ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, Vol. 56(1) (2026), 94-106

SET-VALUED OPTIMIZATION PROBLEMS WITH HIGHER-ORDER (α, β)-δ-CONE CONVEXITY

Koushik Das[1,2]

[1]Officer on Special Duty (OSD), Education Directorate, Department of Higher Education, Government of West Bengal, Bikash Bhawan, Kolkata, West Bengal, India - 700091
and
[2]Assistant Professor, Department of Mathematics, Government General Degree College Singur, Hooghly, West Bengal, India - 712409
Email: koushik@singurgovtcollege.org; koushikdas.maths@gmail.com
(Received: February 22, 2026; In format: February 28, 2026; Revised: March 26, 2026; Accepted: March 29, 2026)

DOI: https://doi.org/10.58250/jnanabha.2026.56111

Abstract

In this paper, we establish the higher-order sufficient Karush-Kuhn-Tucker *(KKT)* optimality conditions for set-valued optimization problems under the higher-order contingent epiderivative and higher-order (α, β)-δ-cone convexity assumptions. We also study the higher-order Mond-Weir (MWD) type of duality and establish the corresponding higher-order strong, weak and converse duality theorems in between the primal (P) and corresponding dual problem. An example is also developed to demonstrate that higher-order (α, β)-δ-cone convexity is more generalized than higher-order cone convexity.
2020 Mathematical Sciences Classification: 26B25; 49N15.
Keywords and Phrases: Convex cone; Set-valued map; Contingent epiderivative; Duality; Arcwisely connectedness.

1 Introduction

Convex analysis plays a vital role when investigating the vector optimization problems, or *VOPs* for short. Several generalized convexity concepts have been proposed to relax the convexity assumption. The notion of invexity in *VOPs* was first introduced by Craven [9], Ben-Israel and Mond [5], Hanson and Mond [19], and numerous others. Weir and Mond [32] and Weir and Jeyakumar [31] introduced preinvexity of scalar-valued maps as a generalization of invexity.

Zalmai [33] introduced the idea of δ-(η, θ)-invexity as a further generalization of invexity. Another type of additional development of invexity is the concept of (α, β)-invexity, which was introduced by Antczak [1, 2]. He presented the optimality conditions required for multiobjective programming problems and used the (α, β)-invexity assumptions to deduce the duality results. The idea of (α, β)-δ-(η, θ)-invexity in *VOPs* was introduced by Mandal and Nahak [26]. They employed the (α, β)-δ-(η, θ)-invexity assumption to establish the required optimality conditions and the Mond-Weir type duality results.

In recent years, a lot of authors have been interested in set-valued optimization problems (*SVOPs* for short), in which the functions related to constraints and the objective function are set-valued mappings (*SVMs* for short). Anbin and Frankowska [4] were the first to introduce the idea of contingent derivative of *SVMs*. Corley [8] applied contingent derivatives to develop the generalized necessary optimality conditions of Fritz John type for the existence of maximum elements of *SVOPs* in 1987. Additionally, he explored the assumption of cone concavity to construct the generalized Fritz John sufficient optimality conditions for *SVOPs*.

Subsequently, in 1991, Sach and Craven [28, 29] established Mond-Weir type duality theorems and developed the concept of invex *SVMs*. Luc and Malivert [25] coined the term "invexity" in set-valued optimization theory. The necessary and sufficient optimality conditions of *SVOPs* have also been investigated using contingent derivatives. The generalized Farkas-Minkowski type theorem for convex *SVMs* and the Mond-Weir and Wolfe types duality results were later derived by Lin [24]. Bhatia and Mehra [6] introduced the concept of preinvex *SVMs*. As an additional notion of differentiability of *SVMs*, Jahn and Rauh [20] proposed the concept of contingent epiderivative.

Sheng and Liu [30] used generalized contingent epiderivative assumptions and the concept of preinvexity to study the *KKT* conditions of *SVOPs*. The existence, uniqueness, and characteristics of contingent epiderivative were studied by Rodriguez-Marin and Sama [27]. Using a higher-order contingent derivative, Li *et al.* [22, 23] established the necessary and sufficient optimality conditions. They also investigated the duality theorems using convexity assumptions and developed the Mond-Weir dual of higher-order for *SVOP*.

In this paper, we propose the concept of higher-order (α, β)-δ-cone convexity of *SVMs* as a generalization of cone convex *SVMs* of higher-order. For $\alpha, \beta, \delta = 0$, we get the standard notion of higher-order cone convexity of *SVMs*. Additionally, we construct a higher-order (α, β)-δ-cone convex *SVM* example that is not cone convex of higher-order.

Our main goal is to determine the higher-order sufficient optimality conditions of the *SVOP* (P) in more generalized cases by applying higher-order generalized contingent epiderivative and higher-order (α, β)-δ-cone convexity assumptions on the constraints and objective functions. We also study the higher-order strong, weak and converse duality theorems of the Mond-Weir (MWD) type via higher-order (α, β)-δ-cone convexity assumption.

2 Definitions and preliminaries

Let S be a real normed space (in short, *RNS*) and $\emptyset \neq D \subseteq S$. Then D is said to be a cone if $\lambda b \in D$, for all $b \in D$ and $\lambda \geq 0$. Additionally, D is said to be proper if $D \neq S$, pointed if $D \cap (-D) = \{\theta_S\}$, solid if $\operatorname{int}(D) \neq \emptyset$, closed if $\overline{D} = D$, and convex if $\lambda D + (1-\lambda)D \subseteq D$, for all $\lambda \in [0,1]$, where $\operatorname{int}(D)$ and $\overline{D}$ represent the interior and closure of D, respectively and θ_S denotes the zero of S.

Let us consider the non-negative orthant $\mathbb{R}^j_+$ of j-dimensional Euclidean space $\mathbb{R}^j$ by

$$\mathbb{R}^j_+ = \{b = (b_1, ..., b_j) \in \mathbb{R}^j : b_i \geq 0, \forall i = 1, 2, ..., j\}.$$

Then $\mathbb{R}^j_+$ is a solid as well as pointed closed convex cone and $\operatorname{int}(\mathbb{R}^j_+) \cup \{0_{\mathbb{R}^j}\}$ is a solid as well as pointed convex cone in $\mathbb{R}^j$, where $0_{\mathbb{R}^j}$ denotes the zero element of $\mathbb{R}^j$.

With respect to (w.r.t.) $\mathbb{R}^j_+$, there exist mainly two types of cone-orders in $\mathbb{R}^j$. For $b_1, b_2 \in \mathbb{R}^j$,

$$b_1 \leq b_2 \text{ if } b_2 - b_1 \in \mathbb{R}^j_+$$

and

$$b_1 < b_2 \text{ if } b_2 - b_1 \in \operatorname{int}(\mathbb{R}^j_+).$$

We say $b_2 \geq b_1$, if $b_1 \leq b_2$ and $b_2 > b_1$, if $b_1 < b_2$. For $\emptyset \neq B, B' \subseteq \mathbb{R}^j$,

$$B \leq 0_{\mathbb{R}^j} \Longleftrightarrow b \leq 0_{\mathbb{R}^j}, \forall b \in B,$$
$$B < 0_{\mathbb{R}^j} \Longleftrightarrow b < 0_{\mathbb{R}^j}, \forall b \in B,$$
$$B \leq B' \Longleftrightarrow b \leq b', \forall b \in B \text{ and } \forall b' \in B',$$

and

$$B < B' \Longleftrightarrow b < b', \forall b \in B \text{ and } \forall b' \in B'.$$

For $\emptyset \neq B, B' \subseteq \mathbb{R}^j$ and $b^*, b'^* \in \mathbb{R}^j$, define

$$b^*B + b'^*B' = \bigcup_{\substack{b \in B \\ b' \in B'}} \{b^*b + b'^*b'\}.$$

The ordering of two subsets of $\mathbb{R}^j$ w.r.t. $\mathbb{R}^j_+$ is defined as

$$B \geq B' \Longleftrightarrow b \geq b', \text{ for all } b \in B \text{ and } b' \in B'.$$

The most commonly used minimality notions w.r.t. a solid as well as pointed convex cone D in a *RNS* S are as follows.

Definition 2.1. *Let $\emptyset \neq B \subseteq S$. The weakly minimal, minimal, and ideal minimal points of B are defined as*

(i) *$b' \in B$ is said to be a weakly minimal point of B if there exists no $b \in B$, satisfying $b < b'$.*
(ii) *$b' \in B$ is said to be a minimal point of B if there exists no $b \in B \setminus \{b'\}$, satisfying $b \leq b'$.*
(iii) *$b' \in B$ is said to be an ideal minimal point of B if $b' \leq b$, for all $b \in B$.*

The sets of weakly minimal, minimal, and ideal minimal points of B as denoted by w-min(B), min(B), and I-min(B), respectively and are being described as

$$\text{w-min}(B) = \{b' \in B : (b' - \text{int}(D)) \cap B = \emptyset\},$$

$$\min(B) = \{b' \in B : (b' - D) \cap B = \{b'\}\},$$

and

$$\text{I-min}(B) = \{b' \in B : B \subseteq \{b'\} + D\}.$$

Let $b = (b_1, ..., b_j) \in \mathbb{R}^j$. The logarithm and exponential of b are defined w.r.t. $\mathbb{R}^j_+$ based on componentwise as

$$\log b = (\log b_1, ..., \log b_j)^T, \text{ for } b > 0_{\mathbb{R}^j}$$

and

$$e^b = (e^{b_1}, ..., e^{b_j})^T, \text{ for any } b.$$

Let $\emptyset \neq B \subseteq \mathbb{R}^j$. Additionally, let us define two sets $\log B$ and e^B w.r.t. $\mathbb{R}^j_+$ as

$$\log B = \{\log b : b \in B\}, \text{ for } B > 0_{\mathbb{R}^j}$$

and

$$e^B = \{e^b : b \in B\}, \text{ for any } B.$$

Likewise, we may define the element $b^{\frac{1}{\alpha}}$ and the set $B^{\frac{1}{\alpha}}$ for any nonzero real number α.

Aubin and Frankowska [3, 4] introduced the notion of higher-order contingent set in set-valued optimization theory.

Definition 2.2. ([3, 4]). *Suppose that S is a RNS and $\emptyset \neq B \subseteq S$, $b' \in \overline{B}$, $k \in \mathbb{N}$, in addition to $k \geq 2$, and $b_1, ..., b_{k-1} \in S$. Then the contingent set of k-th order $T^{(k)}(B, b', b_1, ..., b_{k-1})$ to B at $(b', b_1, ..., b_{k-1})$ is defined as*

$b \in T^{(k)}(B, b', b_1, ..., b_{k-1})$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$ and $\{b_n\}$ in S, in addition to $\lambda_n \to 0^+$ and $b_n \to b$, so that

$$b' + \lambda_n b_1 + ... + {\lambda_n}^{k-1} b_{k-1} + {\lambda_n}^k b_n \in B, \forall n \in \mathbb{N},$$

equivalently, $b \in T^{(k)}(B, b', b_1, ..., b_{k-1})$ if there exist sequences $\{\lambda_n\}$ in $\mathbb{R}$ and $\{b'_n\}$ in S, in addition to $\lambda_n \to 0^+$, $b'_n \in B$, and $b'_n \to b'$, so that

$$\frac{b'_n - b' - \lambda_n b_1 - ... - {\lambda_n}^{k-1} b_{k-1}}{{\lambda_n}^k} \to b, \text{ as } n \to \infty.$$

Let R, S be *RNSs*, 2^S be the set of all subsets of S, and D be a solid as well as pointed convex cone in S. Let $F : R \to 2^S$ be a *SVM* from R to S, i.e., $F(a) \subseteq S$, $\forall a \in R$. The domain, image, graph, and epigraph of F are defined by

$$\text{Dom}(F) = \{a \in R : F(a) \neq \emptyset\},$$

$$F(A) = \bigcup_{a \in A} \{F(a)\}, \text{ for any } \emptyset \neq A \subseteq R,$$

$$\text{Gr}(F) = \{(a, b) \in R \times S : b \in F(a)\},$$

and

$$\text{Epi}(F) = \{(a, b) \in R \times S : b \in F(a) + D\}.$$

The concept of higher-order contingent derivative of *SVMs* was developed by Aubin and Frankowska [4].

Definition 2.3. ([4]). *Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, R, S are RNSs, $F : R \to 2^S$ is a SVM along with* $\mathrm{Dom}(F) = R$, *$(a', b') \in \mathrm{Gr}(F)$, and $(a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$. Then the contingent derivative of k-th order $d^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})$ of F at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ is the SVM from R to S defined by*

$$\mathrm{Gr}(d^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})) \\ = T^{(k)}(\mathrm{Gr}(F), (a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1})).$$

Let R, S be *RNSs*, D be a solid as well as pointed convex cone in S, and $F : R \to 2^S$ be a *SVM*. Let us consider a *SVM* $F + D : R \to 2^S$ by

$$(F + D)(a) = F(a) + D, \forall a \in \mathrm{Dom}(F).$$

The concept of higher-order generalized contingent epi-derivative of *SVMs* was developed by Li and Chen [21].

Definition 2.4. ([21]). *Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, R, S are RNSs, $F : R \to 2^S$ is a SVM along with* $\mathrm{Dom}(F) = R$, *$(a', b') \in \mathrm{Gr}(F)$, and $(a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$. Then the generalized contingent epi-derivative of k-th order of F at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$, denoted by $\overrightarrow{d}_g^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})$, is the SVM from R to S defined by*

$$\overrightarrow{d}_g^{(k)}F(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})(a) \\ = \min\{b \in B : (a, b) \in T^{(k)}(\mathrm{Epi}(F), (a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1}))\}, \\ a \in \mathrm{Dom}(d^{(k)}(F + D)(a', b', a_1, b_1, ..., a_{k-1}, b_{k-1})).$$

The concept of cone convexity of *SVMs* was developed by Borwein [7].

Definition 2.5. ([7]). *Let A be a nonempty convex subset of a RNS R. A SVM $F : R \to 2^S$, with $A \subseteq \mathrm{Dom}(F)$, is said to be D-convex on A if $\forall a_1, a_2 \in A$ and $\lambda \in [0, 1]$,*

$$\lambda F(a_1) + (1 - \lambda)F(a_2) \subseteq F(\lambda a_1 + (1 - \lambda)a_2) + D.$$

The following result was developed by Li *et al.* [23] in terms of the higher-order contingent derivative of *SVMs*.

Proposition 2.1. ([23]). *Suppose that R, S are RNSs and F is D-convex on a nonempty convex subset A of R, then $\forall a, a' \in A$ and $\forall b' \in F(a')$,*

$$F(a) - b' \subseteq d^{(k)}F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')(a - a'),$$

where $a_1, ..., a_{k-1} \in A$, $b_1 \in F(a_1) + D, ..., b_{k-1} \in F(a_{k-1}) + D$.

Das and Nahak [15] developed the notion of higher-order δ-cone convexity of *SVMs*.

Suppose that $F : R \to 2^S$ is a *SVM*, in addition to $A \subseteq \mathrm{Dom}(F)$. Suppose that $k \in \mathbb{N}$, in addition to $k \geq 2$, $(a', b'), (a_1, b_1), ..., (a_{k-1}, b_{k-1}) \in R \times S$ in addition to $a', a_1, ..., a_{k-1} \in A$, $b' \in F(a')$, $b_1 \in F(a_1) + D, ..., b_{k-1} \in F(a_{k-1}) + D$.

Definition 2.6. ([15]). *Suppose that R, S are RNSs, $\emptyset \neq A \subseteq R$, D is a pointed solid convex cone of S, $\delta \in \mathbb{R}$, $e \in \mathrm{int}(D)$, and $F : R \to 2^S$ is a SVM, in addition to $A \subseteq \mathrm{Dom}(F)$. Suppose that F is generalized contingent epiderivable of k-th order at (a', b') for $(a_1 - a', b_1 - b'), ..., (a_{k-1} - a', b_{k-1} - b')$. Then F is said to be δ-D-convex of k-th order w.r.t. e at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ on A if*

$$F(a) - b' \subseteq \overrightarrow{d}_g^{(k)}F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\ (a - a') + \delta\|a - a'\|^2 e + D, \forall a \in A.$$

Das [10, 11] developed the notion of δ-cone convexity of *SVMs*.

Definition 2.7. ([10, 11]). *Let A be a convex subset of a RNS R, $e \in \mathrm{int}(D)$, and $F : R \to 2^S$ be a SVM, with $A \subseteq \mathrm{Dom}(F)$. Then F is said to be δ-D-convex w.r.t. e on A if there exists $\delta \in \mathbb{R}$, satisfying*

$$(1 - \lambda)F(a_1) + \lambda F(a_2) \subseteq F((1 - \lambda)a_1 + \lambda a_2) + \delta\lambda(1 - \lambda)\|a_1 - a_2\|^2 e + D, \\ \forall a_1, a_2 \in A \text{ and } \forall \lambda \in [0, 1].$$

3 Higher-order (α, β)-δ-cone convexity

Das and Nahak [14, 15, 16, 17] and Das [10, 12] introduced the notion of δ-cone convexity of *SVMs*. They established sufficient *KKT* optimality conditions and used the contingent epiderivative and δ-cone convexity assumptions to formulate the duality results for various types of *SVOPs*. We get the standard conception of cone convex *SVMs* for $\delta = 0$, developed by Borwein [7].

Let A be a convex subset of $\mathbb{R}^i$, $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ be a *SVM* with $A \subseteq \mathrm{Dom}(F)$, and $(a', b') \in \mathrm{Gr}(F)$. In the paper, we make the assumption that $\mathbf{0} = (0, ..., 0) \in \mathbb{R}^i$, $\mathbf{0}' = (0, ..., 0) \in \mathbb{R}^j$, $\mathbf{0}'' = (0, ..., 0) \in \mathbb{R}^k$, $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^i$, $\mathbf{1}' = (1, ..., 1) \in \mathbb{R}^j$, and $\mathbf{1}'' = (1, ..., 1) \in \mathbb{R}^k$.

We introduce the notion of higher-order (α, β)-δ-cone convex *SVMs*. For $\alpha = 0$ and $\beta = 0$, we have the usual notion of δ-cone convex *SVMs*.

Definition 3.1. *Let $k \in \mathbb{N}$, in addition to $k \geq 2$, A be a convex subset of $\mathbb{R}^i$, $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ be a SVM with $A \subseteq \mathrm{Dom}(F)$, $a, a' \in A$, and $(a', b') \in Gr(F)$. Let F be generalized contingent epiderivable of k-th order at (a', b') for $(a_1 - a', b_1 - b'), ..., (a_{k-1} - a', b_{k-1} - b')$ and $\alpha, \beta \in \mathbb{R}$. Then F is said to be (α, β)-δ-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ on A if there exists $\delta \in \mathbb{R}$ with*

$$
\begin{aligned}
&(e^{\alpha(a-a')} - \mathbf{1})/\alpha \\
&\in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')), \\
&\qquad for\ \alpha \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&a - a' \in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')), \\
&\qquad for\ \alpha = 0,
\end{aligned}
$$

satisfying ,

$$
\begin{aligned}
(e^{\beta(F(a)-b')} - \mathbf{1}')/\beta &\subseteq \overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha \neq 0, \beta \neq 0), \\
F(a) - b' &\subseteq \overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, \\
&\quad (\alpha \neq 0, \beta = 0), \\
(e^{\beta(F(a)-b')} - \mathbf{1}')/\beta &\subseteq \overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad (a - a') + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha = 0, \beta \neq 0), \\
F(a) - b' &\subseteq \overrightarrow{d}_g^{(k)} F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b') \\
&\quad (a - a') + \delta \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+, (\alpha = 0, \beta = 0).
\end{aligned}
\tag{3.1}
$$

A higher-order (α, β)-δ-cone convex *SVM* example that is not cone convex is developed.

Example 3.1. Let us consider a subset A of $\mathbb{R}^2$ by

$$
A = \left\{ a = (a_1, a_2) \mid a_1 + a_2 \geq \frac{1}{7},\ a_1 \geq 0,\ a_2 \geq 0 \right\}.
$$

Here A is a convex subset of $\mathbb{R}^2$. Define a SVM $F : \mathbb{R}^2 \to 2^{\mathbb{R}}$ as follows:

$$
F(a) = \begin{cases} [0, 5], & \text{if } a_1 + a_2 \geq \frac{1}{7},\ a_1 \neq 5a_2, a = (a_1, a_2), \\ [6, 8], & \text{otherwise.} \end{cases}
$$

Here,

$$
\begin{aligned}
\mathrm{Epi}(F) = &\left\{ ((a_1, a_2), b) : a_1 + a_2 \geq \frac{1}{7},\ a_1 \neq 5a_2, a = (a_1, a_2), b \geq 0 \right\} \\
&\cup \{((a_1, a_2), b) : (a_1, a_2) \in \mathbb{R}^2, b \geq 6\}.
\end{aligned}
$$

Let $a' = (5, 0)$ and $b' = 0$. So, $b' \in F(a')$. Let $k \in \mathbb{N}$, in addition to $k \geq 2$, $a'_1 = (1, 0), ..., a'_{k-1} = (k - 1, 0)$ and $b'_1 = ... = b'_{k-1} = 1$. As a result,

$$
\begin{aligned}
&T^{(k)}(\mathrm{Epi}(F), ((5, 0), 0), (a'_1 - a', b'_1 - b'), ..., (a'_{k-1} - a', b'_{k-1} - b')) \\
&= \left\{ ((a_1, a_2), b) : a_1 + a_2 + \frac{1}{7} \geq 0,\ a_1 \neq 5a_2, a = (a_1, a_2), b \geq 0 \right\} \\
&\qquad \cup \{((a_1, a_2), b) : (a_1, a_2) \in \mathbb{R}^2, b \geq 6\}.
\end{aligned}
$$

Hence,

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a_1,a_2)$$
$$= \text{I-min}\{b : ((a_1,a_2),b)$$
$$\in T^{(k)}(\text{Epi}(F),((5,0),0),(a_1'-a',b_1'-b'),...,(a_{k-1}'-a',b_{k-1}'-b'))\}.$$

Let $a=(0,1)$. Hence,

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$= \overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(5,-1) = 6.$$

Let $\lambda = \frac{1}{2}$. Then,

$$\frac{1}{2}a + \frac{1}{2}a' = \frac{1}{2}(0,1) + \frac{1}{2}(5,0) = \Big(\frac{5}{2},\frac{1}{2}\Big),$$

and

$$\frac{1}{2}F(0,1) + \frac{1}{2}F(5,0) = \frac{1}{2}[0,5] + \frac{1}{2}[0,5] = [0,5] \nsubseteq [6,8] + \mathbb{R}_+ = F\Big(\frac{5}{2},\frac{1}{2}\Big) + \mathbb{R}_+.$$

Hence, it is clear that F is not $\mathbb{R}_+$-convex w.r.t. 1. But on the other side, by choosing $\alpha = 0$, $\beta = 1$, and $\delta = -2$, we get that

$$(e^{\beta(F(a)-b')} - 1)/\beta = e^{F(a)-b'} - 1 = e^{[0,5]} - 1.$$

and

$$\overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$+\delta\|a-a'\|^2 1 = 6 - 52 = -46.$$

Hence,

$$(e^{\beta(F(a)-b')} - 1)/\beta \subseteq \overrightarrow{d}_g^{(k)} F((5,0),0,a_1'-a',b_1'-b',...,a_{k-1}'-a',b_{k-1}'-b')(a'-a)$$
$$+\delta\|a-a'\|^2 1 + \mathbb{R}_+ = -46 + \mathbb{R}_+.$$

As a consequence, F is $(5,0)$-(-2)-$\mathbb{R}_+$-convex of k-th order w.r.t. 1 at $((5,0),0)$ for the elements $(a_1',b_1'),...,(a_{k-1}',b_{k-1}')$ on A.

4 Higher-order optimality conditions

Let A be a nonempty subset of $\mathbb{R}^i$. Suppose that $F : \mathbb{R}^i \to 2^{\mathbb{R}^j}$ and $G : \mathbb{R}^i \to 2^{\mathbb{R}^k}$ are two *SVMs*, with

$$A \subseteq \text{Dom}(F) \cap \text{Dom}(G).$$

We consider a *SVOP* (P) as follows

$$\begin{aligned} &\underset{a\in A}{\text{minimize}} && F(a), \\ &\text{subject to} && G(a) \cap (-\mathbb{R}^k_+) \neq \emptyset. \end{aligned} \tag{P}$$

The feasible set of (P) is provided by

$$X = \{a \in A : G(a) \cap (-\mathbb{R}^k_+) \neq \emptyset\}.$$

Definition 4.1. *A point $(a_0,b_0) \in \mathbb{R}^i \times \mathbb{R}^j$, along with $a_0 \in X$ and $b_0 \in F(a_0)$, is said to be a feasible point of (P).*

Definition 4.2. *A point $(a_0,b_0) \in \mathbb{R}^i \times \mathbb{R}^j$, with $a_0 \in X$ and $b_0 \in F(a_0)$, is said to be a minimizer of the problem (P) if for all $(a,b) \in \mathbb{R}^i \times \mathbb{R}^j$, with $a \in X$ and $b \in F(a)$,*

$$b - b_0 \notin (-\mathbb{R}^j_+) \setminus \{0_{\mathbb{R}^j}\}.$$

Definition 4.3. *A point $(a_0,b_0) \in \mathbb{R}^i \times \mathbb{R}^j$, with $a_0 \in X$ and $b_0 \in F(a_0)$, is said to be a weak minimizer of the problem (P) if for all $(a,b) \in \mathbb{R}^i \times \mathbb{R}^j$, with $a \in X$ and $b \in F(a)$,*

$$b - b_0 \notin (-\text{int}(\mathbb{R}^j_+)).$$

Suppose that $F:\mathbb{R}^i\to 2^{\mathbb{R}^j}$ and $G:\mathbb{R}^i\to 2^{\mathbb{R}^k}$ are two *SVMs*, with

$$A\subseteq \mathrm{Dom}(F)\cap\mathrm{Dom}(G).$$

Suppose that $(a_0,b_0),(a_1,b_1),...,(a_{k-1},b_{k-1})\in\mathbb{R}^i\times\mathbb{R}^j$ with $a_0,a_1,...,a_{k-1}\in A$, $b_0\in F(a_0)$, $b_1\in F(a_1)+\mathbb{R}^j_+,...,b_{k-1}\in F(a_{k-1})+\mathbb{R}^j_+$.

Assume that $(a_0,c_0),(a_1,c_1),...,(a_{k-1},c_{k-1})\in\mathbb{R}^i\times\mathbb{R}^k$ in addition to $c_0\in G(a_0)$, $c_1\in G(a_1)+\mathbb{R}^k_+,...,c_{k-1}\in G(a_{k-1})+\mathbb{R}^k_+$.

Das and Nahak [16] established that the Fritz John optimality conditions are sufficient under the (α,β)-δ-(η,θ)-invexity assumption. The following theorem employs higher-order (α,β)-δ-cone convexity assumption to establish the higher-order sufficient KKT optimality conditions of the problem (P).

Theorem 4.1. ***(Higher-order sufficient optimality conditions)*** *Let A be a convex subset of $\mathbb{R}^i$, (a_0,b_0) be a feasible point of the problem (P), and $c_0\in G(a_0)\cap(-\mathbb{R}^k_+)$. Suppose that F is (α,β)-δ_1-cone convex of k-th order w.r.t. $\boldsymbol{1}'$ at (a_0,b_0) for $(a_1,b_1),...,(a_{k-1},b_{k-1})$ and G is (α,β)-δ_2-cone convex of k-th order w.r.t. $\boldsymbol{1}''$ at (a_0,c_0) for $(a_1,c_1),...,(a_{k-1},c_{k-1})$ on A and*

$$\delta_1(b^*\boldsymbol{1}')+\delta_2(c^*\boldsymbol{1}'')\geq 0.$$

Suppose that there exists $(b^*,c^*)\in\mathbb{R}^j_+\times\mathbb{R}^k_+$, with $b^*\neq 0_{\mathbb{R}^j}$, satisfying

$$\begin{aligned}&b^*\overrightarrow{d}^{(k)}_gF(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)((e^{\alpha(a_0-a')}-\boldsymbol{1})/\alpha)\\&+c^*\overrightarrow{d}^{(k)}_gG(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)((e^{\alpha(a_0-a')}-\boldsymbol{1})/\alpha)\\&\geq 0,\forall a\in X,(for\ \alpha\neq 0),\\&b^*\overrightarrow{d}^{(k)}_gF(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)(a_0-a')\\&+c^*\overrightarrow{d}^{(k)}_gG(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)(a_0-a')\\&\geq 0,\forall a\in X,(for\ \alpha=0),\end{aligned}\tag{4.1}$$

and

$$c^*c_0=0.\tag{4.2}$$

Then (a_0,b_0) is a weak minimizer of the problem (P).

Proof. By using the contradiction approach, we prove the theorem for the instance when $\alpha\neq 0$. For $\alpha=0$, we can prove likewise. Suppose that (a_0,b_0) is not a weak minimizer of the problem (P). Then,

$$(b_0-\mathrm{int}(\mathbb{R}^j_+))\cap F(X)\neq\emptyset.$$

As a result, there exist $a\in X$, $b\in F(a)$ satisfying

$$b<b_0.$$

Hence,

$$e^b<e^{b_0}\Rightarrow\frac{1}{\beta}e^{\beta b}<\frac{1}{\beta}e^{\beta b-0}\Rightarrow(e^{\beta(b-b_0)}-\mathbf{1}')/\beta<\mathbf{0}.$$

As $b^*\neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b-b_0)}-\mathbf{1}')/\beta<0.$$

As $a\in X$, there exists an element $c\in G(a)\cap(-\mathbb{R}^k_+)$. Let $c^*=(c^*_1,...,c^*_k)$, $c=(c_1,...,c_k)$, and $c_0=({c_1}',...,{c_k}')$. As $c\in-\mathbb{R}^k_+$, we have

$$c\leq\mathbf{0}''\Rightarrow e^c\leq\mathbf{1}''\Rightarrow\frac{1}{\beta}e^{\beta c}\leq\frac{1}{\beta}\mathbf{1}''\Rightarrow(e^{\beta c}-\mathbf{1}'')/\beta\leq 0.$$

Now, $c^*c_0=0$, $c_0\in-\mathbb{R}^k_+$, and $c^*\in\mathbb{R}^k_+$. As a result, if ${c_n}'<0$ for some n, where $1\leq n\leq k$, then ${c_n}^*=0$. So,

$${c_n}^*(e^{\beta(c_n-{c_n}')}-1)/\beta=0.$$

Again, if ${c_n}'=0$ for some n, where $1\leq n\leq k$, then

$${c_n}^*(e^{\beta(c_n-{c_n}')}-1)/\beta={c_n}^*(e^{\beta c_n}-1)/\beta\leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c-c_0)}-\mathbf{1}'')/\beta\leq 0.$$

Hence, we have

$$b^*(e^{\beta(b-b_0)} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta < 0. \tag{4.3}$$

As F is (α,β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a_0,b_0) for (a_1,b_1),...,(a_{k-1},b_{k-1}) and G is (α,β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a_0,c_0) for (a_1,c_1),...,(a_{k-1},c_{k-1}) on A, we have

$$\begin{aligned}(e^{\beta(b-b_0)} - \mathbf{1}')/\beta \in \overrightarrow{d}_g^{(k)} F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)&\\ ((e^{\alpha(a_0-a')} - \mathbf{1})/\alpha) + \delta_1\|a-a_0\|^2\mathbf{1}' + \mathbb{R}^j_+&\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta \in \overrightarrow{d}_g^{(k)} G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)&\\ ((e^{\alpha(a_0-a')} - \mathbf{1})/\alpha) + \delta_2\|a-a_0\|^2\mathbf{1}'' + \mathbb{R}^k_+.&\end{aligned} \tag{4.5}$$

As a result, from (4.1), (4.4), (4.5), and the condition

$$\delta_1(b^*\mathbf{1}') + \delta_2(c^*\mathbf{1}'') \geq 0,$$

we have

$$b^*(e^{\beta(b-b_0)} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c_0)} - \mathbf{1}'')/\beta \geq 0.$$

It contradicts (4.3). Hence, (a_0,b_0) is a weak minimizer of the problem (P). □

5 Higher-order Mond-Weir type duality

We prove the results of higher-order duality under higher-order (α,β)-δ-cone convexity assumptions. Let $a' \in \mathbb{R}^i$, $b' \in F(a')$, and $c' \in G(a')$.

Suppose that $(a',b'),(a_1,b_1),...,(a_{k-1},b_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^j$ with $a',a_1,...,a_{k-1} \in A$, $b' \in F(a')$, $b_1 \in F(a_1) + \mathbb{R}^j_+,...,b_{k-1} \in F(a_{k-1}) + \mathbb{R}^j_+$.

Also, assume that $(a',c'),(a_1,c_1),...,(a_{k-1},c_{k-1}) \in \mathbb{R}^i \times \mathbb{R}^k$ in addition to $c' \in G(a')$, $c_1 \in G(a_1) + \mathbb{R}^k_+,...,c_{k-1} \in G(a_{k-1}) + \mathbb{R}^k_+$.

We suppose that F is generalized contingent epiderivable of k-th order at (a',b') for $(a_1-a',b_1-b'),...,(a_{k-1}-a',b_{k-1}-b')$ and G is generalized contingent epiderivable of k-th order at (a',c') for $(a_1-a',c_1-c'),...,(a_{k-1}-a',c_{k-1}-c')$ with,

$$\begin{aligned}&((e^{\alpha(a-a')} - \mathbf{1})/\alpha)\\ &\in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')))\\ &\cap \mathrm{Dom}(\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')),\\ &for\ \alpha \neq 0, a \in \mathbb{R}^i\end{aligned}$$

and

$$\begin{aligned}a'-a \in \mathrm{Dom}(\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b'))&\\ \cap \mathrm{Dom}(\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')),&\\ for\ \alpha = 0, a \in \mathbb{R}^i.&\end{aligned}$$

A higher-order dual problem of the Mond-Weir type, (MWD), is under consideration.

$$\begin{aligned}&\text{maximize } b' \qquad\qquad (MWD)\\ &\text{subject to } b^*\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')\\ &\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha)\\ &\quad + c^*\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')\\ &\quad ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) \geq 0, \forall a \in X, (for\ \alpha \neq 0),\\ &\quad b^*\overrightarrow{d}_g^{(k)} F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')\\ &\quad (a-a')\\ &\quad + c^*\overrightarrow{d}_g^{(k)} G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')\end{aligned}$$

$$
\begin{aligned}
&(a-a') \geq 0, \forall a \in X, (for\ \alpha = 0),\\
&c^* c' \geq 0,\\
&b^* \mathbf{1}' = 1,\ b^* \in \mathbb{R}^j_+ \setminus \{0_{\mathbb{R}^j}\},\ \text{and } c^* \in \mathbb{R}^k_+.
\end{aligned}
$$

Let us define the feasible set W of the problem (MWD) by

$$W = \{b' : (a', b', c', b^*, c^*) \text{ is a feasible point of } (MWD)\}.$$

Definition 5.1. *A feasible point (a', b', c', b^*, c^*) of the problem (MWD) is said to be a weak maximizer of (MWD), if*

$$(b' + \operatorname{int}(\mathbb{R}^j_+)) \cap W = \emptyset.$$

In the following theorems, we formulate higher-order duals of the Mond-Weir (MWD) type and prove the higher-order strong, weak and converse duality theorems using higher-order (α, β)-δ-cone convexity assumption.

Theorem 5.1. ***(Higher-order weak duality)*** *Let A be a convex subset of $\mathbb{R}^i$. Let (a_0, b_0) and (a', b', c', b^*, c^*) be feasible points for the problems (P) and (MWD), respectively, with $c' \in G(a') \cap (-\mathbb{R}^k_+)$. Suppose that F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A and*

$$\delta_1 + \delta_2(c^* \mathbf{1}'') \geq 0. \tag{5.1}$$

Then,

$$b_0 \not< b'.$$

Proof. Using the contradiction approach, we demonstrate the theorem. Suppose that

$$b_0 < b'.$$

Hence,

$$e^{b_0} < e^{b'} \Rightarrow \frac{1}{\beta} e^{\beta b_0} < \frac{1}{\beta} e^{\beta b'} \Rightarrow (e^{\beta(b_0 - b')} - \mathbf{1}')/\beta < \mathbf{0}.$$

As $b^* \neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b_0 - b')} - \mathbf{1}')/\beta < 0.$$

As $a_0 \in X$, there exists an element

$$c_0 \in G(a_0) \cap (-\mathbb{R}^k_+).$$

Let $c^* = (c_1^*, ..., c_k^*)$, $c_0 = (c_1, ..., c_k)$, and $c' = (c_1{}', ..., c_k{}')$. As $c_0 \in -\mathbb{R}^k_+$, we have

$$c_0 \leq \mathbf{0}'' \Rightarrow e^{c_0} \leq \mathbf{1}'' \Rightarrow \frac{1}{\beta} e^{\beta c_0} \leq \frac{1}{\beta} \mathbf{1}'' \Rightarrow (e^{\beta c_0} - \mathbf{1}'')/\beta \leq 0.$$

As $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$, we have

$$c^* c' \leq 0.$$

From duality constraints,

$$c^* c' \geq 0.$$

As a result,

$$c^* c' = 0.$$

Now, $c^* c' = 0$, $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$. As a consequence, if $c_n{}' < 0$ for some n, where $1 \leq n \leq k$, then $c_n{}^* = 0$. So,

$$c_n{}^*(e^{\beta(c_n - c_n{}')} - 1)/\beta = 0.$$

Again, if $c_n{}' = 0$ for some n, where $1 \leq n \leq k$, then

$$c_n{}^*(e^{\beta(c_n - c_n{}')} - 1)/\beta = c_n{}^*(e^{\beta c_n} - 1)/\beta \leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c_0 - c')} - \mathbf{1}'')/\beta \leq 0.$$

So, we have

$$b^*(e^{\beta(b_0-b')}-\mathbf{1}')/\beta+c^*(e^{\beta(c_0-c')}-\mathbf{1}'')/\beta<0. \tag{5.2}$$

As F is (α,β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a',b') for $(a_1,b_1),...,(a_{k-1},b_{k-1})$ and G is (α,β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a',c') for $(a_1,c_1),...,(a_{k-1},c_{k-1})$ on A, we have

$$\begin{aligned}(e^{\beta(b-b')}-\mathbf{1}')/\beta\in \overrightarrow{d}_g^{(k)}F(a',b',a_1-a',b_1-b',...,a_{k-1}-a',b_{k-1}-b')\\ ((e^{\alpha(a-a')}-\mathbf{1})/\alpha)+\delta_1\|a-a'\|^2\mathbf{1}'+\mathbb{R}^j_+\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}(e^{\beta(c-c')}-\mathbf{1}'')/\beta\in \overrightarrow{d}_g^{(k)}G(a',c',a_1-a',c_1-c',...,a_{k-1}-a',c_{k-1}-c')\\ ((e^{\alpha(a-a')}-\mathbf{1})/\alpha)+\delta_2\|a-a'\|^2\mathbf{1}''+\mathbb{R}^k_+.\end{aligned} \tag{5.4}$$

Hence, from (5.3), (5.4), and the conditions $b^*\mathbf{1}'=1$ and $\delta_1+\delta_2(c^*\mathbf{1}'')\geq 0$, we have

$$b^*(e^{\beta(b_0-b')}-\mathbf{1}')/\beta+c^*(e^{\beta(c_0-c')}-\mathbf{1}'')/\beta\geq 0.$$

It contradicts (5.2). Subsequently,

$$b_0\not< b'.$$

□

Theorem 5.2 (**Higher-order strong duality).** *Let A be a convex subset of $\mathbb{R}^i$ and (a_0,b_0) be a weak minimizer of (P). Suppose that for some $(b^*,c^*)\in\mathbb{R}^j_+\times\mathbb{R}^k_+$, with $b^*\boldsymbol{1}'=1$, Eqs.* (4.1) *and* (4.2) *are satisfied for some element $c_0\in G(a_0)\cap(-\mathbb{R}^k_+)$. Then (a_0,b_0,c_0,b^*,c^*) is a feasible solution for (MWD). At this point if Theorem* 5.1 *between (P) and (MWD) is satisfied, then (a_0,b_0,c_0,b^*,c^*) is a weak maximizer of (MWD).*

Proof. As the Eqs. (4.1) and (4.2) are satisfied, we have

$$\begin{aligned}&b^*\overrightarrow{d}_g^{(k)}F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)\\&((e^{\alpha(a_0-a')}-\mathbf{1})/\alpha)\\&+c^*\overrightarrow{d}_g^{(k)}G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)\\&((e^{\alpha(a_0-a')}-\mathbf{1})/\alpha)\geq 0,\forall a\in X,(for\ \alpha\neq 0),\\&b^*\overrightarrow{d}_g^{(k)}F(a_0,b_0,a_1-a_0,b_1-b_0,...,a_{k-1}-a_0,b_{k-1}-b_0)\\&(a_0-a')\\&+c^*\overrightarrow{d}_g^{(k)}G(a_0,c_0,a_1-a_0,c_1-c_0,...,a_{k-1}-a_0,c_{k-1}-c_0)\\&(a_0-a')\geq 0,\forall a\in X,(for\ \alpha=0),\end{aligned}$$

and

$$c^*c_0=0.$$

Hence, (a_0,b_0,c_0,b^*,c^*) is a feasible solution for (MWD). Next, we show that

$$(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W=\emptyset.$$

Let

$$b'\in(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W.$$

As a result,

$$b'-b_0\in\mathrm{int}(\mathbb{R}^j_+)\Rightarrow b_0<b'.$$

It contradicts Theorem 5.1 between (P) and (MWD). As a result,

$$(b_0+\mathrm{int}(\mathbb{R}^j_+))\cap W=\emptyset.$$

Hence, (a_0,b_0,c_0,b^*,c^*) is a weak maximizer for (MWD). □

Theorem 5.3 (Higher-order converse duality). *Let A be a convex subset of $\mathbb{R}^i$ and (a', b', c', b^*, c^*) be a weak maximizer of the problem (MWD) with*

$$c' \in G(a') \cap (-\mathbb{R}^k_+).$$

Suppose that F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A satisfying (5.1). Then (a', b') is a weak minimizer of (P).

Proof. Here (a', b') is a feasible solution of the problem (P). Let (a', b') be not a weak minimizer of the problem (P). Then,

$$(b' - \operatorname{int}(\mathbb{R}^j_+)) \cap F(X) \neq \emptyset.$$

So, there exist $a \in X$ and $b \in F(a)$, satisfying

$$b' - b \in \operatorname{int}(\mathbb{R}^j_+) \text{ i.e., } b < b'.$$

Hence,

$$e^b < e^{b'} \Rightarrow \frac{1}{\beta} e^{\beta b} < \frac{1}{\beta} e^{\beta b'} \Rightarrow (e^{\beta(b-b')} - \mathbf{1}')/\beta < \mathbf{0}'.$$

As $b^* \neq 0_{\mathbb{R}^j}$, we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta < 0.$$

As $a \in X$, there exists an element

$$c \in G(a) \cap (-\mathbb{R}^k_+).$$

Let $c^* = (c_1^*, ..., c_k^*)$, $c = (c_1, ..., c_k)$, and $c' = ({c_1}', ..., {c_k}')$. As $c \in -\mathbb{R}^k_+$, we have

$$c \leq \mathbf{0}'' \Rightarrow e^c \leq \mathbf{1}'' \Rightarrow \frac{1}{\beta} e^{\beta c} \leq \frac{1}{\beta} \mathbf{1}'' \Rightarrow (e^{\beta c} - \mathbf{1}'')/\beta \leq 0.$$

As $c' \in -\mathbb{R}^k_+$ and $c^* \in \mathbb{R}^k_+$, we have

$$c^* c' \leq 0.$$

Further, from duality constraints, we have

$$c^* c' \geq 0.$$

As a result,

$$c^* c' = 0.$$

Now, $c^* c' = 0$, $c' \in -\mathbb{R}^k_+$, and $c^* \in \mathbb{R}^k_+$. As a result, if ${c_n}' < 0$ for some n, where $1 \leq n \leq k$, then ${c_n}^* = 0$. So,

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = 0.$$

Again, if ${c_n}' = 0$ for some n, where $1 \leq n \leq k$, then

$${c_n}^*(e^{\beta(c_n - {c_n}')} - 1)/\beta = {c_n}^*(e^{\beta c_n} - 1)/\beta \leq 0.$$

Combining the two instances, we have

$$c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta \leq 0.$$

As a result, we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta < 0. \tag{5.5}$$

As F is (α, β)-δ_1-cone convex of k-th order w.r.t. $\mathbf{1}'$ at (a', b') for $(a_1, b_1), ..., (a_{k-1}, b_{k-1})$ and G is (α, β)-δ_2-cone convex of k-th order w.r.t. $\mathbf{1}''$ at (a', c') for $(a_1, c_1), ..., (a_{k-1}, c_{k-1})$ on A, we have

$$\begin{aligned}(e^{\beta(b-b')} - \mathbf{1}')/\beta \in \overrightarrow{d}^{(k)}_g F(a', b', a_1 - a', b_1 - b', ..., a_{k-1} - a', b_{k-1} - b')& \\ ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta_1 \|a - a'\|^2 \mathbf{1}' + \mathbb{R}^j_+&\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}(e^{\beta(c-c')} - \mathbf{1}'')/\beta \in \overrightarrow{d}^{(k)}_g G(a', c', a_1 - a', c_1 - c', ..., a_{k-1} - a', c_{k-1} - c')& \\ ((e^{\alpha(a-a')} - \mathbf{1})/\alpha) + \delta_2 \|a - a'\|^2 \mathbf{1}'' + \mathbb{R}^k_+.&\end{aligned} \tag{5.7}$$

Hence, from (5.6), (5.7), and the conditions $b^* \mathbf{1}' = 1$ and

$$\delta_1 + \delta_2(c^* \mathbf{1}'') \geq 0,$$

we have

$$b^*(e^{\beta(b-b')} - \mathbf{1}')/\beta + c^*(e^{\beta(c-c')} - \mathbf{1}'')/\beta \geq 0.$$

It contradicts (5.5). Hence, (a', b') is a weak minimizer of (P). □

6 Conclusion

In this study, we introduce the notion of higher-order (α, β)-δ-cone convexity in set-valued optimization problems. We establish the sufficient optimality conditions for $SVOP$ (P) under the assumption of higher-order (α, β)-δ-cone convexity. With regard to the problem (P), we formulate the higher-order duals of the Mond-Weir (MWD) type. Under the assumed higher-order (α, β)-δ-cone convexity, we additionally establish the higher-order duality results of Mond-Weir type. In order to prove that the notion of higher-order (α, β)-δ-cone convexity is more generalized than higher-order cone convexity, an example is developed.

Future research may explore several promising directions, including the incorporation of fractional-order convexity to further generalize the model, the study of fuzzy set-valued optimization to address uncertainty, and stability analysis of solutions under perturbations. These extensions may enhance both the theoretical depth and practical applicability of the present work.

Acknowledgement. The author is very thankful to the editor and referee for their valuable comments which improved the presentation of the paper.

References

[1] T. Antczak, (p, r)-invex sets and functions, *J. Math. Anal. Appl.*, **263**(2) (2001), 355–379.
[2] T. Antczak, (p, r)-invexity in multiobjective programming, *Eur. J. Oper. Res.*, **152**(1) (2004), 72–87.
[3] J. P. Aubin, Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, in*: Mathematical Analysis and Applications, Part A, L. Nachbin (ed.)*, Academic Press, New York, 1981, pp. 160–229.
[4] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birhäuser, Boston, 1990.
[5] A. Ben-Israel and B. Mond, What is invexity? *J. Aust. Math. Soc. Ser. B*, **28**(1) (1986), 1–9.
[6] D. Bhatia and A. Mehra, Lagrangian duality for preinvex set-valued functions, *J. Math. Anal. Appl.*, **214**(2) (1997), 599–612.
[7] J. Borwein, Multivalued convexity and optimization: a unified approach to inequality and equality constraints, *Math. Program.*, **13**(1) (1977), 183–199.
[8] H. W. Corley, Optimality conditions for maximizations of set-valued functions, *J. Optim. Theory Appl.*, **58**(1), (1988), 1–10.
[9] B. Craven, *Generalized concavity and duality, Generalized concavity in optimization and economics*, Academic Press, New York, (1981), 473–489.
[10] K. Das, Set-valued optimization problems with (p, r)-ρ-cone arcwise connectedness, *Jñānābha*, **51**(2) (2021), 249–260.
[11] K. Das, On constrained set-valued optimization problems with ρ-cone arcwise connectedness, *Sema J.*, **80**(3) (2023), 463–478.
[12] K. Das, Set-valued minimax fractional programming problems with higher-order ρ-cone arcwise connectedness, *Jñānābha*, **54**(2) (2024), 120–130.
[13] K. Das, Set-valued minimax programming problems under higher-order σ-arcwisely connectivity, *SeMA J.*, (2025), 1–23. DOI: https://doi.org/10.1007/s40324-025-00390-y.
[14] K. Das and C. Nahak, Sufficient optimality conditions and duality theorems for set-valued optimization problem under generalized cone convexity, *Rend. Circ. Mat. Palermo*, **63**(3) (2014), 329–345.
[15] K. Das and C. Nahak, Sufficiency and duality of set-valued optimization problems via higher-order contingent derivative, *J. Adv. Math. Stud.*, **8**(1) (2015), 137–151.
[16] K. Das and C. Nahak, Sufficiency and duality in set-valued optimization problems under (p, r)-ρ-(η, θ)-invexity, *Acta Univ. Apulensis*, **62** (2020), 93–110.
[17] K. Das and C. Nahak, Parametric set-valued optimization problems under generalized cone convexity, *Jñānābha*, **51**(1) (2021), 1–11.
[18] K. Das and C. Nahak, Sufficiency and duality for set-valued optimization problems with κ-cone arcwise connectedness of higher-order, *Opsearch*, **63**(1) (2026), 104–128.
[19] M. Hanson and B. Mond, Self-duality and invexity, *FSU Statistics Report M*, (1986), 716.
[20] J. Jahn and R. Rauh, Contingent epiderivatives and set-valued optimization, *Math. Method Oper. Res.*, **46**(2) (1997), 193–211.
[21] S. J. Li and C. R. Chen, Higher order optimality conditions for Henig efficient solutions in set-valued optimization, *J. Math. Anal. Appl.*, **323**(2) (2006), 1184–1200.
[22] S. J. Li, K. L. Teo and X. Q. Yang, Higher-order Mond–Weir duality for set-valued optimization, *J. Comput. Appl. Math.*, **217**(2) (2008), 339–349.

[23] S. J. Li, K. L. Teo and X. Q. Yang, Higher-order optimality conditions for set-valued optimization, *J. Optim. Theory Appl.*, **137**(3) (2008), 533–553.
[24] L. Lin, Optimization of set-valued functions, *J. Math. Anal. Appl.*, **186**(1) (1994), 30–51.
[25] D. T. Luc and C. Malivert, Invex optimisation problems, *B. Aust. Math. Soc.*, **46**(1) (1992), 47–66.
[26] P. Mandal and C. Nahak, $(p, r) - \rho - (\eta, \theta)$-invexity in multiobjective programming problems, *Int. J. Optim. Theory Methods Appl.*, **2**(4) (2010), 273–282.
[27] L. Rodríguez-Marín and M. Sama, About contingent epiderivatives, *J. Math. Anal. Appl.*, **327**(2) (2007), 745–762.
[28] P. H. Sach and B. D. Craven, Invex multifunctions and duality, *Numer. Func. Anal. Opt.*, **12**(5-6) (1991), 575–591.
[29] P. H. Sach and B. D. Craven, Invexity in multifunction optimization, *Numer. Func. Anal. Opt.*, **12**(3-4) (1991), 383–394.
[30] B. Sheng and S. Liu, Kuhn-Tucker condition and Wolfe duality of preinvex set-valued optimization, *Appl. Math. Mech.-Engl.*, **27** (2006), 1655–1664.
[31] T. Weir and V. Jeyakumar, A class of nonconvex functions and mathematical programming, *Bull. Austral. Math. Soc.*, **38**(02) (1988), 177–189.
[32] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, *J. Math. Anal. Appl.*, **136**(1) (1988), 29–38.
[33] G. J. Zalmai, Generalized sufficiency criteria in continuous-time programming with application to a class of variational-type inequalities, *J. Math. Anal. Appl.*, **153**(2) (1990), 331–355.