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A NOTE ON INEQUALITIES AND BOUNDS FOR LAMBERT W FUNCTION, AND ITS RELATIONSHIPS WITH GAMMA AND PSI FUNCTIONS

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Abstract

The Lambert W function, also known as the product logarithm, is an important special function with significant applications in mathematics, probability, combinatorics, and physics. This paper investigates inequalities and bounds for the Lambert W function and explores its relationships with the Gamma and Psi functions. Derivations, approximations, and bounds are provided. Figures demonstrate the accuracy of bounds and approximations.

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1 Introduction

The Lambert W function, denoted as $W(x)$, is defined as the inverse of the function $f(W) = W{*}e^W$, such that $W(x) * e^{W(x)} = x$. It has found applications in many fields such as mathematics, physics, chemistry, biology, probability, and statistics (cf. Georg [9]). The Lambert W function also arises in asymptotic approximations, the analysis of algorithms, combinatorics, quantum physics, and risk theory, among others. The Gamma function $\Gamma(x)$, which generalizes the factorial function, and the Psi function $\psi(x)$, the logarithmic derivative of $\Gamma(x)$, are two of the most studied special functions in mathematics. Recent studies have sought to develop inequalities that connect these functions to one another, since such relationships yield powerful tools in both theoretical and applied mathematics. For details on the Lambert W function $W(x)$, the Gamma function $\Gamma(x)$, and the Psi function $\psi(x)$, the interested readers are referred to Abramowitz and Stegun [2], Amdeberhan and Moll [3], Artin [4], Brito *et al.* [5], Corless *et al.* [6], Dence [7], Erdélyi *et al.* [8], Hassani [10], Hoorfar and Hassani [11], Olver *et al.* [14], and Shakil and Singh [16], among others.

In this paper, we aim to establish inequalities and bounds for the Lambert W function and explore its connections with the Gamma and Psi functions. These relationships will provide new insights into analytic inequalities, asymptotic expansions, and applications in probability theory.

The organization of paper is as follows: In Section 2, we provide some definitions and preliminaries. Section 3 contains main results. In Section 4, we present applications and results. Some conclusions are drawn in Section 5.

2 Definitions and Preliminaries

In what follows, we present some preliminaries of the three central functions: the Lambert $W(x)$, Gamma $\Gamma(x)$, and Psi $\psi(x)$ functions.

2.1 Definitions

- Lambert $\boldsymbol{W}$ function: defined implicitly by $W(x) * e^{W(x)} = x$.
- Gamma function: $\Gamma(x) = \int_0^\infty t^{(x-1)} e^{-t} dt$, for $\mathrm{Re}(x) > 0$.
- Psi function: $\psi(x) = \frac{d}{dx}(ln\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$.

2.2 Fundamental Asymptotics

The following are some fundamental asymptotics that will also be used in our main results:

- $W(x) \sim ln(x) - ln(ln(x))$ as $x \to \infty$.
- $\Gamma(x+1) \sim \sqrt{(2\pi x)}(x/e)^x$ as $x \to \infty$.
- $\psi(x) \sim ln(x) - 1/(2x)$ for large x.

2.3 Basic Properties of the Lambert $\boldsymbol{W}$ Function

Here, we present some basic properties of the Lambert W function (cf. Brito *et al.* [5], Corless *et al.* [6], and Dence [7]). As stated above, the Lambert W function, denoted as $W(x)$, is defined as the inverse of the function

$$f(W) = W * e^W.$$

Thus, it satisfies the fundamental relation:

$$W(x) * e^{W(x)} = x, x \in \left[-\frac{1}{e}, \infty\right).$$

It is a multivalued function, meaning that for some values of x, there are multiple values of $W(x)$. It is commonly used to solve transcendental equations such as:

$$xe^x = a \Rightarrow x = W(a).$$

The following are some important properties of the Lambert W function:

I. Real Branches: For real values of x, the Lambert W function has two real branches:

(a) Principal Branch $\boldsymbol{W_0(x)}$: (i) Defined for: $x \geq -\frac{1}{e}$; (ii) Range: $W_0(x) \geq -1$; and (iii) Passes through: $W_0(0) = 0$, and $W_0(e) = 1$.

(b) Lower Branch $\boldsymbol{W_{-1}(x)}$: (i) Defined for: $-\frac{1}{e} \leq x < 0$; and (ii) Range: $W_{-1}(x) \leq -1$.

(c) Branch Point: Both branches meet at: $x = -\frac{1}{e}, W\left(-\frac{1}{e}\right) = -1$.

(d) It is obvious from the above (cf. Olver *et al.* [14]) that on the interval $[\ 0, \infty\ )$, there is one real solution, and it is nonnegative and increasing. On the interval $\left(-\frac{1}{e}, 0\right)$, there are two real solutions, one increasing and the other decreasing. We call the increasing solution for which $W(x) \geq W\left(-\mathrm{e}^{-1}\right) = -1$ is the principal branch and we denote it by $W_0(x)$. The decreasing solution can be identified as $W_{\pm 1}(x \mp 0\mathrm{i})$.

(e) Graph of the Lambert $\boldsymbol{W}$ *Function:* Below is a visual (Figure 2.1) of the two real branches:

- $W_0(x)$: principal branch (upper curve).
- $W_{-1}(x)$: lower branch.

Illustration of the Graph: The graph illustrates multiple branches of the Lambert W function, specifically $\boldsymbol{W_0(x)}, \boldsymbol{W_{\pm 1}(x)}$, and $\boldsymbol{W_{\pm 2}(x)}$, plotted as functions of x. The vertical axis represents the real part of the function, $\mathrm{Re}(W(x))$, while the horizontal axis represents x.

Principal Branch $\boldsymbol{W_0(x)}$: It is defined for $x \geq -1/e$ and passes through $(0,0)$. It increases smoothly and grows slowly for large x.

Lower Branch $\boldsymbol{W_{-1}(x)}$: It is defined for $-1/e \leq x < 0$. It decreases sharply and approaches negative infinity as x approaches 0 from the left. It meets $\boldsymbol{W_0(x)}$ at $(-1/e, -1)$.

Higher Branches $\boldsymbol{W_1(x)}$, and $\boldsymbol{W_{+2}(x)}$: These branches are complex-valued. The graph shows only their real parts. They drop sharply near $x = 0$ and then increase gradually, remaining below the principal branch.

Behavior Near $x = 0$: Non-principal branches tend toward large negative values near $x = 0$, while $\boldsymbol{W_0(x)}$ remains smooth.

Branch Point: At $x = -1/e$, multiple branches meet, showing the multi-valued nature of the function. So, we can conclude that the Lambert W function consists of multiple branches. $\boldsymbol{W_0(x)}$ is the main real branch, while others provide additional solutions, often complex, useful in advanced mathematical applications.

II. Some Other Properties: The following properties of the Lambert W function are also noteworthy:

(a) Fundamental Identity: $W(x) * e^{W(x)} = x$.

(b) Logarithmic Form: For $x > 0$: $W(x) = ln(x) - ln(W(x))$.

(c) Derivative: $W'(x) = \frac{W(x)}{x(1+W(x))}, x \neq 0$.

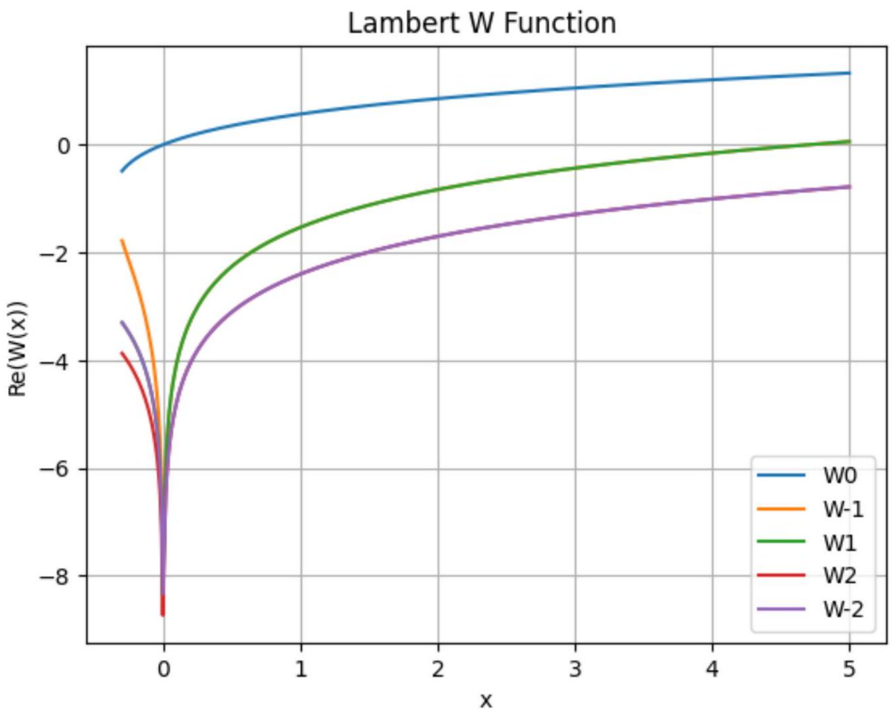

***Figure* 2.1:** *Branches $W_0(x), W_{\pm1}(x)$, and $W_{\pm2}(x)$ of the Lambert W function*

(d) Integral: $\int W(x)dx = xW(x) - x + C$.
(e) Special Values:

$$W_0(0) = 0,$$
$$W_0(e) = 1,$$
$$W\left(-\frac{1}{e}\right) = -1$$

(f) Omega Constant: The Omega constant Ω is defined as: $\Omega = W(1)$.
Numerically: $\Omega \approx 0.567143$.

3 Main Results: Inequalities and Bounds

We now present the inequalities and bounds derived for the Lambert W function and its connections to the Gamma and Psi functions.

3.1 Inequalities for Lambert W Function

3.1.1. For $x \geq e$, the following inequalities hold:
Lower bound:

$$W(x) \geq ln(x) - ln(ln(x)), \tag{3.1}$$

Upper bound:

$$W(x) \leq ln(x). \tag{3.2}$$

Proof. For proof, see Hassani [10], and Hoorfar and Hassani [11]. □

3.1.2. Combining (3.1) and (3.2), we obtain

$$ln(x) - ln(ln(x)) \leq W(x) \leq ln(x). \tag{3.3}$$

Proof. For proof, see Hassani [10], and Hoorfar and Hassani [11]. □

3.1.3. The following refined inequality provided is an approximation for the Lambert W function, $W(x)$, for large values of x :

$$ln(x) - ln(ln(x)) + \left(\frac{ln(ln(x))}{ln(x)}\right) \leq W(x) \leq ln(x) - (1 - ln(ln(x)))/ln(x). \tag{3.4}$$

Proof. A formal proof involves analyzing the asymptotic behavior of $W(x)$ and comparing it to the given bounds, as follows.

Asymptotic Expansion of $\boldsymbol{W(x)}$: For large x, the Lambert W function can be approximated by its asymptotic expansion, with the leading terms given by:

$$W(x) \approx ln(x) - ln(ln(x)) + \frac{ln(ln(x))}{ln(x)} - \frac{(ln(ln(x)))2 - 2ln(ln(x)) + 2}{2(ln(x))^2} + \cdots . \tag{3.5}$$

The upper bound in (3.4) is given by

$$W(x) \leq ln(x) - (1 - ln(ln(x)))/ln(x),$$

which can be written as

$$W(x) \leq ln(x) - \frac{1}{ln(x)} + \frac{ln(ln(x))}{ln(x)} . \tag{3.6}$$

Comparing (3.6) to the asymptotic expansion (3.5), it is observed that the upper bound corresponds to the first few terms of the expansion, with the higher-order terms being negative for sufficiently large x, from which the inequality (3.6) evidently follows. Similarly, the lower bound in (3.4) is given by

$$ln(x) - ln(ln(x)) + \frac{ln(ln(x))}{ln(x)} \leq W(x). \tag{3.7}$$

The lower bound in (3.7) directly matches the first three terms of the asymptotic expansion (3.5) of $W(x)$. This completes the proof of the inequality (3.4). □

3.2 Relationships with the Gamma Function

By exploiting Stirling's approximation and the asymptotics of W, we establish:

$$\Gamma(W(x) + 1) \approx \sqrt{(}2\pi W(x)) \cdot (W(x)/e)^{W(x)}.$$

This gives rise to inequalities of the form:

$$\Gamma(W(x) + 1) < x^{W(x)} < \Gamma(W(x) + 2).$$

3.3 Relationships with the Psi Function

Since $\psi(x) = \frac{d}{dx}(ln\Gamma(x))$, applying it, $W(x)$ yields approximations:

$$\psi(W(x)) \approx ln(W(x)) - 1/(2W(x)).$$

This connects Lambert W with logarithmic asymptotics and allows for inequalities of the form:

$$ln(W(x)) - 1/W(x) < \psi(W(x)) < ln(W(x)) - 1/(2W(x)).$$

3.4 Relationships with Entropy of Gamma-Distributed Variables

If a random variable X follows a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$, its differential entropy $h(X)$ is given by (Johnson *et al.* [12]),

$$h(X) = \alpha + ln(\beta) + ln\Gamma(\alpha) + (1 - \alpha)\psi(\alpha) \tag{3.8}$$

Special Case (Exponential Distribution): For $\alpha = 1$, the gamma distribution becomes an exponential distribution, and the entropy simplifies to

$$h(X) = 1 + ln(\beta).$$

Now, using the inequality (3.1) in (3.8), we obtain

$$\begin{aligned} h(X) &= \alpha + ln(\beta) + ln\Gamma(\alpha) + (1 - \alpha)\psi(\alpha) \\ &\leq \alpha + W(\beta) + ln(ln(\beta) \cdot \Gamma(\alpha)) + (1 - \alpha)\psi(\alpha) \end{aligned} \tag{3.9}$$

Further, the following inequality is a well-known result in the theory of special functions, Alzer [1],

$$ln(x) - \frac{1}{x} < \psi(x) < ln(x), x > 0. \tag{3.10}$$

which, on using in (3.9), we obtain

$$\begin{aligned} \alpha + ln(ln(\beta) \cdot \Gamma(\alpha))(1 - \alpha)\left[ln(\alpha) - \frac{1}{\alpha}\right] &< h(X) \\ &< \alpha + W(\beta) + ln(ln(\beta) \cdot \Gamma(\alpha)) + (1 - \alpha)ln(\alpha). \end{aligned} \tag{3.11}$$

3.5 Relationships to Entropy Power of Gamma Distribution

When random variables follow a Gamma distribution, their entropy power (or "effective variance") is characterized by the Γ function and its logarithmic derivative, the ψ function (digamma function). For a random variable X with differential entropy $h(X)$, the entropy power $N(X)$ is defined as the unique value satisfying

$$N(X) = \frac{1}{2\pi e} * e^{2h(X)}, \tag{3.12}$$

see Kapur [13].

Also, for a random variable $X \sim \text{Gamma}(\alpha,\ \text{b})$, (shape α, rate $\text{b} = 1/\beta$), where β is the scale parameter, the differential entropy is given by

$$h(X) = \alpha - ln(b) + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha). \tag{3.13}$$

Thus, combining (3.11) and (3.12), the entropy power of gamma distribution is obtained as follows:

$$N(X) = \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha)]}}{2\pi e b^2}, \tag{3.14}$$

from which, on using (3.10), we obtain

$$N(X) = \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)\psi(\alpha)]}}{2\pi e b^2} < \frac{e^{2[\alpha + ln\Gamma(\alpha) + (1-\alpha)ln(\alpha)]}}{2\pi e b^2}. \tag{3.15}$$

3.6 Relationships with Cramer-Rao Inequality

The Cramér-Rao inequality provides a lower bound on the variance of an estimator $\hat{\theta}$ of a parameter θ, defined as

$$\text{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)},$$

where $I(\theta)$ is the Fisher information, Rohatgi and Saleh [15].

As an example, for a sample size of n from a gamma distribution $\Gamma(\alpha, \beta)$ with shape α and rate β, we have

A. Cramer-Rao Lower Bound for α :

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi'(\alpha)}, \text{ or } \text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)}, \tag{3.16}$$

where $\psi(\alpha) = \frac{dln\Gamma(\alpha)}{d\alpha}$ denotes the Digamma function, and $\psi_1(\alpha) = \psi'(\alpha)$ denotes the Trigamma function.

Further, using the following inequalities of Alzer [1],

$$\frac{1}{\psi_1(x)} > \frac{x^2}{1+x}, x > 0$$

and

$$\frac{1}{\psi_1(x)} > \frac{1}{e^{1/x} - 1}, x > 0,$$

respectively in (3.15), we obtain the following sharp bounds

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)} > \frac{\alpha^2}{n(1+\alpha)}$$

and

$$\text{Var}(\hat{\alpha}) \geq \frac{1}{n\psi_1(\alpha)} > \frac{1}{n\left(e^{\frac{1}{\alpha}} - 1\right)}.$$

B. Cramer-Rao Lower Bound for β :

$$\text{Var}(\hat{\beta}) \geq \frac{\beta^2}{n\alpha}.$$

3.7 Expressing Cramer-Rao Inequality in Terms of the Lambert W Function

The Lambert W function $W(x)$ is the inverse of $f(w) = w * e^w$. It appears in Cramer-Rao bounds when the Fisher Information involves a transcendental equation, often arising in maximum likelihood estimations for distributions with exponential components (for example, in finding the value of θ that maximizes the log-likelihood function $L(\theta)$).

For example, assume a specific problem where the Fisher information has been derived as $I(\theta) = a * e^{b\theta}$ for constants a and b. To find the specific variance when the estimator is a function of a Lambert W-related variable, we substitute $I(\theta) = a * e^{b\theta}$ so that we obtain

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{a * e^{b\theta}} = \frac{1}{a} e^{-b\theta}.$$

If the parameter θ itself is determined by a maximum likelihood estimation involving an exponential, for example, $\theta = ln($ something $)$, this can be represented using $W(x)$.

In situations where $\psi(x)$ is approximated by $ln(x)$ (for large x), or when analyzing the asymptotic behavior of the estimator, we can use the identity that solutions to $x * e^x = y$ allow us to invert logarithmic equations. If $I(\theta)$ is derived to be a function given by

$$I(\theta) = \frac{\theta}{1 + W(\theta)}$$

then the Cramer-Rao lower bound (CRLB) is obtained as follows:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1 + W(\theta)}{\theta}.$$

Hence, in view of the above, it follows that the general Cramer-Rao inequality in terms of the Psi and Lambert W functions can be expressed as

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{n\psi_1(\theta)} \approx \frac{1}{nI(\mathrm{LambertW}(\theta))}.$$

4 Applications and Results

The following Figures 4.1-4.3 demonstrate the accuracy of bounds and approximations.

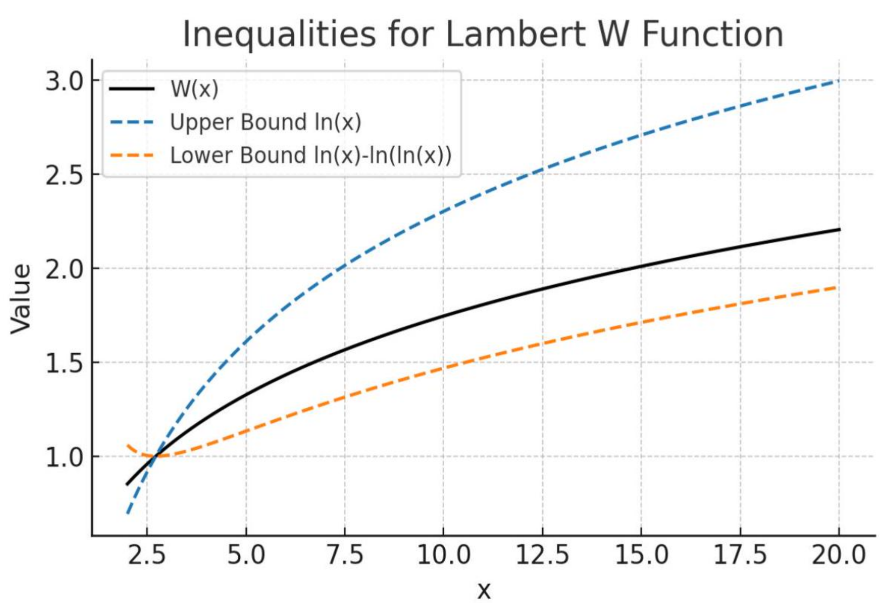

Figure* 4.1*: *Comparison of Lambert W with upper and lower bounds.*

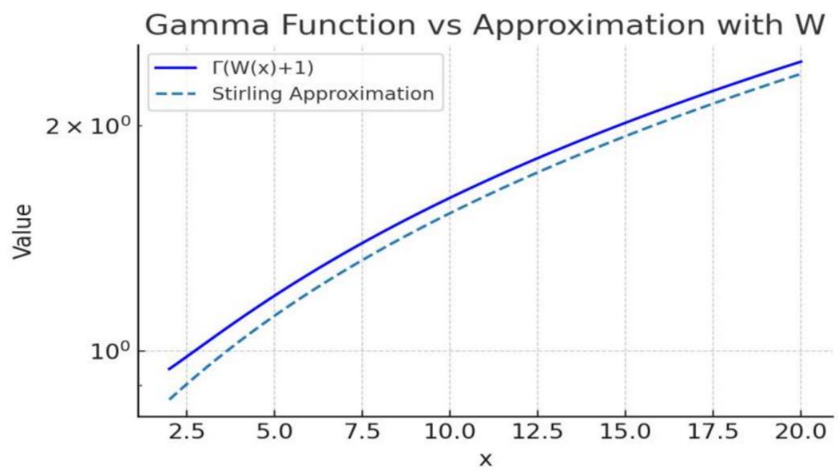

***Figure* 4.2:** *Comparison of* $\Gamma(W(x)+1)$ *with Stirling's approximation.*

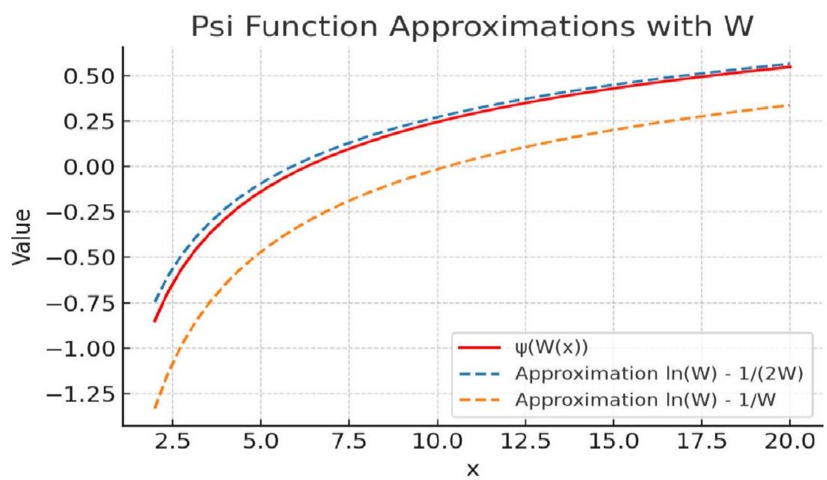

***Figure* 4.3:** *Comparison of* $\psi(W(x))$ *with logarithmic approximation.*

4.1 Lambert W Bounds:

As shown in Figure 4.1, the Lambert W function is compared with its upper and lower bounds. The approximation $W(x) \approx ln(x) - ln(ln(x))$ closely follows the actual function, while $ln(x)$ slightly overestimates and $ln(x) - ln(ln(x))$ slightly underestimates.

Discussion: Figure 4.1 highlights the effectiveness of logarithmic bounds for computationally efficient approximations of $W(x)$.

4.2 Relationships with the Gamma Function:

Figure 4.2 illustrates $\Gamma(W(x)+1)$ versus Stirling's approximation. Stirling's formula provides a very accurate estimate for larger x.

Discussion: Figure 4.2 confirms that Stirling's approximation can simplify calculations involving Gamma functions of $W(x)$.

4.3 Relationships with the Psi Function:

Figure 4.3 compares $\psi(W(x))$ with the logarithmic approximation $ln(W(x)) - 1/(2W(x))$. The approximation is nearly indistinguishable for $x \geq 1$.

Discussion: Figure 4.3 confirms the tightness of the approximation for analytical and computational purposes.

5 Conclusion and Future Directions

The analysis demonstrates that classical logarithmic bounds for Lambert W function are highly effective (Figure 4.1). Stirling's approximation accurately estimates $\Gamma(W(x)+1)$ (Figure 4.2), and the simple logarithmic formula approximates $\psi(W(x))$ well (Figure 4.3). These results collectively support efficient analytical and computational applications involving Lambert W, Gamma, and Psi functions. Future work may explore tighter inequalities for Lambert W function, extend the approximations to complex arguments, and analyze applications in advanced statistical models, information theory, and combinatorial mathematics. Integration with numerical software and symbolic computation could enhance practical utility.

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