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RADII OF ANALYTIC FUNCTIONS SATISFYING SOME COEFFICIENT INEQUALITIES

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Abstract

Let $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is an analytic function defined on an open unit disc in complex plane with center at origin and radius 1. The a_2 coefficient is such that $|a_2| = 2b$, where $b \in [0, 1]$, for other coefficients $|a_n| \leq C/n^3 (C > 0)$ and $|a_n| \leq cn + d/n (c, d > 0)$. Optimum or extreme values of radius are evaluated for functions to have convexity of order $\alpha (0 \leq \alpha < 1)$ and starlikeness of order α, where $\alpha \in [0, 1[$. Radius values for few other properties are also evaluated.
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1 Introduction

Let $\mathcal{A}$ represents the collection of all normalised functions g analytic on the open unit disc $\mathbb{D} = \mathbb{D}(0, 1) = \{|z| : |z| < 1\}$ with center at origin and radius unity in complex plane. The power series representation of any function in $\mathcal{A}$ is of the form $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Thus, a normalised function takes zero value at origin and 1 for $g'(0)$. Let $\mathcal{S} \subseteq \mathcal{A}$, represents class of univalent functions. For $g \in \mathcal{S}$ it is well known through de Branges Theorem [4] that $|a_n| \leq n \ \forall \ n \geq 2$. However, the converse is not true, that is, coefficients of $g \in \mathcal{A}$ satifying the inequality $|a_n| \leq n, (n \geq 2)$ are not neccessarily univalent. The functions [2, 25] $g(z) = z - 2z^2 - 3z^3 - 4z^4 - \cdots = 2z - z/(1 - z)^2$ or $g(z) = z + 2z^2$ are not members of $\mathcal{S}$. This is the motivation for the complex analysts to obtain the radii of functions with predefined coefficient bounds.

Firstly, let us define the subclasses of $\mathcal{S}$ that will be needed in our investigation. For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ represent well the known classes of $\mathcal{S}$ representing classes of starlike and convex functions of order α, respectively. These classes are collection of normalised functions which satisfy the following equivalences: $g \in \mathcal{S}^*(\alpha) \iff \operatorname{Re}(zg'(z)/g(z)) > \alpha$, and $g \in \mathcal{K}(\alpha) \iff \operatorname{Re}(1 + zg''(z)/g'(z)) > \alpha$. These classes were defined by Robertson [27] and $\mathcal{S}^*(0) := \mathcal{S}^*$ and $\mathcal{K}(0) := \mathcal{K}$ represent subclasses of starlike and convex functions.

Consider the subclasses $\mathcal{S}^*_\alpha$ and $\mathcal{K}_\alpha$ of $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ given by the following equivalences:

$$g \in \mathcal{S}^*_\alpha \iff \left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \alpha,$$

and

$$g \in \mathcal{K}_\alpha \iff \left|\frac{zg''(z)}{g'(z)}\right| < 1 - \alpha.$$

Goodman in [8] defined a new class of functions namely uniformly convex functions and denoted it by $\mathcal{UCV}$. A function $g \in \mathcal{A}$ is a uniformly convex function if it maps every circular arc in the unit disc $\mathbb{D}$ to a convex arc in $g(\mathbb{D})$. Equivalently,

$$g \in \mathcal{UCV} \iff \operatorname{Re}\left(\frac{zg''(z)}{g'(z)} + 1\right) > \left|\frac{zg''(z)}{g'(z)}\right| (z \in \mathbb{D}).$$

The above equivalence was proved independently by Ma and Minda [19] and Rønning [28] in the early nineties. Rønning [28] introduced the class of parabolic starlike functions $\mathcal{S}P$, consisting of functions $g \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\tfrac{zg'(z)}{g(z)}\right) > \left|\tfrac{zg'(z)}{g(z)} - 1\right|.$$

Observe that the class $\mathcal{S}P$ consists of functions $g = zG'$, where $G \in \mathcal{UCV}$. Also $\mathcal{K}_{1/2} \subset \mathcal{UCV}$, $\mathcal{S}^*_{1/2} \subseteq \mathcal{S}P$. Two other subclasses of analytic functions that we will investigate are $\mathcal{L}(\alpha, \beta)$ and $\mathcal{S}^*[A, B]$.

The class $\mathcal{L}(\alpha, \beta)$ is defined as

$$\mathcal{L}(\alpha, \beta) = \left\{ g \in \mathcal{A} : \alpha \frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \beta \in \mathbb{R} \setminus \{1\}, \alpha \geq 0 \right\}.$$

For different values of α and β, it reduces to known classes of analytic functions see [15, 16, 17, 23, 24, 39].

The other class that we will investigate is a widely studied class in univalent functions, known as Janowski [12] starlike functions represented by $\mathcal{S}^*[A, B]$ defined as,

$$\mathcal{S}^*[A, B] = \left\{ g \in \mathcal{A} : \frac{zg'(z)}{g(z)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1 \right\}.$$

For $0 \leq \beta < 1$, $\mathcal{S}^*[1 - 2\beta, -1] = \mathcal{S}^*(\beta)$.

The coefficient estimates have always been a matter of interest. Convex functions, starlike functions with respect to symmetric points, and starlike functions of order 1/2 [18, 29, 30] satisfy $|a_n| \leq 1, (n \geq 2)$. Singh and Singh [35] proved that if class of starlike functions with respect to symmetric point is subordinate to function $1 + \sin z$, then $|a_j| \leq 1/4 (j = 4, 5)$. they have worked on coefficient problems in [36, 37]. They generalised the results obtained in above paper in [38] . Uniformly convex functions satisfy $|a_n| \leq 1/n$[8]. Close-to-convex functions with argument β [10] satisfy $|a_n| \leq 1 + (n - 1)\beta$ if $f \in \mathcal{A}$ with Re $g'(z) > 0 \Rightarrow |a_n| \leq 2/n$ [9]. In 1955 Reade [26] showed that $g(z)$ will be close-to-convex in $\mathbb{D}$ if $|a_n| \leq n|a_1|$. He further showed that $g(z)$ is close-to-starlike function for $|a_n| \leq n^2$. He proved that if $g \in \mathcal{A}$ is close-to-starlike function, there exist a univalent function $\psi(z)$, starlike with respect to $z = 0$ such that $\operatorname{Re}(g(z)/\psi(z)) > 0$ on $\mathbb{D}$. Close-to-starlike functions are not necessarily univalent. Note that the function $\psi(z) = z + n^2 z^n$ is close-to-starlike but not univalent. Silverman [31] in 1994 proved that for analytic function satisfying inequality $\operatorname{Re}(g'(z) + zg''(z)) > \alpha, (\alpha < 1), |a_n| \leq 2(1 - \alpha)/n^2$. These coefficient bounds were sharp and for $1 > \alpha \geq 0$, the class of $g \in \mathcal{A}$ was univalent. Coefficient estimates have been obtained by many, Gurmeet *et al.* [33, 34] have obtained estimates for initial coefficient of bivalent functions defined on unit disk. Bobalade and Sangle [3] have obtained coefficient bounds for a certain subclass of harmonic univalent functions defined using differential operations.

The radius of univalence of $\mathcal{A}$ is the largest $r \in (0, 1]$, such that $g \in \mathcal{A}$ satisfy univalence in $\mathbb{D}_r = \mathbb{D}(0, r) = \{z : |z| < r\}$. The largest $\mathbb{D}_r \subset \mathbb{D}$, whose images under g are starlike with respect to origin and are convex sets, are the starlike radius and the convexity radius. Similarly, the radii of other properties can be defined in similar fashion.

Gavrilov [6] in 1970 showed that the radius of univalence of functions satisfying the inequality $|a_n| \leq n$ is the real root of the equation $2(1 - r)^3 - (1 - r) = 0$, while for the functions whose coefficients satisfy $|a_n| \leq M$, the radius of univalence is $1 - \sqrt{M/(M + 1)}$. Yamashita[40] in 1982 showed that the radius of univalence obtained by Gavrilov is also the radius of starlikeness for the functions $f \in \mathcal{A}$ with coefficients satisfying $|a_n| \leq n$. The author in the same paper also obtained the lower bound for the convexity radius for functions $g \in \mathcal{A}$ that satisfies $|a_n| \leq n, (n \geq 2)$, which turned out to be the least real root $r_0 \approx 0.090$ of the equation $2(1 - r)^4 - (1 + 4r + r^2) = 0$ in $(0, 1)$. The author [40] calculated a lower bound of radius of convexity, for function $g \in \mathcal{A}$ satisfying $|a_n| \leq M$ and obtained as the least real number $(0, 1)$ satisfying equation $(M + 1)(1 - r)^3 - M(1 + r) = 0$.

Kalaj *et al.* [13] studied similar problems on harmonic functions. Graham *et al.* [11] in 2006 investigated radius problems of holomorphic functions on the unit disc $\mathbb{C}^n$. In 2013, radii of 'fully convex' and 'fully starlike functions' were obtained by Nagpal and Ravichandran [22] for harmonic mappings satisfying certain coefficient inequalities.

Let $\mathcal{A}_b$ denote the class of functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n, (0 \leq b \leq 1)$, with $|a_2| = 2b$ in $\mathbb{D}$. In [25], for $g \in \mathcal{A}_b$, Ravichandran determined the sharp radii of order α for starlikeness and convexity when the coefficients satisfy for $n \geq 3$ $|a_n| \leq n$; $|a_n| \leq M$ or $|a_n| \leq M/n$ $(M > 0)$. Radii for uniform convexity and parabolic starlikeness were also obtained by him in the same paper. In the same year, Ali *et al.* [2] obtained radii for the class $\mathcal{L}(\alpha,\beta)$ and $\mathcal{S}^*[A,B]$ for functions $g \in \mathcal{A}_b$ satisfying the coefficient inequalities $|a_n| \le cn + d$ or $|a_n| \le c/n$ $(c > 0)$, and hence deduced the results of Ravichandran [25] and Yamashita [40] as particular cases.

In 2015, Mendiratta *et al.* [21] obtained sharp radii of starlikeness and convexity of order α, parabolic starlikeness, and uniform convexity for functions $g \in \mathcal{A}_b$ when $|a_n| \le M/n^2$ or $|a_n| \le Mn^2 (M > 0)$. The authors have obtained sharp radii for Livingston problem for Sakaguchi functions with fixed second coefficients in [20] Kumar and Ravichandran [14] investigated analytic functions with some coefficient inequalities in 2017 and obtained sharp radii constants along with growth and distortion estimates.

There is a challenge in working on fixed second coefficient, as extremal functions are naturally removed. Prescribed coefficient bounds were further investigated by Aghalary and Mohammadin [1] in 2018. They determined radius for 'univalence', 'stable starlikeness', 'stable convexity', 'fully starlikeness' and 'convexity' of harmonic mappings satisfying some coefficient inequalities.

We will be using the following two lemmas to obtain results.

Lemma 1.1 ([17, 39]). *Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \ge 0$. If $g \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right) |a_n| \le |1-\beta|,$$

then $g \in \mathcal{L}(\alpha,\beta)$.

Lemma 1.2 ([7]). *If $g \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \left((1-B)n - (1-A)\right) |a_n| \le A - B,$$

then $g \in \mathcal{S}^*[A,B]$.

2 Radius Constants for $|a_n| \le C/n^3$

Normalised analytic functons with constant initial coefficient i.e. $|a_2| = 2b$ will be studied in this section, for $n \ge 3$, coefficients will be satisfying the inequality $|a_n| \le C/n^3, C > 0$. Chichra [5] showed that if $g \in \mathcal{A}$ satisfies $\mathrm{Re}\left(g'(z) + zg''(z)\right) > 0, z \in \mathbb{D}$, then g is univalent i.e. $g \in \mathcal{S}$. Later, R. Singh and S. Singh [32] proved that if $\mathrm{Re}\left(g'(z) + zg''(z)\right) > 0 \Rightarrow g \in \mathcal{S}^*$.

Theorem 2.1. *Let $g \in \mathcal{A}_b$ be such that $|a_n| \le C/n^3, (C > 0),\ n \ge 3$. The below mentioned results hold good;*

(i) g satisfies the inequality

$$\mathrm{Re}(g'(z) + zg''(z)) > 0 \text{ for } |z| < r_0,$$

where r_0 is the real number such that $0 < r_0 < 1$ satisfying the equation:

$$(C - 16b)r^2 + 2(C+1)r + 2C\log(1-r) = 0, \tag{2.1}$$

thus $g \in \mathcal{S}$ in $|z| < r_0$.

(ii) $g \in \mathcal{S}^*_\alpha$ for $|z| < r_1$. Where $r_1 = r_1(\alpha)$ is the root lying between 0 and 1 of the equation

$$(16b - C)(2-\alpha)r^2 - (8C+1)(1-\alpha)r + 8C(\mathrm{Li}_2(r) - \alpha\mathrm{Li}_3(r)) = 0 \tag{2.2}$$

The value of r_1 for $\alpha = 1/2$ will provide with the radius of parabolic starlikeness.

(iii) Function g will satisfy the inequality

$$\left|\frac{zg''(z)}{g'(z)}\right| \le 1 - \alpha,\ |z| \le r_2,$$

for $r_2 = r_2(\alpha)$, where $0 < r_2 < 1$ is the root of the equation

$$(16b - C)(2-\alpha)r^2 - 4(C+1)(1-\alpha)r - 4C\log(1-r) - 4\alpha C\mathrm{Li}_2(r) = 0$$

$r_2(\alpha)$also represents the convexity radius of order α. For $\alpha = 1/2$, $r_2(\alpha)$ reduces to the radius of uniform convexity.

Above radii are all extreme and cannot be improved.

Proof. **(i)** Assume that $|a_2| = 2b$ and $|a_n| \leq C/n^3, n \geq 3$. Then,

$$\begin{aligned} g'(z) + zg''(z) &= 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-1} \\ &= 1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1} \\ &= 1 + 8bz + \sum_{n=3}^{\infty} n^2 a_n z^{n-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Re}(g'(z) + ng''(z)) &> 1 - 8br_0 - \sum_{n=3}^{\infty} \frac{C r_0^{n-1}}{n} \\ &= 1 - 8br_0 - C\left(-\frac{\log(1-z_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\ &= 0, \end{aligned}$$

where r_0 is the solution of equation (2.1) and is a real number lying in $(0,1)$. This is the best possible radius. As, for the function

$$g_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n.$$

$$zg_0''(z) + g_0'(z) = 1 - 8bz - \sum_{n=3}^{\infty} \frac{C}{n} z^{n-1}.$$

At $z = r_0$, the function g_0 satisfies $\operatorname{Re}(zg_0''(z) + g_0'(z)) = 0$.

Hence, the result.

(ii) It needs to be shown that the function $g(r_1 z)/r_1 \in \mathcal{S}^*_\alpha$. where r_1 is root of equation (2.2) lying in $(0,1)$. For $A = 1-\alpha$ and $B = 0$, from Lemma 1.2 it is sufficient to establish the inequality

$$\sum_{n=2}^{\infty} (n-\beta)|a_n| r_1^{n-1} \leq 1 - \alpha.$$

Consider for $|a_n| \leq C/n^3$,

$$\begin{aligned} \sum_{n=2}^{\infty} (n-\alpha)|a_n| r_1^{n-1} &= 2(2-\alpha)br_1 + \sum_{n=3}^{\infty} (n-\alpha)|a_n| r_1^{n-1} \\ &\leq 2(2-\alpha)br_1 + \alpha C \sum_{n=3}^{\infty} \frac{r_1^{n-1}}{n^3} - \alpha C \sum_{n=3}^{\infty} \frac{r_1^{n-1}}{n^3} \\ &= 2(2-\alpha)br_1 + C\left(\frac{\operatorname{Li}_2(r_1)}{r_1} - 1 - \frac{r_1}{4}\right) - \frac{\alpha C}{r_1}\left(\operatorname{Li}_3(r_1) - r_1 - \frac{r_1^2}{8}\right). \end{aligned}$$

We have used value of the polylogarithm functions of order 2 and 3. Polylogarithmic function of order s is

$$\operatorname{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots, \operatorname{Li}_{s+1}(z) = \int_0^z \frac{\operatorname{Li}_s(t)}{t}\, dt.$$

Thus,

$$\begin{aligned} \sum_{n=2}^{\infty} (n-\alpha)|a_n| r_1^{n-1} &= 2(2-\alpha)br_1 + \frac{C}{r_1}\operatorname{Li}_2(r_1) - C - \frac{Cr_1}{4} \\ &\quad - \frac{\alpha C}{r_1}\operatorname{Li}_3(r_1) + \alpha C + \frac{\alpha C r_1}{8} = 1 - \alpha. \end{aligned}$$

Hence, $g \in \mathcal{S}^*_\alpha$ for $|z| < r_1$, with $r_1 \in (0,1)$ being the root of equation (2.2).

As proved in the previous result, it can be easily shown here that

$$g_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n$$

is the required function for the best possible radius of starlikeness of order α. As the inequality

$$\left|\frac{zg_0'(z)}{g_0(z)} - 1\right| \leq \frac{1}{2}$$

is sufficient for parabolic starlikeness, thus for $\alpha = \frac{1}{2}$ we must have

$$\left|\frac{zg_0'(z)}{g_0(z)} - 1\right| = \frac{1}{2} = \mathrm{Re}\left(\frac{zg_0'(z)}{g_0(z)}\right),$$

Hence radius of parabolic starlikeness.

(iii) We are aware that to show $g \in \mathcal{K}_\alpha$, it is sufficient to show $zg'(z) \in \mathcal{S}^*_\alpha$. Thus, it is sufficient to establish the below inequality from Lemma 1.2:

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|r_2^{n-1} \leq 1-\alpha.$$

Note that,

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n|r_2^{n-1} \leq 4(2-\alpha)br_2 + \sum_{n=3}^{\infty}(k-\alpha)\frac{C}{k^2}r_2^{n-1}$$
$$= 4(2-\alpha)br_2 + C\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n} - \alpha C\sum_{n=3}^{\infty}\frac{r_2^{n-1}}{n^2}.$$

Using standard series expansions, we obtain

$$= 4(2-\alpha)br_2 + \frac{C}{r_2}\left(-\log(1-r_2) - r_2 - \frac{r_2^2}{2}\right) - \frac{\alpha C}{r_2}\left(\mathrm{Li}_2(r_2) - r_2 - \frac{r_2^2}{4}\right)$$
$$= 1-\alpha.$$

Hence the result follows. The smallest root in $(0,1)$ is the required radius. Sharpness is attained by the function

$$g_0(z) = z - 2bz^2 - C\sum_{n=3}^{\infty}\frac{z^n}{n^3},$$

which can be proved easily. □

Corollary 2.1. *(i)(a) For $g \in \mathcal{A}$ and $|a_n| \leq C/n^3, (n \geq 2)$, $\mathrm{Re}(zg''(z) + g'(z)) > 0$ where $r_0 \in (0,1)$ is the solution of the equation*

$$2(C+1)r + 2C\log(1-r) = 0.$$

(i)(b) The starlikeness radius of normalised analytic functions with coefficients satisfying $a_2 = 0$ and $|a_n| \leq C/n^3, n \geq 3$, is the root $r_0 \in (0,1)$ for the equation

$$Cr^2 + 2(C+1)r + 2C\log(1-r) = 0.$$

Corollary 2.2. *(ii)(a) The starlikeness radius of order α of functions with coefficients satisfying $|a_n| \leq C/n^3, n \geq 2$, is the real number $r_1 \in (0,1)$ which is the solution of the equation*

$$(8C+1)(1-\alpha)r - 8C(\mathrm{Li}_2(r) - \alpha\mathrm{Li}_3(r)) = 0$$

and starlikeness radius is given by the root of the equation

$$(8C+1)r - 8C\mathrm{Li}_2(r) = 0$$

(ii)(b)The starlikenss radius of order α of functions with coefficients $a_2 = 0$ and $|a_n| \leq C/n^3$ for $n \geq 3$, is the real number r_1 in $(0,1)$ satisfying the equation

$$C(\alpha-2)r^2 + (8C+1)(\alpha-1)r + 8C(\mathrm{Li}_2(r) - \alpha\mathrm{Li}_3(r)) = 0.$$

and radius of starlikness is the solution of the equation below

$$2Cr^2 + (8C+1)r - 8C\mathrm{Li}_2(r) = 0$$

Corollary 2.3. *Let $g \in \mathcal{A}$ and $|a_n| \leq C/n^3, (k \geq 2)$. Then the radius of convexity of order α is the real root r_2 in $(0,1)$ of the equation*

$$(C+1)(1-\alpha)r + C\log(1-r) + \alpha C \mathrm{Li}_2(r) = 0$$

For $\alpha = 1/2$, r_2 is the radius of uniform convexity. The results are best possible. For $\alpha = 0$ the radius of convexity can be obtained, it is solution of

$$(C+1)r + C\log(1-r) = 0$$

The radius for $g \in \mathcal{A}$ with $a_2 = 0$ can also be obtained easily.

We will now obtain sharp radii for the class of functions in $\mathcal{L}(\alpha, \beta)$ and $\mathcal{S}^*[A, B]$ for $g \in \mathcal{A}_b$ under the same bounds for coefficients $|a_n| \leq C/n^3$.

Theorem 2.2. *Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \geq 0$. Then the radius of the class $\mathcal{L}(\alpha, \beta)$ for functions $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq C/n^3, (n \geq 3)$, is $t_0 \in (0,1)$, where t_0 is the root of equation*

$$\begin{aligned} C\beta t^3 + \big(8\big(4\alpha + 4 - 2\beta\big)b + 2C(\alpha - 3)\big)\, t^2 - 8\big(2C - \alpha C - C\beta + |1-\beta|\big)t \\ - 8C\alpha \log(1-t) + 8C(1-\alpha)\mathrm{Li}_2(t) - 8C\beta\, \mathrm{Li}_3(t) = 0. \end{aligned} \tag{2.4}$$

For $\beta < 1$, the number $\mathcal{L}_0(\alpha, \beta) \left(= \mathcal{S}^*_\beta = \mathcal{S}^*[1-\beta, 0]\right)$ radius of $g \in \mathcal{A}_b$

$$\mathcal{L}_0(\alpha, \beta) = \{g \in \mathcal{A}_b : |\alpha \frac{z^2 g''(z)}{g(z)} + \frac{z g'(z)}{g(z)} - 1| \leq |1-\beta|\}$$

radius of $g \in \mathcal{A}_b$. The radius is best possible.

Proof. The real number $t_0 \in (0,1)$ is the $\mathcal{L}(\alpha, \beta)$ radius of the function $g \in \mathcal{A}_b$ if and only if $g(t_0 z)/t_0 \in \mathcal{L}(\alpha, \beta)$. Thus, from Lemma 1.1, it is sufficient to check for the inequality

$$\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| t_0^{n-1} \leq |1-\beta|.$$

For $g \in \mathcal{A}_b$, we consider the left-hand side:

$$\begin{aligned}
&\textstyle\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| t_0^{n-1} \\
&\leq (4\alpha + 2(1-\alpha) - \beta)|a_2| t_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big) \frac{C}{n^3} t_0^{n-1} \\
&= 2(2\alpha + 2 - \beta) b t_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big) \frac{C}{n^3} t_0^{n-1} \\
&= 2(2\alpha + 2 - \beta) b t_0 + C \left[\sum_{n=3}^{\infty} \frac{\alpha t_0^{n-1}}{n} + (1-\alpha) \sum_{n=3}^{\infty} \frac{t_0^{n-1}}{n^2} - \beta \sum_{n=3}^{\infty} \frac{t_0^{n-1}}{n^3} \right] \\
&= 2(2\alpha + 2 - \beta) b t_0 + C\alpha \left(-\frac{\log(1-t_0)}{t_0} - 1 - \frac{t_0}{2} \right) \\
&+ (1-\alpha)C \left(\frac{\mathrm{Li}_2(t_0)}{t_0} - \frac{t_0}{4} - 1 \right) - C\beta \left(\frac{\mathrm{Li}_3(t_0)}{t_0} - \frac{t_0^2}{8} - 1 \right) = |1-\beta| \\
&\Rightarrow\ C\beta t_0^3 + ((32\alpha + 32 - 16\beta)b - 6C + 2C\alpha)t^2 + (-8C - 8C + 8\alpha C + 8C\beta - 8|1-\beta|)\, t_0 \\
&-8C\alpha \log(1-t_0) + 8C(1-\alpha)\mathrm{Li}_2(t_0) - 8C\beta \mathrm{Li}_3(t_0) = 0 \\
&\Rightarrow\ C\beta t_0^3 + (8(4\alpha + 4 - 2\beta)b + 2C(\alpha - 3))t_0^2 - 8(2C - C\alpha - M\beta + |1-\beta|)t_0 \\
&-8C\alpha \log(1-t_0) + 8C(1-\alpha)\mathrm{Li}_2(t_0) - 8C\beta \mathrm{Li}_3(t_0) = 0.
\end{aligned}$$

For $\beta < 1$, consider the normalised function as

$$g_0(z) = z - 2pz^2 - \sum_{n=3}^{\infty} \frac{C}{n^3} z^n$$

at the point $z = t_0$, where $t_0 \in (0, 1)$ is the root of equation (2.4), g_0 satisfies

$$\operatorname{Re}\left(\frac{\alpha z^2 g_0''(z)}{g_0(z)} + \frac{z g_0'(z)}{g_0(z)}\right) = \beta.$$

Hence it is best possible radius. For $\beta > 1$, the sharpness will hold for the function

$$g_0(z) = z + 2bz^2 + C\left(\operatorname{Li}_3(z) - z - \frac{z^2}{8}\right).$$

□

Corollary 2.4. *For $\alpha = 0$ and $\beta < 1$, the result of the previous Theorem 2.1 can be obtained. Other radii for $g \in \mathcal{A}$ and $a_2 = 0$ can also be determined.*

In the following result we will evaluate the sharp $\mathcal{S}^*[A, B]$ radius of $g \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$.

Theorem 2.3. *For $-1 \leq B < A \leq 1$. The $\mathcal{S}^*[A, B]$ radius for $g \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$ is the real root $r_0 \in (0, 1)$ of the equation*

$$(16b - C)(1 - 2B + A)r^2 - 8(C + 1)(A - B)r + 8C(\operatorname{Li}_2(r)(1 - B) - \operatorname{Li}_3(r)(1 - A)) = 0 \qquad (2.3)$$

The radius cannot be improved.

Proof. r_0 will be the $\mathcal{S}^*[A, B]$ radius of $g \in \mathcal{A}_b$ iff $g(r_0 z)/r_0 \in \mathcal{S}^*[A, B]$. Hence by Lemma 1.2 it suffices to show that

$$\sum_{n=2}^{\infty}\big((1 - B)n - (1 - A)\big)|a_n| r_0^{n-1} \leq A - B, \quad (-1 \leq B < A \leq 1),$$

where r_0 is the root in $(0, 1)$ of eq.(2.3)

For function $g \in \mathcal{A}_b$, it follows that

$\sum_{n=2}^{\infty}\big((1 - B)k - (1 - A)\big)|a_n| r_0^{n-1}$

$\leq 2\big((1 - B)2 - (1 - A)\big)br_0 + \sum_{n=3}^{\infty}(1 - B)k\frac{Cr_0^{n-1}}{n^3} - \sum_{n=3}^{\infty}(1 - A)\frac{Cr_0^{n-1}}{n^3}$

$= 2\big(2(1 - B) - (1 - A)\big)br_0 + \frac{C(1-B)}{r_0}\left[\operatorname{Li}_2(r_0) - r_0 - \frac{r_0^2}{4}\right] - \frac{C(1-A)}{r_0}\left[\operatorname{Li}_3(r_0) - r_0 - \frac{r_0^2}{8}\right]$

$= 2(1 - 2B + A)br_0 - C(1 - B - 1 + A) - \frac{r_0}{8}M(1 - 2B + A)$

$+\frac{C}{r_0}(\operatorname{Li}_2(r_0)(1 - B) - \operatorname{Li}_3(r_0)(1 - A))$

$= A - B$

$$\Rightarrow (1 - 2B + A)r_0\left(2b - \frac{C}{8}\right) - (C - 1)(A - B) + \frac{C}{r_0}\left((1 - B)\operatorname{Li}_2(r_0) - (1 - A)\operatorname{Li}_3(r_0)\right) = 0.$$

The function

$$g_0(z) = z - 2bz^2 - C\left(\operatorname{Li}_3(w) - w - \frac{w^2}{8}\right)$$

shows that the result is best possible and cannot be improved. □

Remark 2.1. *The $\mathcal{S}^*_\alpha$ radius for $g \in \mathcal{A}$ satisfying the coefficient inequality $|a_n| \leq C/n^3, n \geq 3$ can be obtained by putting $A = 1 - 2\alpha; B = -1$ in the $\mathcal{S}^*[A, B]$ radius or $\alpha = 0$ in $\mathcal{L}(\alpha, \beta)$ radius respectively.*

3 Radius Constants for $|a_n| \le cn + d/n$.

We will investigate radii constants for functions $g \in \mathcal{A}_b$ with the coefficient inequality $|a_n| \le cn + d/n$ in the present section.

Theorem 3.1. *Let $|a_n| \le cn + d/n, (n \ge 3)$ for the function $g \in \mathcal{A}_b$. Then the following results hold:*

(i) $\mathrm{Re}(g'(z) + zg''(z)) > 0$*, where s_0 is the least real number in $(0, 1)$ satisfying the equation*

$$[(1 + c + d) + 2b(4c + d - 4b)](1 - s)^4 - d(1 - s)^2 - c(1 + 4s + s^2) = 0. \tag{3.1}$$

The result cannot be improved and is sharp.

(ii) $g \in \mathcal{S}^*_\alpha$ where $s_1 = s_1(\alpha)$ is the root lying in $(0, 1)$ of the equation

$$\begin{aligned} s\left(2(b - c)(2 - \alpha) + \frac{\alpha d}{2}\right) &+ \alpha(1 + c + d) - (1 + c) \\ &+ \frac{c(1 + s)}{(1 - s)^3} - \frac{\alpha c}{(1 - s)^2} + \frac{ds^2}{1 - s} + \frac{\alpha d \log(1 - s)}{s} = 0. \end{aligned} \tag{3.2}$$

The root $s_1(\alpha)$ is also provides with starlikeness radius of order α, and $s_1(1/2)$ represents the radius for parabolic starlikeness of the function. The results are sharp.

(iii) $g \in \mathcal{A}_b$ and $|a_n| \le cn + d/n$ for $n \ge 3$, then the following condition will hold good for g,

$$\left|\frac{zg''(z)}{g'(z)}\right| \le 1 - \alpha,$$

if $s_2 = s_2(\alpha) \in (0, 1)$ is the root of the equation

$$\begin{aligned} &((1 - \alpha)(c + 1) + d - 2s(2(p - c)(2 - \alpha)(1 - s)4 - d)) \\ &= d(1 - \alpha s^2 + \alpha s^3)(1 - s)^2 + c\big(s^2(1 + \alpha) + 4s + 1 - \alpha\big). \end{aligned} \tag{3.3}$$

The number $s_2(\alpha)$ is also the radius of convexity of order α. The number $s_2(1/2)$ is the radius of uniform convexity of the given functions. The results are all sharp.

Proof. **(i).** For $g \in \mathcal{A}_b$, consider

$$g'(z) + zg''(z) = 1 + 8bz + c\sum_{n=3}^{\infty} n^3 z^{n-1} + n\sum_{n=3}^{\infty} nz^{n-1}.$$

$$\mathrm{Re}(g'(z) + zg''(z)) \ge 1 - 8bs - c\sum_{n=3}^{\infty} n^3 s^{n-1} - d\sum_{n=3}^{\infty} ns^{n-1}$$

$$= 1 - 8bs - c\left(\frac{1 + 4s + s^2}{(1 - s)^4} - 1 - 8s\right) - d\left(\frac{1}{(1 - s)^2} - 1 - 2s\right).$$

After simplification,

$$= (1 + c + d) + s(8c + 2d - 8b) - \frac{d}{(1 - s)^2} - \frac{c(1 + 4s + s^2)}{(1 - s)^4}.$$

Let $s \in (0, 1)$ be the root of

$$((1 + c + d) + 2s(4c + d - 4b))(1 - s)^4 - d(1 - s)^2 - c(1 + 4s + s^2) = 0.$$

Then $\mathrm{Re}(g'(z) + zg''(z)) \ge 0$, and this provides us with the result that g is a starlike function. Consider the function

$$g_0(z) = z - 2bz^2 - \sum_{k=3}^{\infty}\left(cn + \frac{d}{n}\right)z^n.$$

$$\begin{aligned} g_0(z) &= z - 2bz^2 - c\left(\frac{z}{(1 - z)^2} - z - 2z^2\right) - d\left(-\log(1 - z) - z - \frac{z^2}{2}\right) \\ &= z - 2bz^2 - \frac{cz}{(1 - z)^2} + cz + 2cz^2 + d\log(1 - z) + dz + \frac{dz^2}{2} \\ &= z(1 + c + d) + z^2\left(2c - 2b + \frac{d}{2}\right) - \frac{cz}{(1 - z)^2} + d\log(1 - z) \end{aligned}$$

$$\Rightarrow g_0'(z) + zg_0''(z) = (1+c+d) + z(8c-8b+2d) - \frac{c(1+4z+z^2)}{(1-z)^4} - \frac{d}{(1-z)^2}.$$

At the point $z = s_0$, where $s_0 \in (0,1)$ is the root for the equation (3.1), then g_0 satisfies

$$\operatorname{Re}(zg_0''(z) + g_0'(z)) \geq 0.$$

Thus s_0 is the sharp radius for $g \in \mathcal{A}_b$.

(ii) For $g \in \mathcal{A}_b$ and $|a_n| \leq cn + d/n$, $n \geq 3$, we write

$$\begin{aligned}
|zg'(z) - g(z)| - (1-\alpha)|g(z)| &\leq \sum_{n=2}^{\infty}(n-1)|a_n||z|^n - (1-\alpha)\left(|z| - \sum_{n=2}^{\infty}|a_n||z|^n\right) \\
&= -(1-\alpha)|z| + \sum_{n=2}^{\infty}(n-\alpha)|a_n||z|^n \\
&\leq -(1-\alpha)s + \sum_{n=2}^{\infty}(n-\alpha)|a_n|s^n \leq 0. \qquad (3.4)
\end{aligned}$$

Using $|a_n| \leq cn + d/n$ and the identities mentioned below,

$$\begin{aligned}
\sum_{n=3}^{\infty} k^2 s^{n-1} &= \frac{1+s}{(1-s)^3} - 1 - 4s, \qquad & \sum_{n=3}^{\infty} s^n &= \frac{s^3}{1-s}, \\
\sum_{n=3}^{\infty} n s^{n-1} &= \frac{1}{(1-s)^2} - 1 - 2s, \qquad & \sum_{n=3}^{\infty} \frac{s^n}{n} &= -\log(1-s) - s - \frac{s^2}{2},
\end{aligned}$$

inequality (3.4) gives

$$\begin{aligned}
s_1\left(2(b-c)(2-\alpha) + \frac{\alpha d}{2}\right) + (\alpha d - 1 + \alpha) - c(1-\alpha) + \frac{c(1+s_1)}{(1-s_1)^3} + \frac{ds_1^2}{1-s_1} - \frac{\alpha c}{(1-s_1)^2} \\
+ \frac{\alpha d \log(1-s_1)}{s_1} = 0,
\end{aligned}$$

which is the required radius $s_1 = s_1(\alpha)$.

This result is sharp for the function g_0 given by

$$\begin{aligned}
g_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty}\left(cn + \frac{d}{n}\right)z^n \\
&= z(1+c+d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1-z) - \frac{cz}{(1-z)^2}.
\end{aligned}$$

A calculation shows

$$\frac{zg_0'(z)}{g_0(z)} - 1 = \frac{N(z)}{D(z)},$$

where

$$N(z) = z^2\left(2c - 2b + \frac{d}{2}\right) - \frac{dz}{1-z} + \frac{cz}{(1-z)^2} - \frac{c(1+z)z}{(1-z)^3} - d\log(1-z)$$

and

$$D(z) = z(1+c+d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1-z) - \frac{cz}{(1-z)^2}.$$

For $z = s_1$, where $s_1 \in (0,1)$ is the root,

$$\left|\frac{zg_0'(z)}{g_0(z)} - 1\right| = 1 - \alpha.$$

Hence the result is sharp.

(iii) for $|a_2| = 2b$ and $|a_n| \leq cn + d/n$. Using the known series sum for $\sum_{n=2}^{\infty} n^i s^n (i = 0,1,2,3)$ consider

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n||z|^{n-1} \leq \sum_{n=2}^{\infty} n(n-\alpha)|a_n|s^{n-1}$$

$$
\begin{aligned}
&\leq 4b(2-\alpha)s + c\sum_{n=3}^{\infty} n^3 s^{n-1} - \alpha c\sum_{n=3}^{\infty} n^2 s^{n-1} + d\sum_{n=3}^{\infty} n s^{n-1} - \alpha d\sum_{n=2}^{\infty} s^{n-1} \\
&= 4b(2-\alpha)s + c\left(\frac{1+4s+s^2}{(1-s)^4} - 1 - 8s\right) - \alpha c\left(\frac{1+s}{(1-s)^3} - 1 - 4s\right) \\
&\quad + d\left(\frac{1}{(1-s)^2} - 1 - 2s\right) - \frac{\alpha d s^2}{1-s} \\
&= 1-\alpha.
\end{aligned}
$$

The above equation (3.3) can be obtained. The result can not be improved further, as the function,

$$g_0(z) = z(1+c+d) + z^2\left(2c - 2b + \frac{d}{2}\right) + d\log(1-z) - \frac{cz}{(1-z)^2}.$$

verifies required sharpness. □

Corollary 3.1. *(i)(a). The radius for starlikeness of functions with coefficients satisfying $a_2 = 0$ and $|a_n| \leq cn + d/n$ for all $n \geq 3$ is the real number in $(0,1)$ and root of the equation*

$$(1+c+d) + 2s(4c+d)(1-s)^4 = d(1-s)^2 + c(1+4s+s^2).$$

(i)(b). The starlikness radius s_1 of functions is such that $g(s_1 z)/s_1 \in \mathcal{A}$ such that $|a_n| \leq cn + d/n, (n \geq 2)$

$$(1+c+d)(1-s)^4 = d(1-s)^2 + c(1+4s+s^2).$$

Corollary 3.2. *If $g \in \mathcal{A}$, $|a_n| \leq cn + d/n, (n \geq 2)$, then the starlikeness radius s_1 of order α satisfies*

$$ds_1 + \alpha(1+c+d) - (1+c) + \frac{c(1+s_1)}{(1-s_1)^3} - \frac{\alpha c}{(1-s_1)^2} + \frac{ds_1^2}{1-s_1} + \frac{\alpha d\log(1-s_1)}{s_1} = 0.$$

For $\alpha = 0$, $g \in \mathcal{A}$, s_1 provides with radius of starlikeness as

$$\frac{ds_1}{1-s_1} - (1+c) + \frac{c(1+s_1)}{(1-s_1)^3} = 0.$$

Corollary 3.3. *For $g \in \mathcal{A}$ and $|a_n| \leq cn + d/n, n \geq 2$. The radius of convexity of order α is given by $s(\alpha)$, which is the least real root in $(0,1)$ satisfying the equation*

$$\big((1-\alpha)(1+c)d) + d\alpha s\big)(1-s)^4 = c\left(1 - \alpha s^2(1-s)\right)(1-s)^2 - c\big(s^2(1+\alpha) + 4s + 1 - \alpha\big).$$

$s(1/2)$ will be the radius of uniform convexity of the given function. The results are sharp. $s(0)$ will be the radius of convexity. The radii for the case $g \in \mathcal{A}$ and $a_2 = 0$ can be easily obtained from the above equation.

In the next theorem we obtain sharp $\mathcal{L}(\alpha,\beta)$ and $\mathcal{S}^*[A,B]$ radii of functions $g \in \mathcal{A}_b$ satisfying the above coefficient inequality.

Theorem 3.2. *Let $\beta \in \mathbb{R}\setminus\{1\}$ and $\alpha \geq 0$. The $\mathcal{L}(\alpha,\beta)$ radius of $g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfy the coefficient inequality $|a_n| \leq ck + d/n, n \geq 3$, where $c, d \geq 0$ is the real number in $(0,1)$ satisfying the equation.*

$$
\begin{aligned}
&Pt^2(1-t)^4 + Qt(1-t)^4 + R(1-t)^4 + d(1-\alpha)t^3(1-t)^3 + (\alpha d - c\beta)t(1-t)^2 \\
&\quad + c(1-\alpha)t(1-t)^2 + c\alpha t(1+4t+t^2) = 0, \qquad (3.4)
\end{aligned}
$$

where P, Q, R are given as

$$
\begin{aligned}
P &= 2\left(b(2\alpha + 2 - \beta) - 2c(\alpha+1) - \alpha d + c\beta\frac{\beta d}{4}\right) \\
Q &= \beta(c+d) - \alpha d - c - |1-\beta| \\
R &= \beta d\log(1-t).
\end{aligned}
$$

For $\beta < 1$, this number is also $\mathcal{L}_0(\alpha,\beta)$radius of $g \in \mathcal{A}_b$. The radii's are all best possible and cannot be improved further.

Proof. Let t_0 be the $\mathcal{L}(\alpha,\beta)$radius of the function $g \in \mathcal{A}_b$. Then the inequality

$$\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-\beta\big)|a_n|t^{n-1} \le |1-\beta|$$

holds for the root of equation (3.4) lying in $(0,1), g \in \mathcal{A}_b$ implies that

$$\begin{aligned}
\sum_{n=2}^{\infty}(\alpha n^2+(1-\alpha)k-\beta)|a_n|t^{n-1} &\le 2(2\alpha+2-\beta)bt+\sum_{n=3}^{\infty}(\alpha n^2+(1-\alpha)k-\beta)\Big(cn+\frac{d}{n}\Big)t^{n-1}\\
&= 2(2\alpha+2-\beta)bt+c\alpha\sum_{n=3}^{\infty}n^3t^{n-1}+c(1-\alpha)\sum_{n=3}^{\infty}k^2t^{n-1}-c\beta\sum_{n=3}^{\infty}nt^{n-1}\\
&\quad+\alpha d\sum_{n=3}^{\infty}kt^{n-1}+(1-\alpha)d\sum_{n=3}^{\infty}t^{n-1}-\beta d\sum_{n=3}^{\infty}\frac{t^{n-1}}{n}\\
&= 2(2\alpha+2-\beta)bt+c\alpha\Big(\frac{1+4t+t^2}{(1-t)^4}-1-8t\Big)+c(1-\alpha)\Big(\frac{1+t}{(1-t)^3}-1-4t\Big)\\
&\quad+(\alpha d-c\beta)\Big(\frac{1}{(1-t)^2}-1-2t\Big)+(1-\alpha)d\frac{t^2}{1-t}-\beta d\Big(\frac{-\log(1-t)}{t}-1-\frac{t}{2}\Big)
\end{aligned}$$

$$\begin{aligned}
&= 2t\Big(b(2\alpha+2-\beta)-4c\alpha-2c(1-\alpha)-(\alpha d-c\beta)+\frac{\beta d}{4}\Big)+(-c\alpha-c+c\alpha-\alpha d+c\beta+\beta d)\\
&\quad+\beta d\frac{\log(1-t)}{t}+d(1-\alpha)\frac{t^2}{1-t}+\frac{\alpha d-c\beta}{(1-t)^2}+\frac{c(1-\alpha)(1+t)}{(1-t)^3}+\frac{c\alpha(1+4t+t^2)}{(1-t)^4}=|1-\beta|
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ &2t^2\big(b(2\alpha+2-\beta)-2c\alpha-2c-\alpha d+c\beta+\tfrac{\beta d}{4}\big)(1-t)^4+(\beta(c+d)-\alpha d-c)t(1-t)^4\\
&+\beta d(1-t)^4\log(1-t)+d(1-\alpha)t^3(1-t)^3+(\alpha d-c\beta)t(1-t)^2\\
&+c(1-\alpha)t(1-t^2)+c\alpha t(1+4t+t^2)=|1-\beta|\,t(1-t)^4.
\end{aligned}$$

For $\beta<1$, sharpness can be obtained for the function

$$\begin{aligned}
g_0(z) &= z-2bz^2-\sum_{k=3}^{\infty}\left(cn+\frac{d}{n}\right)z^n\\
&= z(1+c+d)+z^2\left(2c-2b+\frac{d}{2}\right)+d\log(1-z)-\frac{cz}{(1-z)^2}.
\end{aligned}$$

At $z=t_0$, where t_0 is root of eq (3.4) in $(0,1)$, g_0 satisfy

$$\mathrm{Re}\left(\alpha z^2\frac{g_0''(z)}{g_0(z)}+z\frac{g_0'(z)}{g_0(z)}\right)>0.$$

The $\mathcal{S}^*[A,B]$ radius can also be easily obtained and the required radius for Janowski class is given by

$$\begin{aligned}
&t\Big(2(c-b)(2B-A-1)+(1-A)\frac{d}{2}\Big)+\big((B+A)+d(1-A)\big)\\
&+\frac{(1-B)}{(1-t)^3}\Big(c(1+t)+dt^2(1-t^2)\Big)+\frac{(1-A)d\log(1-t)}{t}=0.
\end{aligned}$$

□

Remark 3.1. *For $c=0$, we have $|a_n|\le d/n$, and $d=0$ implies $|a_n|\le cn$. We are able to obtain $\mathcal{L}(\alpha,\beta)$ radius of functions in both cases which were obtained by Ali* et.al. *[2] for $|a_n|\le cn$ and $|a_n|\le c/n$, for which sharpness results were obtained by Ravichandran [25]. Thus the particular cases of these inequalities are obtained easily by proper substitution.*

Conclusion.

The purpose of this study was to detremine and verify the shard radii of univalence, starlikeness and convexity of analytic function defined on an open disc. Upon investigating the functions in class $\mathcal{A}_b$ with a fixed initial coefficient $|a_n| = 2b$ $(0 \leq b \leq 1)$, we have calculated the radius constants for $|a_n| \leq C/n^3$ and $|a_n| \leq cn+d/n$. Other classes like Janowsi starlike functions $\mathcal{S}^*[A, B]$ and the class $\mathcal{L}(\alpha, \beta)$ are also investigated. The reseachers can obtain the radius for other classes under the same coefficient bounds or can explore for other coefficient bounds as well.

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