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HYBRID FIXED POINT THEORY FOR SUM OF TWO OPERATORS IN A LATTICE ORDERED BANACH SPACE WITH APPLICATIONS TO NONLINEAR DISCONTINUOUS INTEGRAL EQUATIONS

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Abstract

We prove a hybrid fixed point theorem for sum of two operators in a lattice ordered Banach space and apply to nonlinear discontinuous hybrid integral equations of mixed type for proving the existence of maximal and minimal integrable solutions under certain mixed Lipschitz and monotonicity conditions of the nonlinear functions. Our main existence result is illustrated with a numerical example as well as with an application to initial value problems of nonlinear first and second order discontinuous hybrid linearly perturbed ordinary differential equations for proving the existence of maximal and minimal solutions.

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1 Introduction

Most of the problems of universe are nonlinear in nature and so governed by dynamic systems of nonlinear equations. Analysis of such problems for different aspects of their solutions is the nonlinear analysis and so its importance in the subject of mathematical analysis. The existence theory of nonlinear problems constitutes the major and core part of the stream of nonlinear analysis. There are several approaches to deal with nonlinear problems for existence and other aspects of their study. One of these approaches is the operator theoretic tools from the subject of nonlinear functional analysis. Therefore, several operatorial techniques are developed and used from the subject of nonlinear operator theory to accomplish the purpose. The fixed point theory is a part of nonlinear operator theory which provides most powerful tools for resolving the existence theory of nonlinear equations quantitatively and qualitatively. The applications of fixed point theorems to nonlinear problems require a little skill and a clever selection of fixed point theorem gives the cleverer results.

The existence theorems for both continuous and discontinuous standard nonlinear, that is, nonlinearly perturbed linear differential and integral equations are available in the literature and obtained under the continuity, Carathéodory and Chandrabhan type conditions by using the classical analytic, topological and algebraic fixed point principles respectively. See Granas and Dugundji [25], Hekkilä and Lakshmikatham [26], Dhage [7, 8, 13], Dhage *et al.* [16, 17, 18, 19] and references therein. But the case with nonlinear hybrid differential equations is different, where the existence results have been obtained only for continuous hybrid differential and integral equations by using the hybrid fixed point principles on the lines of Krasnoselskii [27] and Dhage [7, 8, 10, 11, 12, 13, 14, 15] and no existence result is so far proved for nonlinear discontinuous hybrid differential and integral equations. Therefore, it is of interest to obtain the existence results for discontinuous hybrid differential and integral equations which is the main motivation of the present work. In the present discussion, we establish a hybrid fixed point theorem for sum of the two operators in a lattice ordered Banach space and apply to nonlinear discontinuous linearly perturbed hybrid integral equations for proving the existence of maximal and minimal integrable solutions. Before going to the main hybrid fixed point theorems, we give some preliminaries which are needed in what follows.

2 Preliminaries

Let $(E, \|\cdot\|)$ denote a Banach space with norm $\|\cdot\|$. We introduce a lattice structure in E with the help of a partial order $\preceq$ in E which is a reflexive, antisymmetric and transitive binary relation on E into itself. See Lipschitz [28] and Maddux [29]. Then $(E, \|\cdot\|, \preceq)$ becomes a partially ordered Banach space provided the partial order is compatible with the linearity in E, that is, (i) if $x_1 \leq y_1$ and $x_2 \preceq y_2$, then $x_1 + x_2 \preceq y_1 + y_2$, and (ii) if $x_1 \preceq y_1$, then $\lambda x_1 \preceq \lambda y_1$ for all $x_1, x_2, y_1, y_2 \in E$ and $\lambda \in \mathbb{R}_+$. We assume that $(E, \preceq)$ is a lattice. Note that a partially ordered set (in short poset) $(E, \preceq)$ is called a **lattice** if $\wedge\{a, b\}$ and $\vee\{a, b\}$ exist for all $a, b \in E$, where $\wedge$ and $\vee$ denote respectively the **meet** and **join** lattice operations in E. Next, the lattice $(E, \preceq)$ is called **complete lattice** if $\wedge S$ and $\vee S$ exist for every subset S of E. The details of the lattice structure of the Banach space appear in Birkkoff [3]. Now, the triplet $(E, \|\cdot\|, \preceq)$ is called a **partially lattice ordered Banach space**. If the order relation $\preceq$ is defined by the cone K in the partially lattice ordered Banach space E, then the triplet $(E, \|\cdot\|, K)$ is simply called a **lattice ordered Banach space**. Note that a non-empty closed convex subset K of the Banach algebra is called a **cone** if it satisfies (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$ with $\lambda > 0$, and (iii) $\{-K\} \cap K = \{0\}$. Furthermore, if the norm $\|\cdot\|$ satisfies the property that $\|x\| \leq \|y\|$ whenever $|x| \preceq |y|$, where $|x| = \max\{x\,,\, -x\}$, then $(E, \|\cdot\|, \preceq)$ is called a **Banach lattice**. In particular, $(E, \|\cdot\|, K)$ is a Banach lattice if the norm $\|\cdot\|$ is monotone on K, that is, if $x, y \in K$ and $x \preceq y$, then $\|x\| \leq \|y\|$. It is known that every Banach lattice is lattice ordered Banach space but the converse is not necessarily true. Below we state a couple of classical fixed point theorems which we need in the sequel. To state the first analytical fixed point theorem, we need the following definitions in what follows.

Definition 2.1. *A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a **$\mathcal{D}$-function** if it is upper semi-continuous and nondecreasing satisfying $\psi(0) = 0$. The set of all $\mathcal{D}$-functions on $\mathbb{R}_+$ is denoted by $\mathfrak{D}$.*

Definition 2.2. *An operator $\mathcal{T} : E \to E$ is said to be a **$\mathcal{D}$-Lipschitzian** if there exists a $\mathcal{D}$-function $\psi \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|)$$

*for all $x, y \in E$. If $\psi(r) = k\,r$, then $\mathcal{T}$ is called a Lipschitz operator on E with **Lipschitz** constant k. Again, if $0 \leq k < 1$, $\mathcal{T}$ is called a **contraction** operator on E with contraction constant k. Furthermore, if $\psi(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a **nonlinear $\mathcal{D}$-contraction** on E with contraction $\mathcal{D}$-function ψ.*

Our first analytical or geometrical fixed point theorem is as follows.

Theorem 2.1 (Dhage [10, 11])**.** *Let E be a Banach space and let $\mathcal{A} : E \to E$ be a nonlinear $\mathcal{D}$-contraction. Then $\mathcal{A}$ has a unique fixed point ξ and the sequence $\{\mathcal{A}^n(x)\}$ of successive iterations converges to ξ for each $x \in E$.*

To state the second algebraic fixed point theorem, we need the following definition.

Definition 2.3. *A mapping $\mathcal{T}$ on a lattice $(L, \preceq)$ into itself is called **isotone increasing** if it preserve the order relation $\preceq$, that is, if $x, y \in L$ with $x \preceq y$, then $\mathcal{T}x \preceq \mathcal{T}y$.*

Theorem 2.2 (Tarski [30])**.** *Let $(L, \preceq)$ be a partially ordered set and let $\mathcal{T} : L \to L$ be a mapping. Suppose that*

(a) $\mathcal{T}$ is isotone increasing,
(b) $(L, \preceq)$ is a complete lattice, and
(c) $F_{\mathcal{T}} = \{u \in L \mid \mathcal{T}u = u\}$.

Then $F_{\mathcal{T}} \neq \emptyset$ and $\big(F_{\mathcal{T}}, \preceq\big)$ is a complete lattice.

In the present paper, we combine Theorems 2.1 and 2.2 and prove a hybrid fixed point theorem involving the sum of two operators satisfying mixed contraction and isotone increasing conditions in a partially lattice ordered Banach space.

3 Hybrid Fixed Point Principles

In this section we discuss the hybrid fixed point principles for sum of the two operators in a partially lattice ordered Banach space. Throughout rest of the paper unless otherwise mentioned, let E denote the partially lattice ordered Banach space. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two operators and consider the operator equation

$$\mathcal{A}x + \mathcal{B}x = x. \tag{3.1}$$

Note that the fixed point of the operator $\mathcal{T} = \mathcal{A} + \mathcal{B}$ in E is the solution of the operator equation (3.1) and vice versa. If E is a complete lattice and $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, then by Theorem 2.2, the operator equation (3.1) has a solution and the set of all solutions is a complete lattice. Similarly, a hybrid fixed point theorem for sum of two discontinuous operators on a non-empty subset of a partially lattice ordered Banach space may be stated as follows.

Theorem 3.1. *Let S be a non-empty subset of a partially lattice ordered Banach space $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}$, $\mathcal{B}$: $S \to E$ be two operators such that*
(a) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing,
(b) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and
(c) $(S, \preceq)$ is complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on S by
$$\mathcal{T}x = \mathcal{A}x + \mathcal{B}x. \tag{3.2}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, the operator $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is also a complete lattice in view if hypothesis (c), the desired conclusion follows by an application of Theorem 2.2. □

Corollary 3.1. *Let S be a non-empty subset of a lattice ordered Banach space $(E, \|\cdot\|, K)$ and let $\mathcal{A}$, $\mathcal{B}$: $S \to E$ be two operators such that*
(a) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing,
(b) $\mathcal{A}x + \mathcal{B}x \in S$ for all $x \in S$, and
(c) $(S, \preceq)$ is complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

The conclusion of above hybrid fixed point results does not remain true if the operators $\mathcal{A}$ and $\mathcal{B}$ are not isotone increasing on E or in particular on S . Therefore, it is of interest to prove the solution of the operator equation (3.1) when both the operators $\mathcal{A}$ and $\mathcal{B}$ are not continuous as well as isotone increasing on E. In the following we prove some results in this direction.

Theorem 3.2. *Let $(E, \|\cdot\|, \preceq)$ be a partially lattice ordered Banach space and let $\mathcal{A}, \mathcal{B} : E \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,
(c) $(E, \preceq)$ is a complete lattice, and
(d) $F_{(\mathcal{A}+\mathcal{B})} = \{u \in E \mid \mathcal{A}u + \mathcal{B}u = u\}$.
Then $F_{(\mathcal{A}+\mathcal{B})} \neq \emptyset$ and $\left(F_{(\mathcal{A}+\mathcal{B})}, \preceq\right)$ is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on E into itself by
$$\mathcal{T} = (I - \mathcal{A})^{-1}\mathcal{B}. \tag{3.3}$$
We show that $\mathcal{T}$ is well defined operator and maps E into itself. Let $y \in E$ be fixed element and define an operator A_y on E by
$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.4}$$

Let $x_1, x_2 \in E$ be arbitrary. Then from (3.3) it follows that
$$\|\mathcal{A}_y(x_1) - \mathcal{A}_y(x_2)\| = \|\mathcal{A}x_1 - \mathcal{A}x_2\| \leq \psi(\|x_1 - x_2\|),$$
where ψ is a $\mathcal{D}$-function satisfying $\psi(r) < r$, $r > 0$. This shows that $\mathcal{A}_y$ is a nonlinear $\mathcal{D}$-contraction operator on E. Now by application of Theorem 2.1, there is a unique point $y' \in E$ such that
$$\mathcal{A}_y(y') = y',$$
which by definition of $\mathcal{A}_y$ implies that
$$\mathcal{A}y' + \mathcal{B}y = y'$$
or
$$(I - \mathcal{A})y' = \mathcal{B}y. \tag{3.5}$$

Now,

$$\begin{aligned}(I-\mathcal{A})^{-1} &= I + A + A^2 + \cdots + \mathcal{A}^n + \ldots. \\ &= \sum_{n=0}^{\infty} \mathcal{A}^n,\end{aligned}$$

which does exists in view of hypothesis (a). Moreover, $(I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on E in view of hypothesis (b). Operating $(I-\mathcal{A})^{-1}$ on both sides of (3.5) we obtain $\mathcal{T}y = y'$. Therefore, the mapping $\mathcal{T}$ is well defined and maps E into itself. By hypotheses (b) and (c), $\mathcal{T}$ is isotone increasing on the complete lattice E. Now the desired conclusion follows by an application of Theorem 2.2. □

Remark 3.1. *We note that the operator $\mathcal{T} = (I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on E if $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on E, however the converse may not be true.*

Next we prove a hybrid fixed point theorem for sum of two operators in a subset of the lattice ordered Banach space which is applicable to nonlinear integral equations of mixed type for proving various aspects of the integrable solutions.

Theorem 3.3. *Let S be a non-empty subset of a partially lattice ordered Banach space $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}: E \to E$ and $\mathcal{B}: S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,
(b) $(I-\mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,
(c) $x = \mathcal{A}x + \mathcal{B}y \Longrightarrow x \in S$ for all $y \in S$, and
(d) $(S, \preceq)$ is a complete lattice.
Then the operator equation (3.1) has a solution in S and the set of all solutions in S is a complete lattice.

Proof. Define an operator $\mathcal{T}$ on S by (3.3). We show that $\mathcal{T}$ is well defined and maps S into itself. Let $y \in S$ be fixed and consider the operator $\mathcal{A}_y$ on E defined by

$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.6}$$

Clearly, $\mathcal{A}$ defines the mapping $\mathcal{A}_y : E \to E$. Then proceeding as in the proof of Theorem 3.1 it is proved that there exists a unique element $y' \in E$ such that

$$\mathcal{A}_y(y') = \mathcal{A}y' + \mathcal{B}y = y'. \tag{3.7}$$

Now, by hypothesis (c), we have that $y' \in S$. Further from (3.6) it follows that

$$(I-\mathcal{A})^{-1}\mathcal{B}y = y' \quad \text{or} \quad \mathcal{T}y = y'.$$

This shows that $\mathcal{T}$ is well defined and maps S into itself. Next, by hypothesis (c), $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is a complete lattice in view of hypothesis (d). Hence the desired conclusion follows by an application of Theorem 2.2. □

Definition 3.1. *Let $u, v \in (E, \|\cdot\|, \preceq)$ be any two elements with $u \preceq v$, then the closed order interval $[u, v]$ is a set of points in $(E, \|\cdot\|, \preceq)$ defined by*

$$[u, v] = \{x \in E \,:\, u \preceq x \preceq v\}.$$

The order interval $[u, v]$ is also called a lattice interval in the Banach lattice $(E, \|\cdot\|, \preceq)$.

Lemma 3.1. *A non-empty, closed lattice interval $[u, v]$ of the complete Banach lattice $(E, \|\cdot\|, K)$ is norm-closed and norm-bounded subset of E which is again a complete lattice.*

Proof. To prove $[u, v]$ is norm-closed, we must show that the partial order $\preceq$ is compatible with the norm topology. In any partially ordered Banach space E, the positive cone

$$K = \{x \in E \mid x \succeq 0\}$$

is norm-closed subset of E. Now, we can express the lattice interval $[u, v]$ using the translations of positive cone as

$$[u, v] = \{x \in E \mid u \preceq x \preceq v\} = (u + K) \cap (v - K).$$

Since K is a norm-closed subset of E, the translations $u + K$ and $v - K$ are norm-closed. As $[u, v]$ is he intersection of two norm-closed sets in E, it is norm-closed.

Next, we prove $[u, v]$ is norm-bounded subset of E. Now, in a partially orderd banach space (such as Banach lattice), the norm $\|\cdot\|$ is monotone with respect to the partial ordereing $\preceq$ in E. This means there exists a constant $M > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq M\|y\|$. Now for any element $x \in [u, v]$, we have $u \preceq x \preceq v \Longrightarrow x - u \preceq v - u$. Applying monotonicity of the norm $\|\cdot\|$, we obtain

$$\|x - u\| \leq M\|v - u\|.$$

Using the tiangle inequality property of the nrom, we have

$$\|x\| \leq \|x - u + u\| \leq \|x - u\| + \|u\| \leq M\|v - u\| + \|u\|.$$

Since the bound $M\|v - u\| + \|u\|$ is an upper bound independent of x, $[u, v]$ is a norm-bounded subset of E.

Finally, we show that $[u, v]$ is a complete lattice. Let A be a non-empty subset of the lattice interval $[u, v]$. We must prove that A possesses both $\wedge A$ and $\vee A$ inside $[u, v]$. Since E is given a complete lattice, A has $s_0 = \vee A$ and $t_0 = \wedge A$ evaluated over the entire space E. Because every element $a \in A$ belongs to $[u, v]$, we have $u \preceq a \preceq v$ for all $a \in A$. Since v is an upper bound for A, the least upper bound must satisfy $s_0 \preceq v$. Since A is non-empty, choose nny $a \in A$. Then we have $u \preceq a \preceq s_0$. Hence we must have $u \preceq s_0 \preceq v$ and $s_0 \in [u, v]$. Similarly, by dual reasoning we obtain $t_0 \in [u, v]$. Hence $[u, v]$ is a complete lattice. □

As a consequence of Theorem 3.2 and Lemma 3.1 we obtain the following applicable hybrid fixed point result as corollary.

Corollary 3.2. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete Banach lattice $(E, \|\cdot\|, K)$ and let $\mathcal{A} : E \to E$ and $\mathcal{B} : S \to E$ be two operators. Suppose that*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing, and

(b) $x = \mathcal{A}x + \mathcal{B}y \Longrightarrow x \in S$ for all $y \in S$.

Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing operators on the closed and bounded subset S of E, the operator $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing on S in view of Remark 3.1. Furthermore, the lattice completeness of S follows from Lemma 3.1. Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.2. Hence the operator equation (3.1) has a solution in S and the set of all solutions is a complete lattice. □

The above corollary 3.2 is an improvement of the Burton [4] type hybrid fixed point theorem involving the sum of two discontinuous operators in a lattice ordered Banach space. If we restrict the domain of the operator $\mathcal{A}$ to the subset S of the lattice ordered Banach space E, then the hybrid fixed result, Theorem 3.2 also remains true under a stronger hypothesis than hypothesis (c) as shown below.

Theorem 3.4. *Let S be a non-empty closed subset of a complete Banach lattice $(E, \|\cdot\|, \preceq)$ and let $\mathcal{A}, \mathcal{B} : S \to E$ be two operators. Suppose that*

(a) $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction,,

(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E,

(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$, and

(d) $(S, \preceq)$ is a complete lattice.

Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Proof. Define an operator $\mathcal{T}$ on S by (3.2). We show that $\mathcal{T}$ is well defined and maps S into itself. Let $y \in S$ be fixed and consider the operator $\mathcal{A}_y$ on S defined by

$$\mathcal{A}_y(x) = \mathcal{A}x + \mathcal{B}y. \tag{3.8}$$

Clearly, $\mathcal{A}_y$ defines the mapping $\mathcal{A}_y : S \to S$ in view of hypothesis (c). Then proceeding as in the proof of Theorem 3.1, it is proved that there exists a unique element $y' \in S$ such that

$$\mathcal{A}_y(y') = \mathcal{A}y' + \mathcal{B}y = y'. \tag{3.9}$$

Now, from (3.6) it follows that

$$(I - \mathcal{A})^{-1}\mathcal{B}y = y' \quad \text{or} \quad \mathcal{T}y = y'.$$

This shows that $\mathcal{T}$ is well defined and maps S into itself. Next, by hypothesis (c), $\mathcal{T}$ is isotone increasing on S and $(S, \preceq)$ is a complete lattice in view of hypothesis (d). Hence the desired conclusion follows by an application of Theorem 2.2. □

As a consequence of above hybrid fixed point theorem, we obtain a few more results of hybrid fixed point theory in view of Lemma 3.1. Since the proofs of these results are obvious, we omit the details.

Corollary 3.3. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete lattice partially ordered Banach space $\big(E, \|\cdot\|, \preceq\big)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is a contraction,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E, and
(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$.
Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Corollary 3.4. *Let $S = [u, v]$ be a non-empty closed lattice interval of a complete Banach lattice $(E, \|\cdot\|, K)$ and let $\mathcal{A}$, $\mathcal{B} : S \to E$ be two operators. Suppose that*
(a) $\mathcal{A}$ is a contraction,,
(b) $(I - \mathcal{A})^{-1}\mathcal{B}$ is isotone increasing, where I is the identity operator on E, and
(c) $\mathcal{A}x + \mathcal{B}y \in S$ for all $x, y \in S$.
Then the operator equation (3.1) *has a solution in S and the set of all solutions in S is a complete lattice.*

Remark 3.2. *If we compare all the above Corollaries 3.2 to 3.4, we observe that the hypothesis (a) of Corollary 3.2 is more general than than hypothesis (a) of Corollary 3.4, whereas the hypothesis (b) of Corollary 3.4 is is more general than hypothesis (b) of Corollary 3.2. Again hypothesis (c) of Corollary 3.2 is weaker than hypothesis (c) of Corollary 3.4. However, all the above hybrid fixed point results are applicable to nonlinear hybrid differential and integral equations under suitable conditions depending upon the prevailing situation and the circumstances of the problems for proving the existence of extremal solutions.*

4 Nonlinear Hybrid Integral Equations

In this section we present the statement of the proposed nonlinear hybrid integral equation of this paper, preliminary definitions, abstract spaces and some fundamental results which are useful in the subsequent development of the paper.

Given a closed and bounded interval $J = [a, b]$ in $\mathbb{R}$, the set of real numbers, we consider the nonlinear hybrid integral equation (in short *HIE*) of mixed type,

$$x(t) = q(t) + \lambda \int_a^t v(t, s) f(s, x(s))\, ds + \mu \int_a^b k(t, s) g(s, x(s))\, ds \tag{4.1}$$

for all $t \in J$, where $\lambda, \mu \in \mathbb{R}$, $\lambda > 0$, $\mu > 0$ and the functions $q : J \to \mathbb{R}$, $v, k : J \times J \to \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain hybrid conditions of integrability, Lipschitz, Carathéodory and monotonicity on their respective domains of definition to be specified later.

Definition 4.1. *A function $x \in L^1(J, \mathbb{R})$ is said to be an integrable solution of the HIE* (4.1) *if it satisfies the equation* (4.1) *on J, where $L^1(J, \mathbb{R})$ is the space of Lebesgue integrable real-valued functions defined on J.*

The *HIE* (4.1) is well-known and includes both nonlinear Volterra and Fredholm integral equations given respectively by

$$x(t) = q(t) + \lambda \int_a^t v(t, s) f(s, x(s))\, ds, \ \ t \in J, \tag{4.2}$$

and

$$x(t) = q(t) + \mu \int_a^b k(t, s) g(s, x(s))\, ds, \ \ t \in J, \tag{4.3}$$

as the particular cases. All the nonlinear integral equations given above have already been discussed in the literature extensively for various aspects of the integrable solutions. See Banas [1], Banas and Sayed [2] Emmanuel [23] and the references therein. The existence of continuous solution of the *HIE* (4.1) can be proved by using hybrid fixed point theorem of Krasnoselskii or its improvement given by Dhage (see [13] and references therein). Here we discuss the *HIE* (4.1) for extremity aspects of integrable solutions on J. In particular, we discuss the maximal and minimal integrable solutions via analytic and lattice theoretic hybrid fixed point theorem developed in this paper.

We place the problem of FrIE (4.1) in the function space $L^\infty(J,\overline{\mathbb{R}})$ of essentially bounded Lebesgue measurable extended real-valued functions defined on J, where $\overline{\mathbb{R}} = [-\infty,+\infty]$. It is clear that every essentially bounded and Lebesgue measurable function is Lebesgue integrable, so $L^1(J,\mathbb{R}) \supset L^\infty(J,\overline{\mathbb{R}})$. We define a standard norm $\|\cdot\|_\infty$ in $L^\infty(J,\overline{\mathbb{R}})$ by

$$\begin{aligned}\|x\|_\infty &= \operatorname*{ess\,sup}_{t\in J} |x(t)| \\ &= \inf\{C>0 : |x(t)| \le C \ \text{ a.e. } t \in J\} \\ &= \inf\{C>0 : -C \le x(t) \le C \ \text{ a.e. } t\in J\}.\end{aligned} \tag{4.4}$$

Clearly, $L^\infty(J,\overline{\mathbb{R}})$ becomes a Banach space w.r.t. the norm $\|\cdot\|_\infty$ defined above. Next, we introduce a partial order relation $\preceq$ in $L^\infty(J,\overline{\mathbb{R}})$ by the order cone K given by

$$K = \{x \in L^\infty(J,\overline{\mathbb{R}}) \mid x(t) \ge 0 \ \text{ a.e. } \ t\in J\}. \tag{4.5}$$

Thus,

$$x \preceq y \iff y - x \in K,$$

or equivalently,

$$x \preceq y \iff x(t) \le y(t) \ \text{ a.e. } \ t \in J. \tag{4.6}$$

The few details of order cones and their properties appear in the monographs of Guo and Lakshmikathham [24] and Granas and Dugundji [25]. We need the following results from lattice theory in what follows. See Dhage [10] and references therein.

Lemma 4.1 (Birkhoff [3]). *The partially ordered Banach space $\big(L^\infty(J,\overline{\mathbb{R}}), \|\cdot\|_\infty, \preceq\big)$ is a complete Banach lattice.*

Lemma 4.2. *A non-empty, norm-closed and norm-bounded subset of the form*

$$S = \{x \in L^\infty(J,\overline{\mathbb{R}}) \mid \|x\|_\infty \le r\}$$

for some real number $r > 0$, is a closed lattice interval in $L^\infty(J,\overline{\mathbb{R}})$.

Proof. Let S be a non-empty, norm-closed and norm-bounded subset of the complete Banach lattice $L^\infty(J,\overline{\mathbb{R}})$ defined by

$$S = \{x \in L^\infty(J,\overline{\mathbb{R}}) \mid \|x\|_\infty \le r\}$$

for some real number $r > 0$. Then, we have

$$\begin{aligned}S &= \{x \in L^\infty(J,\overline{\mathbb{R}}) \ \mid \ |x(t)| \le r \ \text{ a.e. } t \in J\} \\ &= \{x \in L^\infty(J,\overline{\mathbb{R}}) \ \mid \ -r \le x(t)| \le r \ \text{ a.e. } t\in J\} \\ &= [-r, r].\end{aligned}$$

If we let $u_0(t) = -r$ and $v_0(t) = r$ for almost all $t \in J$, then u_0 and v_0 are constant real-valued functions defined on J. So $u_0, v_0 \in L^\infty(J,\overline{\mathbb{R}})$ and $u_0 \preceq v_0$. As a result, the lattice interval $[u_0, v_0]$ is well defined and $S = [u_0, v_0]$. The proof of the lemma is complete. □

In the following section we prove our main existence result for the *HIE* (4.1) under suitable conditions on the functions involved in it.

5 Existence of Extremal Integrable Solutions

We consider the following definitions in the sequel.

Definition 5.1. *A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be* ***Chandrabhan***

(i) the map $t \mapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \mapsto f(t,x)$ is monotone nondecreasing for almost every $t \in J$.

*Furthermore, a Chandrabhan function $f(t,x)$ is called L^1_r-**Chandrabhan** if*

(iii) there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|f(t,x)| \le h_r(t) \ \ \text{a. e.} \ \ t \in J,$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

*Again, a Chandrabhan function $f(t,x)$ is called $L^1_{\mathbb{R}}$-**Chandrabhan** if* *(iv)* *there exists a function $h \in L^1(J,\mathbb{R})$ such that*

$$|f(t,x)| \leq h(t) \quad a.\,e. \quad t \in J,$$

for all $x \in \mathbb{R}$.

*Finally, a Chandrabhan function $f(t,x)$ is called $\mathbb{R}$-**Chandrabhan** if*

(iv) *there exists a number $M_f \in \mathbb{R}_+$ such that*

$$|f(t,x)| \leq M_f \quad a.\,e. \quad t \in J,$$

for all $x \in \mathbb{R}$.

Remark 5.1. *Note that the relation between the above types of functions goes as follows:*

$$\text{“}\mathbb{R}\text{-Chandrabhan} \Rightarrow L^1_{\mathbb{R}}\text{-Chandrabhan} \Rightarrow L^1_r\text{-Chandrabhan,”}$$

however, the reverse implication may not hold.

Similarly, we have the following definition.

Definition 5.2. *A function $k : J \times J \to \mathbb{R}$ is said to satisfy **integrability condition** if*

(i) *the map $(t,s) \mapsto k(t,s)$ is jointly measurable, and*

(iii) *there exists a function $\gamma_k \in L^1(J,\mathbb{R})$ such that*

$$|k(t,s)| \leq \gamma_k(s) \quad a.\,e.\ s \in J,$$

for all $t \in J$.

Definition 5.3. *A function $f : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if*

(i) *the map $t \mapsto f(t,x)$ is measurable for each $x \in \mathbb{R}$, and*

(ii) *the map $x \mapsto f(t,x)$ is continuous for almost everywhere $t \in J$.*

Further, a Carathéodory function is called L^1-Carathéodory if

(iii) *there exists a function $h \in L^1(J,\mathbb{R})$ such that*

$$|f(t,x)| \leq h(t) \ a.\,e.\ t \in J,$$

for all $x \in \mathbb{R}$.

Remark 5.2. *It is clear that every continuous real-valued function $k(t,s)$ on $J \times J$ is L^1-Carathéodory and every such L^1-Carathéodory function k satisfies the integrability condition on $J \times J$, however the converse may not be true. Moreover, if a real-valued function k satisfies the integrability condition on $J \times J$, then it also satisfies the integrability condition on every subset $J' \times J'$ of $J \times J$ for some $J' \subset J$. The details of L^1-Carathéodory functions appear in Banas [1], Banas and El-Sayed [2], Dhage [10] and Heikkilä and Lakshmikantham [26].*

Proposition 5.1. *Let $f(t,x)$ be a Chandrabhan function on $J \times \mathbb{R}$. Then the superposition operator $\mathcal{F}$ given by $\mathcal{F}x(t) = f(t,x(t))$, $t \in J$, monotonically maps the space $L^1(J,\mathbb{R})$ into itself if and only if f satisfies the growth condition*

$$|f(t,x)| \leq a(t) + b|x| \tag{5.1}$$

for almost all $t \in J$ and $x \in \mathbb{R}$, where $a \in L^1(J,\mathbb{R})$ and $b \geq 0$ is a real number.

Proof. The proof of the proposition is similar to a result that given in Krasnoselskii [27] for Caratheódory functions f on $J \times \mathbb{R}$ into $\mathbb{R}$. But for the sake of completeness we give the details of proof.

Part I (Sufficiency): Assume first that the growth condition (5.1) hold. We first prove the measurability of the superposition operator $\mathcal{F}$. Let $x \in L^1(J,\mathbb{R})$ be a measurable function. Since f is Chandrabhan function, $t \mapsto f(t,x)$ is measurable and $x \mapsto f(t,x)$ is nondecreasing on $\mathbb{R}$. It is known that every monotonic function is almost everywhere continuous and has at most countably many jump discontinuities J with finite values. Therefore, the composition function $\mathcal{F}x(t) = f(t,x(t))$ remains a measurable function on J. Taking the Lebesgue integration on both sides of the inequality (5.1), we obtain

$$\|\mathcal{F}x\|_{L^1} = \int_J |f(t,x(t))|\,dt \leq \int_J \big(a(t) + b|x(t)|\big)\,dt = \|a\|_{L^1} + b\|x\|_{L^1} < \infty.$$

As a result, we have that $\mathcal{F}x \in L^!(J,\mathbb{R})$. Next, let $x, y \in L^1(J,\mathbb{R})$ be such that $x(t) \leq y(t)$ for almost every $t \in J$. As $f(t,\cdot)$is nondecreasing for almost all $t \in J$, it follows that

$$\mathcal{F}x(t) = f(t,x(t)) \leq f(t,y(t)) = \mathcal{F}y(t)$$

for almost everywhere $t \in J$. Therefore $\mathcal{F}$ is a monotone nondecreasing operator on $L^1(J,\mathbb{R})$ into itself.

Part II (Necessity): Assume that $\mathcal{F}$ monotonically maps the space $L^1(J,\mathbb{R})$ into itself. We show that f must satisfy the growth condition (5.1. Suppose not and for each $n \in \mathbb{N}$, there exist elements $x_n \in \mathbb{R}$ and a subset $J_n \subset J$ of positive measure such that

$$|f(t,x_n)| > a(t) + n|x_n|$$

for $t \in J_n$. Now, choose a disjoint sequence of subintervals/subsets $J_n \subset J$ and construct a function $x^*(t)$ defined by piecewise values from x_n on J_n ensuring that

$$\int_{J_n} |x_n|\,dt < \frac{1}{2^n}.$$

This ensures that $x^*(t) = \sum_{n=1}^{\infty} x_n \chi_{J_n}(t)$ which is an element of $L^1(J,\mathbb{R})$. However, when evaluating the superposition operator $\mathcal{F}$ on this function, we obtain

$$\|\mathcal{F}x^*\|_{L^1} = \int_J |f(t,x^*(t))|\,dt = \sum_{n=1}^{\infty} \int_{J_n} |f(t,x_n)|\,dt > \sum \int_{J_n} \big(a(t) + n|x_n|\big)\,dt = \infty.$$

This contradicts to our assumptions that $\mathcal{F}x \in L^1(J,\mathbb{R})$. Hence the growth condition (5.1) must hold. This completes the proof. □

In the special case when $b = 0$ in above Proposition 5.1, we obtain the following useful result for our further discussion.

Lemma 5.1 (Dhage [9, 10]). *If $f(t,x)$ is Chandrabhan, then the function $t \mapsto f(t,x(t))$ is measurable for each $x \in L^1(J,,\mathbb{R})$. Moreover, if $f(t,x)$ is $L^1_{\mathbb{R}}$-Chandrabhan, then $f(\cdot,x(\cdot))$ is Lebesgue integrable on J for each $x \in L^1(J,\mathbb{R})$.*

Definition 5.4. *An integrable solution $x_M \in L^1(J,\mathbb{R})$ of the HIE (4.1) is said to be maximal if x is any other integrable solution, then $x(t) \leq x_M(t)$ for almost every $t \in J$. Similarly, a minimal integrable solution x_m of the HIE (4.1) is defined on J.*

We need the following hypotheses in what follows.

(H_1) The function $q : J \to \mathbb{R}$ is Lebesgue measurable and bounded.

(H_2) The functions v and k are nonnegative and satisfy integrability conditions on $J \times J$.

(H_3) There exists a constant $\alpha > 0$ such that

$$|f(t,x) - f(t,y)| \leq \alpha\,|x - y| \text{ a. e. t} \in \text{J},$$

for all $x, y \in \mathbb{R}$. Moreover, $\lambda\alpha(b-a) < 1$.

(H_4) The function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$.

(H_5) The function g are L^1_r-Chandrabhan on $J \times \mathbb{R}$.

Theorem 5.1. *Assume that hypotheses (H_1) through (H_5) hold. Then the HIE (4.1) has a maximal and a minimal integrable solution defined on J.*

Proof. Here set $E = L^\infty(J,\overline{\mathbb{R}})$. Define a subset S of the complete lattice $\big(L^\infty(J,\overline{\mathbb{R}}), \preceq\big)$ by

$$S = \big\{x \in L^\infty(J,\overline{\mathbb{R}}) \,\big|\, \|x\|_\infty \leq r\big\}, \tag{5.2}$$

where,

$$r = \|q\|_\infty + \big[\lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}\big](b-a),$$

and

$$H_{vf}(t) = \gamma_v(t)h(t), \quad H_{kg}(t) = \gamma_k(t)h_r(t),\ t \in J.$$

By Lemmas 3.1, 4.1 and 4.2, $(S,\preceq)$ is a complete Banach lattice. Define two operators $\mathcal{A}$ on $L^\infty(J,\overline{\mathbb{R}})$ and $\mathcal{B}$ on S by

$$\mathcal{A}x(t) = q(t) + \lambda \int_a^t v(t,s)f(s,x(s))\,ds,\ \ t \in J, \tag{5.3}$$

and

$$\mathcal{B}x(t) = \mu \int_a^b k(t,s)g(s,x(s))\,ds, \;\; t \in J. \tag{5.4}$$

Then the *HIE* (4.1) is transformed into the hybrid operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \;\; t \in J. \tag{5.5}$$

We show that $\mathcal{A}$ defines a mapping $\mathcal{A} : L^\infty(J,\overline{\mathbb{R}}) \to L^\infty(J,\overline{\mathbb{R}})$. Now by hypothesis (H$_4$), the function f is $L^1_{\mathbb{R}}$-Chandrabhan on $J \times \mathbb{R}$ and by hypothesis (H$_2$), the function v satisfies the integrability condition on $J \times J$. Therefore, the integral on right hand of the equation (5.3) exists. Moreover, the integral is continuous and hence Lebesgue integrable. Again the sum of two Lebesgue integrable functions is again Lebesgue integrable on J. Hence $\mathcal{A}x \in L^\infty(J,\overline{\mathbb{R}})$. Similarly, we have $\mathcal{B}x \in L^\infty(J,\overline{\mathbb{R}})$ for each $x \in S$. Next, we show that $\mathcal{A}$ and $\mathcal{B}$ are isotone increasing on their respective domains of definition. Let $x, y \in L^\infty(J,\overline{\mathbb{R}})$ be such that $x \preceq y$. Then we have

$$\begin{aligned}
\mathcal{A}x(t) &= q(t) + \lambda \int_a^t v(t,s)f(s,x(s))\,ds \\
&\leq q(t) + \lambda \int_a^t v(t,s)f(s,y(s))\,ds \\
&= \mathcal{A}y(t)
\end{aligned}$$

for almost every $t \in J$. This shows that $\mathcal{A}x \preceq \mathcal{A}y$ on J. Consequently, $\mathcal{A}$ is an isotone increasing operator on E. Similarly it can be proved that the operator $\mathcal{B}$ is an isotone increasing on S.

Next, we show that $\mathcal{A}$ is a contraction on $L^\infty(J,\overline{\mathbb{R}})$. Let $x, y \in L^\infty(J,\overline{\mathbb{R}})$. Then, by hypothesis (H$_3$), we get

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \lambda \int_a^t |f(s,x(s)) - f(s,y(s))|\,ds \\
&\leq \lambda \int_a^t \alpha\,|x(s) - y(s)|\,ds \\
&\leq \lambda\,\alpha \int_a^b |x(s) - y(s)|\,ds \\
&= \lambda\,\alpha\,(b-a)\|x - y\|_\infty .
\end{aligned}$$

Taking the essential supremum on both sides of the above inequality over t, we get

$$\|\mathcal{A}x - \mathcal{A}y\|_\infty \leq \lambda\alpha(b-a)\|x-y\|_\infty$$

for all $x, y \in S$, where $0 \leq \lambda\alpha(b-a) < 1$. This shows that $\mathcal{A}$ is a contraction on $L^\infty(J,\overline{\mathbb{R}})$ and so satisfies the condition (a) of Theorem 3.4. Next we show that condition (b) of Theorem 3.4 is satisfied. Let $y \in S$ be arbitrary and consider the operator equation $x = \mathcal{A}x + \mathcal{B}y$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathcal{A}x(t)| + |\mathcal{B}y(t)| \\
&\leq |q(t)| + \lambda \int_0^t |v(t,s)|\,|f(s,x(s))|\,ds + \mu \int_a^b |k(t,s)|\,|g(s,y(s))|\,ds \\
&\leq |q(t)| + \lambda \int_a^t \gamma_v(s)h(s)\,ds + \mu \int_a^b \gamma_k(s)h_r(s)\,ds \\
&\leq |q(t)| + \lambda \int_a^b H_{vf}(s)\,ds + \mu \int_a^b H_{kg}(s)\,ds \\
&\leq \|q\|_\infty + \lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}.
\end{aligned}$$

Taking the essential supremum on both sides of the above inequality over t, we obtain

$$\|x\|_\infty \leq \|q\|_\infty + (b-a)\big[\lambda\|H_{vf}\|_{L^1} + \mu\|H_{kg}\|_{L^1}\big] \leq r$$

which shows that $x \in S$. Thus condition (b) of Theorem 3.4 is satisfied. Thus all the conditions of Theorem 3.4 hold, whence the operator equations $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S and the set $F_{\mathcal{A}+\mathcal{B}}$ of all solutions is a complete lattice. Consequently $x_m = \wedge F_{\mathcal{A}+\mathcal{B}}$ and $x_M = \vee F_{\mathcal{A}+\mathcal{B}}$ both exist and are respectively the minimal and maximal integrable solutions of the *HIE* (4.1) defined on J. This completes the proof. $\square$

Remark 5.3. *The existence theorem, Theorem 5.1 may be extended under similar hypotheses to generalized nonlinear discontinuous hybrid functional integral equations of the type,*

$$x(t) = q(t) + \lambda \int_a^{\alpha(t)} v(t,s) f(s,x(s))\,ds + \mu \int_a^{\beta(t)} k(t,s) g(s,x(s))\,ds, \ \ t \in J, \tag{5.6}$$

with appropriate modifications, where $\alpha, \beta : J \to J$ are continuous functions.

Remark 5.4. *Notice that Theorem 5.1 also remains true if we replace the function $v(t,s)$ in hypothesis (H_2) by a weaker condition that $v(t,s)$ is nonnegative and satisfies the integrablity condition on the closed subinterval $[a,t]$ for every $t \in J$. As t is a variable in the interval $[a,b]$, we have to check the validity of this condition for every $t \in [a,b]$. To avoid this cumbersome job, we have assumed that the function $v(t,s)$ satisfies the nonnegativity and integrability condition on whole of the interval $[a,b] \times [a,b]$. The situation is useful while dealing with the functional integral equation of the type* (5.6) *defined on J.*

Example 5.1. Let $J = [0,1] \subset \mathbb{R}$ and consider the nonlinear HIE of mixed type,

$$x(t) = t^2 + \frac{1}{2} \int_0^t (t-s) \tan^{-1} x(s)\,ds + \int_0^1 (t+s) \tanh x(s)\,ds, \ t \in [0,1]. \tag{5.7}$$

Here, $q(t) = t^2$ which is a continuous and hence Lebesgue measurable and bounded function on $[0,1]$. Again, $v(t,s) = t-s$, $t \geq s$, and $k(t,s) = t+s$ for $t,s \in [0,1]$. Obviously, v and k are continuous and nonnegative real-valued functions defined on $[0,1] \times [0,1]$, so they satisfy the nonnegativity condition on the domain $[0,1] \times [0,t]$ and $[0,1] \times [0,1]$ respectively. The integrability condition also holds for the functions v and k on their domain of definition in view of Remark 5.2. Next, $f(t,x) = \tan^{-1} x$. Clearly, f is a $L^1_{\mathbb{R}}$-Chandrabhan function on $[0,1] \times \mathbb{R}$ with $h(t) = \frac{\pi}{2}$ for all $t \in [0,1]$ and satisfies the hypothesis (H_3) with with Lipschitz constant $\alpha = \frac{1}{2}$. Finally, $g(t,x) = \tanh x$ which is also is a $L^1_{\mathbb{R}}$-Chandrabhan function on $[0,1] \times \mathbb{R}$ with $h(t) = 1$ for all $t \in [0,1]$. Thus, all the conditions (H_1) through (H_5) of Theorem 5.1 are satisfied for $t \in [0,1]$ and $x \in \mathbb{R}$. Therefore, the HIE (5.7) has minimal and maximal integrable solutions defined on the interval $[0,1]$.

6 The Applications

In this section we apply our main existence theorem of the previous section to a couple of the IVPs of nonlinear hybrid first and second order ordinary linearly perturbed differential equations of first kind for proving the existence of maximal and minimal solutions under suitable conditions. Note that such hybrid perturbed differential equations are well-known and have already been discussed for existence of solution via fixed point theorems of Krasnoselskii [27] and Dhage [14] and for existence of extremal solutions via hybrid fixed point theorems due to Dhage [12]. The existence of both lower and upper solutions is assumed and the solution is obtained under continuity and discontinuity of Caratheodory type conditions. Here, we discuss the HPDE (6.1) for existence of maximal and minimal solutions without assuming the existence of both lower and upper solutions as well as relaxing the continuity conditions of any type.

6.1 First order hybrid differential equations

Let $J = [0,T]$ be a closed and bounded interval in $\mathbb{R}$ and consider the hybrid perturbed differential equation (in short *HPDE*) of the type

$$\left.\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t)) \ \text{ a. e. } t \in J, \\ x(0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{6.1}$$

where the functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain Lipschitz and Chandrabhan type hybrid conditions repectively.

Definition 6.1. *A function $x \in AC(J,\mathbb{R})$ is said to be a solution of the HPDE* (6.1) *if x satisfies the equations in* (6.1)*, where $AC(J,\mathbb{R})$ is the space of absolutely continuous real-valued functions defined on J.*

Lemma 6.1. *If $h \in L^1(J,\mathbb{R})$, then the differential equation*

$$\left.\begin{aligned} x'(t) &= h(t) \ \ a.\,e.\ t \in J, \\ x(0) &= x_0 \in \mathbb{R}_+, \end{aligned}\right\} \tag{6.2}$$

is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t h(s)\,ds, \ \ t \in J. \tag{6.3}$$

Theorem 6.1. *Assume that hypotheses* (H_1)*,* (H_3)*,* (H_4) *and* (H_5) *hold. If* $\alpha < 1$*, then the HPDE* (6.1) *has maximal and minimal solutions defined on* J*.*

Proof. By Lemma 6.1, the *HPDE* (6.1) is equivalent to the nonlinear hybrid perturbed integral equation (in short *HPIE*)

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds, \quad t \in J. \tag{6.4}$$

Set $E = L^\infty(J, \overline{\mathbb{R}})$. Let $q(t) = x_0$. Then q is a constant real function which is Lebesgue measurable and bounded on J. Now the desired conclusion follows by an application of Theorem 5.1 together with Remark 5.3 satisfying $v \equiv 1 \equiv k$ on $J \times J$ and $\alpha(t) = t = \beta(t)$ for all $t \in J$. □

Example 6.1. Given a closed interval $J = [0, 1]$ of the real line $\mathbb{R}$, consider the IVP of ordinary first order nonlinear HPDE,

$$\left.\begin{aligned} x'(t) &= \frac{1}{2}\tan^{-1} x + \tanh x \ \text{ a. e. } t \in J, \\ x(0) &= 1. \end{aligned}\right\} \tag{6.5}$$

It is easy to verify that the functions $f(t, x) = \tan^{-1} x$ and $g(t, x) = \tanh x$ satisfy all hypotheses (H_1), (H_3), (H_4) and (H_5) of Theorem 6.1 and hence the *HPDE* (6.5) has maximal and minimal solutions defined on J.

6.2 Second order hybrid differential equations

Let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$ and consider the hybrid perturbed differential equation (in short *HPDE*) of the type

$$\left.\begin{aligned} x''(t) &= f(t, x(t)) + g(t, x(t)) \ \text{ a. e. } t \in J, \\ x(0) &= x_0, \ \ x'(0) = x_1, \end{aligned}\right\} \tag{6.6}$$

where $x_0, x_1 \in \mathbb{R}$ and the functions $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ satisfy certain Lipschitz and Chandrabhan type hybrid conditions receptively.

Definition 6.2. *A function* $x \in AC^1(J, \mathbb{R})$ *is said to be a solution of the HPDE* (6.6) *if* x *satisfies the equations in* (6.1)*, where* $AC^1(J, \mathbb{R})$ *is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on* J*.*

Lemma 6.2. *If* $h \in L^1(J, \mathbb{R})$*, then the differential equation*

$$\left.\begin{aligned} x''(t) &= h(t) \ \text{ a. e. } t \in J, \\ x(0) &= x_0, \ \ x'(0) = x_1, \end{aligned}\right\} \tag{6.7}$$

is equivalent to the integral equation

$$x(t) = x_0 + x_1 t + \int_0^t (t - s)\, h(s)\, ds, \quad t \in J. \tag{6.8}$$

Theorem 6.2. *Assume that hypotheses* (H_1)*,* (H_3)*,* (H_4) *and* (H_5) *hold. If* $\alpha < 1$*, then the HPDE* (6.6) *has maximal and minimal solutions defined on* J*.*

Proof. By Lemma 6.1, the *HPDE* (6.1) is equivalent to the nonlinear hybrid perturbed integral equation (in short HPIE)

$$x(t) = x_0 + x_1 t + \int_0^t (t - s)\, f(s, x(s))\, ds + \int_0^t (t - s)\, g(s, x(s))\, ds, \quad t \in J. \tag{6.9}$$

Set $E = L^\infty(J, \overline{\mathbb{R}})$. Let $q(t) = x_0 + x_1 t$. Then q is a continuous real function which is Lebesgue measurablea and bounded on J. Now the desired conclusion follows by an application of Theorem 5.1 together with Remark 5.3 satisfying $v \equiv (t - s) \equiv k$ on $J \times J$ and $\alpha(t) = t = \beta(t)$ for all $t \in J$. □

Example 6.2. Given a closed interval $J = [0,1]$ of the real line $\mathbb{R}$, consider the IVP of ordinary second order nonlinear HPDE,

$$\left.\begin{aligned} &x''(t) = f_1(t, x(t)) + \tanh x(t) \ \text{ a.e. } t \in J, \\ &x(0) = 1, \ \ x'(0) = 2, \end{aligned}\right\} \tag{6.10}$$

where

$$f_1(t,x) = \begin{cases} 1 & \text{if } \ x \le 0, \\ \dfrac{1}{2} \cdot \dfrac{x}{1+x} & \text{if } \ x > 0. \end{cases}$$

It is easy to verify that the functions $f(t,x) = f_1(t,x)$ and $g(t,x) = \tanh x$ satisfy all hypotheses (H_1), (H_3), (H_4) and (H_5) of Theorem 6.2 and hence the HPDE (6.10) has maximal and minimal solutions defined on J.

Remark 6.1. *The study of the nonlinear first and second order discontinuous HPDEs* (6.1) *and* (6.6) *of this section may be extended to the higher n^{th} order nonlinear ordinary HPDEs in an analogous way under similar conditions with appropriate modifications for proving the existence of maximal and minimal solutions. Some of the results in this direction will be reported elsewhere.*

Open Problem

Finally, while concluding, we mention an open problem: whether the hybrid fixed point results, Theorems 3.1 and 3.2 of this paper can be extended with appropriate modifications to the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ involving the product of two continuous and discontinuous nonlinear operators $\mathcal{A}$ and $\mathcal{B}$ on a lattice ordered Banach algebra E into itself. A few results along this line under some stringent conditions recently appear in Dhage *et al.* [17], however other related problems are still open.

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