$, define the Picard sequence by*

$$\xi_{n+1} = \mathscr{T}\xi_n, \qquad n \ge 0.$$

Assume that (4.1) and (4.2) hold. Suppose, in addition, that

$$q_{\mathrm{m}} := \lambda + \mu \in [0, 1), \qquad \beta^* q_{\mathrm{m}} < 1. \tag{4.16}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and $\xi_n \to \xi^\star$ as $n \to \infty$.

Proof. For $n \ge 1$, applying (3.3) to (ξ_n, ξ_{n-1}) gives

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\max\{\delta_n, \delta_{n-1}\}}{1 + \delta_{n-1}}. \tag{4.17}$$

If $\delta_n \le \delta_{n-1}$, then

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\delta_{n-1}}{1 + \delta_{n-1}} \le (\lambda + \mu)\delta_{n-1}.$$

If $\delta_n > \delta_{n-1}$, then

$$\delta_n \le \lambda\delta_{n-1} + \mu\frac{\delta_n}{1 + \delta_{n-1}} \le \lambda\delta_{n-1} + \mu\delta_n,$$

and hence

$$\delta_n \le \frac{\lambda}{1 - \mu}\delta_{n-1}.$$

Since $\lambda + \mu < 1$,

$$\frac{\lambda}{1 - \mu} \le \lambda + \mu = q_{\mathrm{m}}.$$

Thus, in either case,

$$\delta_n \le q_{\mathrm{m}}\delta_{n-1} \le q_{\mathrm{m}}^n\delta_0, \qquad n \ge 1. \tag{4.18}$$

Consequently, $\delta_n \to 0$.

For $m > n \geq 0$, Lemma 2.1, together with (4.1) and (4.18), yields

$$\begin{aligned}\mathscr{D}(\xi_n, \xi_m) &\leq \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n} \delta_k \\ &\leq \alpha^* \delta_0 q_{\mathrm{m}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{m}})^r \\ &\leq \frac{\alpha^* \delta_0 q_{\mathrm{m}}^n}{1 - \beta^* q_{\mathrm{m}}}. \end{aligned} \tag{4.19}$$

Since $\beta^* q_{\mathrm{m}} < 1$, the right-hand side tends to zero as $n \to \infty$, uniformly for $m > n$. Hence, $\{\xi_n\}$ is Cauchy. By completeness, there exists $\xi^\star \in \mathscr{X}$ such that

$$\xi_n \longrightarrow \xi^\star.$$

Sequential continuity of $\mathscr{T}$ implies

$$\mathscr{T}\xi_n \longrightarrow \mathscr{T}\xi^\star.$$

Since $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$, it remains to show that these limits coincide. By (DC3),

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) \leq \alpha(\mathscr{T}\xi^\star, \xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) \\ + \beta(\xi_{n+1}, \xi^\star)\mathscr{D}(\xi_{n+1}, \xi^\star).\end{aligned} \tag{4.20}$$

The control sequences in (4.20) are bounded by (4.2), whereas

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star, \mathscr{T}\xi_n) \longrightarrow 0$$

and

$$\mathscr{D}(\xi_{n+1}, \xi^\star) \longrightarrow 0.$$

Letting $n \to \infty$ in (4.20), we obtain

$$\mathscr{D}(\mathscr{T}\xi^\star, \xi^\star) = 0.$$

Therefore, (DC1) gives

$$\mathscr{T}\xi^\star = \xi^\star.$$

For uniqueness, let υ and ϖ be fixed points of $\mathscr{T}$. Then (3.3) gives

$$\mathscr{D}(\upsilon, \varpi) \leq \lambda \mathscr{D}(\upsilon, \varpi),$$

because

$$\mathscr{D}(\upsilon, \mathscr{T}\upsilon) = \mathscr{D}(\varpi, \mathscr{T}\varpi) = 0.$$

Since (4.16) implies $\lambda < 1$, we obtain

$$\mathscr{D}(\upsilon, \varpi) = 0.$$

Thus, $\upsilon = \varpi$ by (DC1), which completes the proof. □

Theorem 4.4. *Let $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ be a complete DCMTS and let $\mathscr{T} : \mathscr{X} \to \mathscr{X}$ satisfy the DC–interpolative rational contractive condition* (3.4). *For an arbitrary $\xi_0 \in \mathscr{X}$, let $\{\xi_n\}_{n\geq 0}$ be the corresponding Picard sequence.*

Assume that (4.1) *and* (4.2) *hold. Suppose that there exists $q_{\mathrm{i}} \in [0, 1)$ such that*

$$\delta_n \leq q_{\mathrm{i}} \delta_{n-1}, \qquad n \geq 1, \qquad \beta^* q_{\mathrm{i}} < 1. \tag{4.21}$$

Then $\{\xi_n\}$ converges to a fixed point $\xi^\star \in \mathscr{X}$ of $\mathscr{T}$.

Proof. Iterating the first inequality in (4.21), we obtain

$$\delta_n \leq q_{\mathrm{i}}{}^n \delta_0, \qquad n \geq 0. \tag{4.22}$$

Since $q_{\mathrm{i}} \in [0, 1)$, it follows that $\delta_n \to 0$ as $n \to \infty$.

Let $m > n$. By Lemma 2.1,

$$\mathscr{D}(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} \left(\prod_{j=n}^{k-1} \beta(\xi_j, \xi_{j+1}) \right) \alpha(\xi_k, \xi_{k+1}) \delta_k. \tag{4.23}$$

Using (4.1) and (4.22), we obtain

$$\begin{aligned}\mathscr{D}(\xi_n,\xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\le \alpha^*\delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{i}}^{\,k} \\ &= \alpha^*\delta_0 q_{\mathrm{i}}^{\,n} \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{i}})^r \\ &\le \frac{\alpha^*\delta_0 q_{\mathrm{i}}^{\,n}}{1-\beta^* q_{\mathrm{i}}}. \end{aligned} \tag{4.24}$$

Since $\beta^* q_{\mathrm{i}} < 1$, the right-hand side of (4.24) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Hence, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$.

By completeness, there exists $\xi^\star \in \mathscr{X}$ such that $\xi_n \to \xi^\star$. We now verify that $\xi^\star$ is a fixed point of $\mathscr{T}$.

By (DC3), using ξ_{n+1} as an intermediate point, we have

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) \le\ & \alpha(\mathscr{T}\xi^\star,\xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \\ & + \beta(\xi_{n+1},\xi^\star)\mathscr{D}(\xi_{n+1},\xi^\star). \end{aligned} \tag{4.25}$$

Since $\xi_{n+1} = \mathscr{T}\xi_n$, applying (3.4) to $(\xi^\star,\xi_n)$ gives

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) &\le \lambda\mathscr{D}(\xi^\star,\xi_n)^\theta \\ &\quad + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)^{1-\theta}\mathscr{D}(\xi_n,\mathscr{T}\xi_n)^{1-\theta}}{1+\mathscr{D}(\xi^\star,\xi_n)} \\ &= \lambda\mathscr{D}(\xi^\star,\xi_n)^\theta + \mu\frac{\mathscr{D}(\xi^\star,\mathscr{T}\xi^\star)^{1-\theta}\delta_n^{1-\theta}}{1+\mathscr{D}(\xi^\star,\xi_n)}. \end{aligned} \tag{4.26}$$

Because $\mathscr{D}(\xi^\star,\xi_n) \to 0$ and $\delta_n \to 0$, relation (4.26) implies that

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \longrightarrow 0 \qquad \text{as } n \to \infty.$$

By (4.2), the sequences $\{\alpha(\mathscr{T}\xi^\star,\xi_{n+1})\}$ and $\{\beta(\xi_{n+1},\xi^\star)\}$ are bounded. Passing to the limit in (4.25), we obtain $\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) = 0$. Therefore, (DC1) yields $\mathscr{T}\xi^\star = \xi^\star$.

Hence, the Picard sequence $\{\xi_n\}$ converges to a fixed point of $\mathscr{T}$. □

Remark 4.1. *Condition* (3.4) *alone does not guarantee uniqueness. Indeed, if υ and ϖ are distinct fixed points of $\mathscr{T}$, then*

$$\mathscr{D}(\upsilon,\varpi) \le \lambda\mathscr{D}(\upsilon,\varpi)^\theta,$$

and consequently

$$\mathscr{D}(\upsilon,\varpi)^{1-\theta} \le \lambda.$$

This inequality does not imply $\mathscr{D}(\upsilon,\varpi) = 0$. Thus, an additional separation condition is required to establish uniqueness.

Theorem 4.5. *Let $(\mathscr{X},\mathscr{D};\alpha,\beta)$ be a complete DCMTS and let $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ be a sequentially continuous mapping satisfying the DC–mixed rational contractive condition* (3.5). *For an arbitrary $\xi_0 \in \mathscr{X}$, let $\{\xi_n\}_{n\ge 0}$ be the corresponding Picard sequence.*

Assume that (4.1) *and* (4.2) *hold. Suppose that there exists $q_{\mathrm{x}} \in [0,1)$ such that*

$$\delta_n \le q_{\mathrm{x}}\delta_{n-1}, \qquad n \ge 1, \qquad \beta^* q_{\mathrm{x}} < 1. \tag{4.27}$$

Then $\mathscr{T}$ has a unique fixed point $\xi^\star \in \mathscr{X}$, and $\{\xi_n\}$ converges to $\xi^\star$.

Proof. Iterating the first inequality in (4.27), we obtain

$$\delta_n \le q_{\mathrm{x}}^{\,n}\delta_0, \qquad n \ge 0. \tag{4.28}$$

Hence, $\delta_n \to 0$ as $n \to \infty$.

Let $m > n$. By Lemma 2.1,

$$\mathscr{D}(\xi_n,\xi_m) \le \sum_{k=n}^{m-1}\left(\prod_{j=n}^{k-1}\beta(\xi_j,\xi_{j+1})\right)\alpha(\xi_k,\xi_{k+1})\delta_k. \tag{4.29}$$

Using (4.1) and (4.28), we get

$$\begin{aligned}\mathscr{D}(\xi_n,\xi_m) &\le \alpha^* \sum_{k=n}^{m-1} (\beta^*)^{k-n}\delta_k \\ &\le \alpha^*\delta_0 \sum_{k=n}^{m-1} (\beta^*)^{k-n} q_{\mathrm{x}}^k \\ &= \alpha^*\delta_0 q_{\mathrm{x}}^n \sum_{r=0}^{m-n-1} (\beta^* q_{\mathrm{x}})^r \\ &\le \frac{\alpha^*\delta_0 q_{\mathrm{x}}^n}{1-\beta^* q_{\mathrm{x}}}. \end{aligned} \tag{4.30}$$

Since $\beta^* q_{\mathrm{x}} < 1$, the right-hand side of (4.30) tends to zero as $n \to \infty$, uniformly with respect to $m > n$. Thus, $\{\xi_n\}$ is a Cauchy sequence in $\mathscr{X}$.

By completeness, there exists $\xi^\star \in \mathscr{X}$ such that $\xi_n \to \xi^\star$. Sequential continuity of $\mathscr{T}$ yields $\mathscr{T}\xi_n \to \mathscr{T}\xi^\star$, whereas $\mathscr{T}\xi_n = \xi_{n+1} \to \xi^\star$.

To identify these limits, apply (DC3) with ξ_{n+1} as the intermediate point:

$$\begin{aligned}\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) \le\ & \alpha(\mathscr{T}\xi^\star,\xi_{n+1})\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) \\ &+ \beta(\xi_{n+1},\xi^\star)\mathscr{D}(\xi_{n+1},\xi^\star). \end{aligned} \tag{4.31}$$

By (4.2), the sequences $\{\alpha(\mathscr{T}\xi^\star,\xi_{n+1})\}$ and $\{\beta(\xi_{n+1},\xi^\star)\}$ are bounded. Moreover,

$$\mathscr{D}(\mathscr{T}\xi^\star,\xi_{n+1}) = \mathscr{D}(\mathscr{T}\xi^\star,\mathscr{T}\xi_n) \longrightarrow 0,$$

while $\mathscr{D}(\xi_{n+1},\xi^\star) \to 0$. Passing to the limit in (4.31), we obtain $\mathscr{D}(\mathscr{T}\xi^\star,\xi^\star) = 0$. Hence, by (DC1), $\mathscr{T}\xi^\star = \xi^\star$.

For uniqueness, let $\upsilon,\varpi \in \mathscr{X}$ be fixed points of $\mathscr{T}$. Applying (3.5) to (υ,ϖ) gives

$$\begin{aligned}\mathscr{D}(\upsilon,\varpi) &= \mathscr{D}(\mathscr{T}\upsilon,\mathscr{T}\varpi) \\ &\le \lambda \max\{\mathscr{D}(\upsilon,\varpi),\mathscr{D}(\upsilon,\mathscr{T}\upsilon)\} + \mu\frac{\mathscr{D}(\varpi,\mathscr{T}\varpi)}{1+\mathscr{D}(\upsilon,\varpi)} \\ &= \lambda\mathscr{D}(\upsilon,\varpi). \end{aligned}$$

Therefore,

$$(1-\lambda)\mathscr{D}(\upsilon,\varpi) \le 0.$$

Since $\lambda < 1$ and $\mathscr{D}(\upsilon,\varpi) \ge 0$, it follows that $\mathscr{D}(\upsilon,\varpi) = 0$. Thus, (DC1) yields $\upsilon = \varpi$.

Hence, $\mathscr{T}$ has a unique fixed point and the Picard sequence converges to it. □

5 Examples

In this section, we present examples illustrating the applicability of the main results established in the previous section.

Example 5.1. Let

$$\mathscr{X} = [0,\infty), \qquad \mathscr{D}(\xi,\eta) = |\xi-\eta|^2,$$

and define the control functions by

$$\alpha(\xi,\eta) \equiv 2, \qquad \beta(\xi,\eta) \equiv 2.$$

As verified above, $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \ln\left(1+\frac{\xi}{2}\right), \qquad \xi \ge 0.$$

Since

$$\mathscr{T}'(\xi) = \frac{1}{2+\xi} \le \frac{1}{2},$$

the Mean Value Theorem gives

$$|\mathscr{T}(\xi)-\mathscr{T}(\eta)| \le \frac{1}{2}|\xi-\eta|,$$

and hence

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \le \frac{1}{4}\, \mathscr{D}(\xi, \eta). \tag{5.1}$$

Therefore, the *DC*–rational contractive condition (3.1) holds with

$$\lambda = \frac{1}{4}, \qquad \mu = 0.$$

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$, (5.1) yields

$$\delta_n \le \frac{1}{4}\, \delta_{n-1},$$

so that

$$\delta_n \le \left(\frac{1}{4}\right)^n \delta_0.$$

Moreover,

$$\alpha^* = \beta^* = 2, \qquad q_{\mathrm{r}} = \frac{1}{4}, \qquad \beta^* q_{\mathrm{r}} = \frac{1}{2} < 1.$$

Thus, all assumptions of Theorem 4.1 are satisfied. Hence, $\mathscr{T}$ possesses a unique fixed point. Since

$$\mathscr{T}(0) = 0,$$

the unique fixed point is

$$\xi^\star = 0,$$

and every Picard iteration converges to $\xi^\star$.

Example 5.2. Let

$$\mathscr{X} = [0,1], \qquad \mathscr{D}(\xi, \eta) = |\xi - \eta|,$$

and define

$$\alpha(\xi, \eta) \equiv 1, \qquad \beta(\xi, \eta) \equiv 1.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \frac{\xi}{2}, \qquad \xi \in [0,1].$$

Since

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) = \frac{1}{2}\, \mathscr{D}(\xi, \eta),$$

the *DC*–quadratic rational contractive condition (3.2) holds with

$$\lambda = \frac{1}{2}, \qquad \mu = 0.$$

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$, we have

$$\xi_n = \frac{\xi_0}{2^n}, \qquad \delta_n = \mathscr{D}(\xi_n, \xi_{n+1}) = \frac{\xi_0}{2^{n+1}} = \frac{1}{2}\, \delta_{n-1}.$$

Hence,

$$\delta_n = \left(\frac{1}{2}\right)^n \delta_0.$$

Moreover,

$$\Delta^* = \delta_0 \le \frac{1}{2}, \qquad \alpha^* = \beta^* = 1, \qquad q_{\mathrm{q}} = \frac{\lambda + \mu\Delta^*}{1 - \mu\Delta^*} = \frac{1}{2},$$

and

$$\beta^* q_{\mathrm{q}} = \frac{1}{2} < 1.$$

Therefore, all assumptions of Theorem 4.2 are satisfied.

Since

$$\mathscr{T}(0) = 0,$$

the mapping $\mathscr{T}$ has the unique fixed point

$$\xi^\star = 0,$$

and every Picard sequence converges to $\xi^\star$.

Example 5.3. Let

$$\mathscr{X} = [0,3], \qquad \mathscr{D}(\xi,\eta) = |\xi-\eta|,$$

and define

$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$

Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \min\left\{1, \frac{\xi}{2}\right\}, \qquad \xi \in [0,3].$$

Since both $\xi \mapsto \xi/2$ and $t \mapsto \min\{1,t\}$ are Lipschitz continuous with constants $1/2$ and 1, respectively,

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta) = |\mathscr{T}\xi - \mathscr{T}\eta| \le \frac{1}{2}\,\mathscr{D}(\xi,\eta).$$

Hence, the DC–max rational contractive condition (3.3) holds with

$$\lambda = \frac{1}{2}, \qquad \mu = \frac{1}{4}.$$

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,

$$\delta_n = \mathscr{D}(\xi_n,\xi_{n+1}) \le \frac{1}{2}\,\delta_{n-1},$$

and therefore

$$\delta_n \le \left(\frac{1}{2}\right)^n \delta_0.$$

Moreover,

$$\alpha^* = \beta^* = 1, \qquad q_{\mathrm{m}} = \lambda + \mu = \frac{3}{4}, \qquad \beta^* q_{\mathrm{m}} = \frac{3}{4} < 1.$$

Thus, all assumptions of Theorem 4.3 are satisfied.

Finally, if $\xi = \mathscr{T}(\xi)$, then

$$\xi = \min\left\{1, \frac{\xi}{2}\right\}.$$

If $0 \le \xi \le 2$, then $\xi = \xi/2$, giving $\xi = 0$; whereas $\xi > 2$ would imply $\xi = 1$, which is impossible. Hence,

$$\xi^\star = 0$$

is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

Example 5.4. Let

$$\mathscr{X} = [0,1], \qquad \mathscr{D}(\xi,\eta) = |\xi-\eta|,$$

and define

$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$

Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \frac{\xi}{4}, \qquad \xi \in [0,1].$$

Choose

$$\theta = \frac{1}{2}, \qquad \lambda = \frac{1}{4}, \qquad \mu = \frac{1}{4}.$$

Since $0 \le |\xi-\eta| \le 1$,

$$|\xi-\eta| \le |\xi-\eta|^{1/2},$$

and therefore

$$\mathscr{D}(\mathscr{T}\xi,\mathscr{T}\eta) = \frac{1}{4}\,\mathscr{D}(\xi,\eta) \le \lambda \mathscr{D}(\xi,\eta)^\theta.$$

Hence, $\mathscr{T}$ satisfies the DC–interpolative rational contractive condition (3.4).

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,

$$\xi_n = \frac{\xi_0}{4^n}, \qquad \delta_n = \frac{1}{4}\,\delta_{n-1},$$

so that (4.21) holds with

$$q_{\mathrm{i}} = \frac{1}{4}.$$

Moreover,

$$\alpha^* = \beta^* = 1, \qquad \beta^* q_{\mathrm{i}} = \frac{1}{4} < 1.$$

Hence, all assumptions of Theorem 4.4 are satisfied.

Finally,

$$\mathscr{T}(\xi) = \xi \iff \frac{\xi}{4} = \xi \iff \xi = 0.$$

Thus,

$$\xi^\star = 0$$

is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

Example 5.5. Let

$$\mathscr{X} = [0,4], \qquad \mathscr{D}(\xi,\eta) = |\xi - \eta|,$$

and define

$$\alpha(\xi,\eta) \equiv 1, \qquad \beta(\xi,\eta) \equiv 1.$$

Then $(\mathscr{X}, \mathscr{D}; \alpha, \beta)$ is a complete *DCMTS*.

Define

$$\mathscr{T}(\xi) = \begin{cases} \dfrac{\xi}{3}, & 0 \le \xi \le 2, \\ \dfrac{\xi+2}{6}, & 2 < \xi \le 4. \end{cases}$$

The mapping is continuous on $[0,4]$ and satisfies

$$\mathscr{T}(\mathscr{X}) \subseteq [0,1].$$

Moreover,

$$|\mathscr{T}\xi - \mathscr{T}\eta| \le \frac{1}{3}|\xi - \eta|, \qquad \xi, \eta \in \mathscr{X},$$

and hence

$$\mathscr{D}(\mathscr{T}\xi, \mathscr{T}\eta) \le \frac{1}{3}\,\mathscr{D}(\xi,\eta).$$

Choosing

$$\lambda = \frac{1}{3}, \qquad \mu = \frac{1}{4},$$

it follows immediately that $\mathscr{T}$ satisfies the *DC*–mixed rational contractive condition (3.5).

For the Picard sequence $\xi_{n+1} = \mathscr{T}\xi_n$,

$$\delta_n = \mathscr{D}(\xi_n, \xi_{n+1}) \le \frac{1}{3}\,\delta_{n-1},$$

so that

$$q_{\mathrm{x}} = \frac{1}{3}.$$

Furthermore,

$$\alpha^* = \beta^* = 1, \qquad \beta^* q_{\mathrm{x}} = \frac{1}{3} < 1.$$

Hence, all assumptions of Theorem 4.5 are fulfilled.

Finally,

$$\mathscr{T}(\xi) = \xi$$

gives

$$\frac{\xi}{3} = \xi \quad (0 \le \xi \le 2),$$

or

$$\frac{\xi+2}{6} = \xi \quad (2 < \xi \le 4).$$

The first equation yields $\xi = 0$, whereas the second gives $\xi = \frac{2}{5}$, which does not belong to $(2,4]$. Therefore,

$$\xi^\star = 0$$

is the unique fixed point, and every Picard sequence converges to $\xi^\star$.

6 Application to a Nonlinear Integral Equation

Let

$$\mathscr{X} = C([0,1],\mathbb{R})$$

be the Banach space of real-valued continuous functions on $[0,1]$ equipped with the supremum norm

$$\|\varphi\|_\infty = \max_{t\in[0,1]} |\varphi(t)|.$$

Define

$$\mathscr{D}(\varphi,\psi) = \|\varphi-\psi\|_\infty, \qquad \alpha(\varphi,\psi) \equiv 1, \qquad \beta(\varphi,\psi) \equiv 1.$$

Then $(\mathscr{X},\mathscr{D};\alpha,\beta)$ is a complete *DCMTS*.

Consider the nonlinear Fredholm integral equation

$$\varphi(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi(s))\,ds, \qquad t\in[0,1]. \tag{6.1}$$

Define the associated integral operator

$$(\mathscr{T}\varphi)(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi(s))\,ds.$$

Theorem 6.1. *Assume that*

(I) $\mathscr{G} \in C([0,1],\mathbb{R})$*;*
(II) $\mathscr{K} \in C([0,1]\times[0,1],\mathbb{R})$*;*
(III) $\mathscr{F} : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous;*
(IV) *there exist* $\lambda,\mu \in [0,1)$ *with* $\lambda+\mu<1$ *such that*

$$\mathscr{D}(\mathscr{T}\varphi,\mathscr{T}\psi) \le \lambda\mathscr{D}(\varphi,\psi) + \mu\frac{\mathscr{D}(\varphi,\mathscr{T}\varphi)\mathscr{D}(\psi,\mathscr{T}\psi)}{1+\mathscr{D}(\varphi,\psi)}, \quad \forall\,\varphi,\psi\in\mathscr{X}. \tag{6.2}$$

Then the integral equation (6.1) *admits a unique solution in* $C([0,1],\mathbb{R})$*.*

Proof. The continuity of $\mathscr{G}$, $\mathscr{K}$, and $\mathscr{F}$ ensures that $\mathscr{T} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is well defined.

Condition (6.2) is precisely the *DC*–rational contractive condition (3.1). Since $\alpha \equiv \beta \equiv 1$, the control boundedness assumptions of Theorem 4.1 are automatically satisfied. Hence, $\mathscr{T}$ possesses a unique fixed point $\varphi^\star \in \mathscr{X}$.

Finally,

$$\mathscr{T}\varphi^\star = \varphi^\star$$

is equivalent to

$$\varphi^\star(t) = \mathscr{G}(t) + \int_0^1 \mathscr{K}(t,s)\,\mathscr{F}(s,\varphi^\star(s))\,ds,$$

which is exactly (6.1). Therefore, (6.1) has a unique continuous solution. □

7 Conclusion

In this paper, we introduced several classes of rational-type contractions in double controlled metric type spaces and established fixed point results for rational, quadratic, maximum, interpolative, and mixed rational contractive mappings. Under suitable control conditions, existence and uniqueness were obtained for the rational, quadratic, maximum, and mixed contractions, while existence together with Picard convergence was proved for the interpolative case. Illustrative examples and an application to a nonlinear Fredholm integral equation demonstrated the applicability of the proposed results. Future work may focus on multivalued mappings, graph and ordered structures, coupled and common fixed points, and applications to fractional and nonlinear evolution equations.

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