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GRAPH THEORY FUNDAMENTALS AND APPLICATIONS

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Abstract

Graphs are mathematical structures composed of nodes and edges representing entity relationships. They help analyze the network properties, and model real-world phenomena to understand the complex and tricky systems. There are several applications of graphs in real-life problems. This article deals with the applications and modeling of graph theory in computer science, network biology, and chemical processes. We can also use graph theory to study the structural properties of network topologies such as star and wheel graphs, and Peterson graphs. It can also be used to learn the graph operations that can help study the infectious diseases network like COVID-19. Also, it can study network optimization by finding the optimal cost, optimal route, shortest distance, and the study of spanning networks. This is a fundamental article specifically for beginners.

2020 Mathematical Sciences Classification: 05Cxx, 05C85, 05C90

Keywords and Phrases: Graphs, networks, adjacency matrix, modelling, complex systems.

1 Introduction

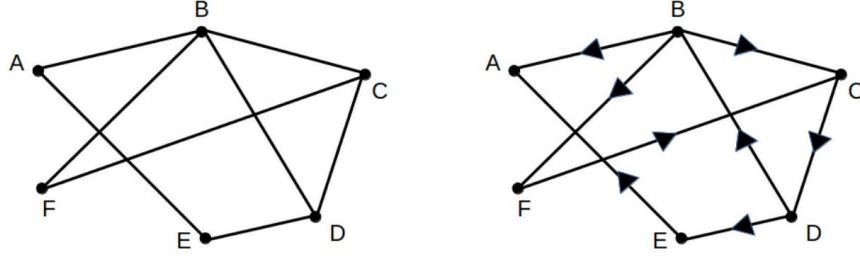
A famous Knigsberg problem was solved by a Swiss mathematician, Leonhard Euler in 1736 and since then he became the father of Graph Theory. In 1847, the theory of trees was introduced by Kirchhoff for solving a system of linear equations. Graphs are mathematical structures composed of nodes and edges representing entity relationships [5]. These nodes and edges can be used to create a mathematical model for solving complex and tricky systems [12]. They are also used to represent multiple types of networks in different domains of science like communication networks, biological networks [7], computer networks, etc. In the next section, we introduce the preliminaries of graph theory that are general and can be used in any type of modeling. In sections 3 and 4, we introduce a new example of graph theoretical algorithms called Breadth First Search and Depth First Search. In section 5 we've explained a basic example of knights moves on a 3×3 grid using graph modeling and conclude with a discussion of the introduced concepts.

2 Preliminaries

We discuss here the fundamental definitions of graphs in the section. These definitions lay the foundation of what we discuss in the further sections.

2.1 Graph

A graph G is a diagrammatic representation of showing relationships/connections between entities. Entities are denoted using nodes, V and relationships among them are depicted using edges, E between the nodes. Mathematically, $G := (V, E)$ where V and E are a finite set of nodes and edges. It is important to note that V must be non-empty. E in G are of two types, directed and undirected. Fig. 2.1(a) shows an undirected graph with 6 nodes and 8 edges and Fig. 2.1(b) shows a directed graph with 6 nodes and 8 edges.



Example of (a) An Undirected Graph

(b) Directed Graph

Figure 2.1: Example of (a) An Undirected Graph (b) Directed Graph

2.2 Loop

A loop is an edge of a graph whose starting and ending nodes are the same. While considering the definition of a simple graph, a loop is not allowed.

2.3 Degree

A degree is associated with the nodes of a graph. For an undirected graph, the number of edges associated with a vertex is called the degree of that vertex. But in a directed graph, every vertex has an in-degree and an out-degree which is defined as the number of incoming edges and the number of outgoing edges to any vertex. For example, vertex B in Fig. 2.1(a) has degree 4 and vertex B in Fig. 2.1(b) has indegree 1 and out-degree 3.

2.4 Handshaking lemma

The Handshaking lemma states that the sum of the degrees of all the vertices in an undirected graph is always equal to twice the number of edges in the graph.

2.5 Adjacency Representation

We represent a simple graph using an Adjacency Matrix. The matrix is such that all of its entries are Boolean and order is $n \times n$ if n is the number of vertices of the graph. We assign value 1 to a_{ij} entry of an adjacency matrix A if there is an edge between the i^{th} and j^{th} node of the graph. For eg. if we represent the graphs in Fig. 2.1. Using the Adjacency matrix then they can be represented as follows.:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

represents adjacency matrix for undirected graph (see Fig 2.1(a)) and

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

represents the adjacency matrix for the directed graph (see Fig 2.1(b)).

2.6 Walk

In a graph, the walk is defined as a sequence containing edges and vertices alternatively. It is like we are walking on a graph. In walk, vertices' and edges can be repeat any number of times. For eg., $C \rightarrow B \rightarrow F \rightarrow C \rightarrow D \rightarrow B \rightarrow C$ is a walk of Fig. 2.1(a).

2.7 Trail

The trail is a walk such that we move on an edge only once. For eg., In Fig. 2.1(a) $C \rightarrow B \rightarrow F \rightarrow C \rightarrow D$ is an open trail while $A \rightarrow B \rightarrow D \rightarrow E \rightarrow A$ is a closed trail. A closed trail is also called a circuit.

2.8 Path

A Path in a graph is defined as a walk in which we neither repeat a vertex nor an edge. For eg., $B \rightarrow F \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ is a path in Fig. 2.1(a).

2.9 Tree

We call a connected graph a tree when there is no cycle present in an undirected graph. In other words, we say when any two vertices are connected by a single edge only in a graph then it is a tree.

2.10 Eulerian Graph

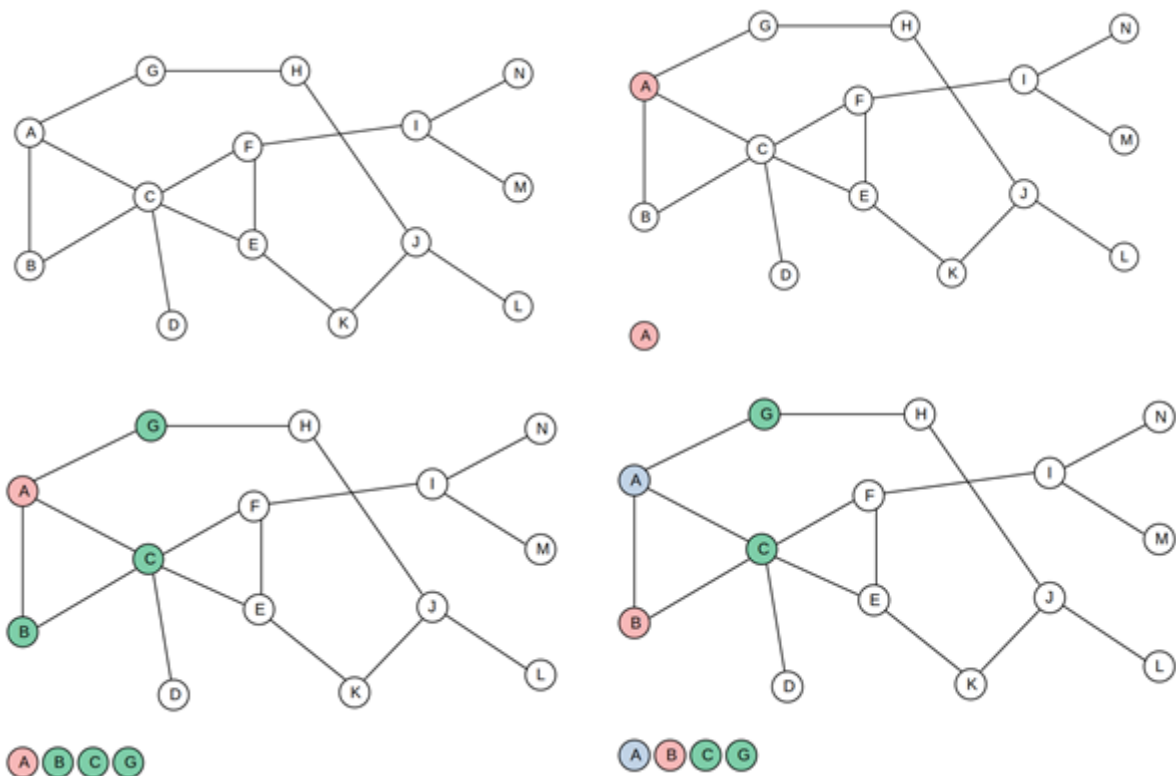
A connected graph on which we can walk on all the edges without repeating any edge and come back to the same vertex where we started then that type of graph is called the Eulerian graph. For eg., Fig. 2.1(a) is not an Eulerian Graph.

2.11 Hamiltonian Graph

A connected graph on which we can walk on all the vertices without repeating any vertex and come back to the same vertex where we started then that type of graph is called a Hamiltonian graph. For e.g., we have Hamiltonian circuit $B \rightarrow F \rightarrow C \rightarrow D \rightarrow E \rightarrow A \rightarrow B$ in Fig. 2.1(a).

3 Breadth First Search Algorithm

Breadth First search is a graph traversal technique that is used to explore nodes and edges of a graph. It is used as a fundamental algorithm for many tasks like shortest path finding, cycle detection, topological sorting, etc. It begins at a vertex and covers all the nodes by covering the neighborhood of nodes at each level. An example is shown below to explain the steps of the algorithm.



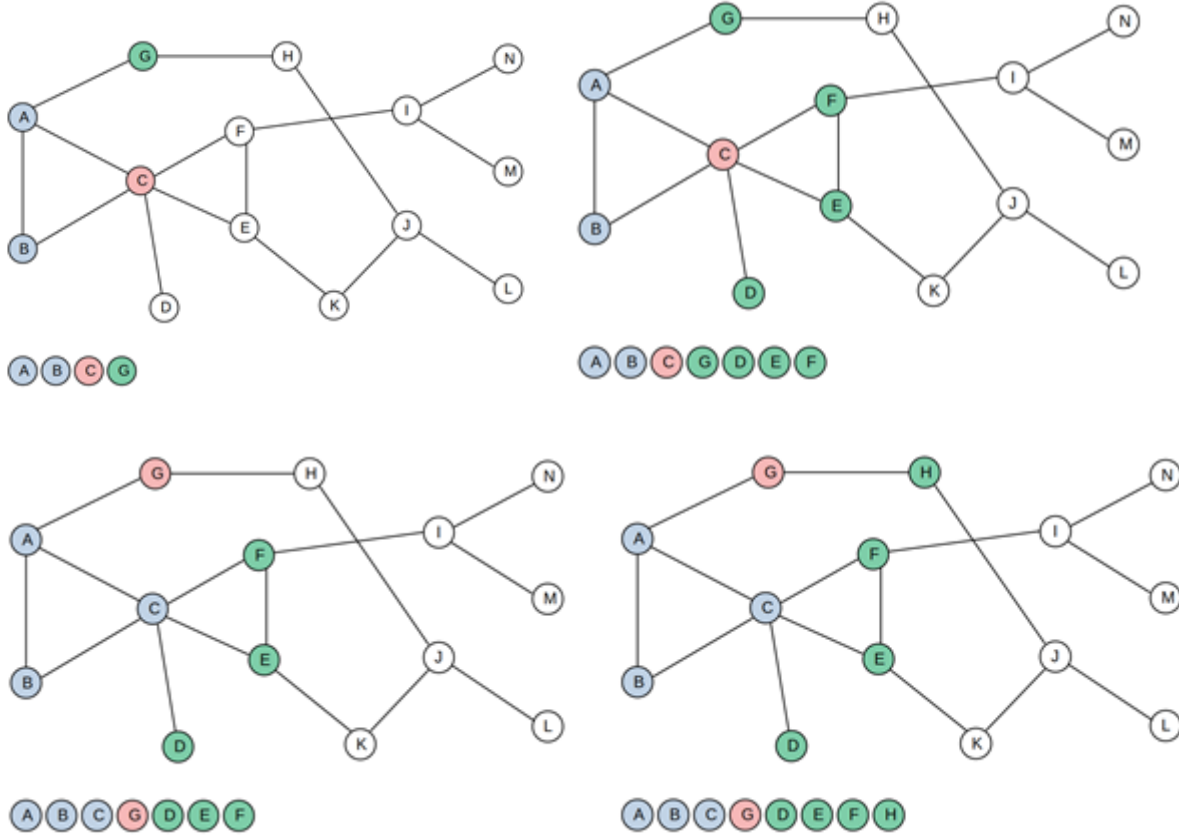


Figure 3.1: (a) (b) (c) (d) (e) (f) (g) (h) An illustrative example of Depth First Search Algorithm

In Fig. 3.1, an example of a graph with 14 vertices and 16 edges is considered. We apply the Breadth First Search on the graph and a few initial steps have been explained in the figure. The initial vertex chosen to start from is shown in red color. Suppose, we start with node A and look for its neighbors that are depicted using the green color of the nodes. After covering all the neighbors of a node and adding them to the queue we color it with blue, move on to the next vertex of the queue, and repeat the process using the next red vertex. Proceeding in the same way, we have the following sequence of vertices as our result of the algorithm.



Figure 3.2: Solution to the Breadth First Search of the example shown in Fig. 3.1.

4 Depth First Search

It is another graph traversal technique used to explore every node of a graph or a tree. We start with an initial vertex and explore each vertex as long as possible along every branch and then move backward after reaching the end of the branch. We illustrate the algorithm in Fig. 4.1. with the help of the same graph used to explain the Breadth First Search Algorithm.

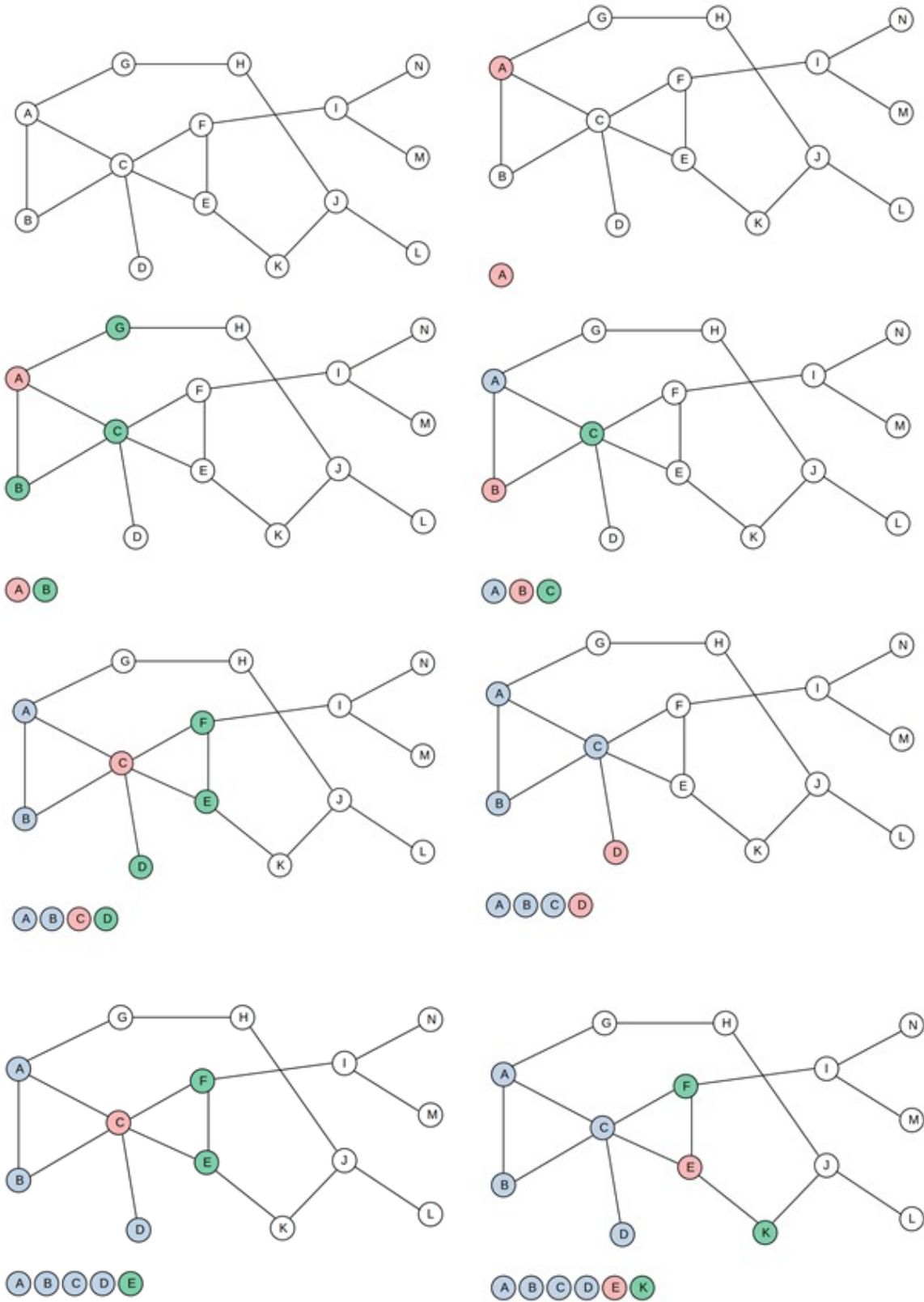


Figure 4.1: (a) (b) (c) (d) (e) (f) (g) (h) An illustrative example of Depth First Search Algorithm

The algorithm continues and when all the vertices are covered then we have the following sequence of vertices shown in Fig. 4.2.



Figure 4.2: Solution to the Depth First Search of the example shown in Fig. 4.1.

5 Applications of Graphs

Graph theory is used in many ways [3]. Theoretically, while studying communication networks, biological networks, computer networks, etc. we use graphs to study the topological properties of these networks. In real-life problems [10], we use graph theory, like in Airline scheduling, and directions in Google Maps, Sudoku's puzzles, Social Media marketing, etc. These problems use different graph theory topics like graph coloring, shortest path problems, etc [1,6].

We illustrate the application of graphs using an example. A very interesting example is studying the knight's moves on a chess board. Consider we have a 3×3 grid and we wish to understand the minimum number of moves required from the initial position to the final position of the 4 knights (2 white and 2 black) as shown in the Fig. 5.1.

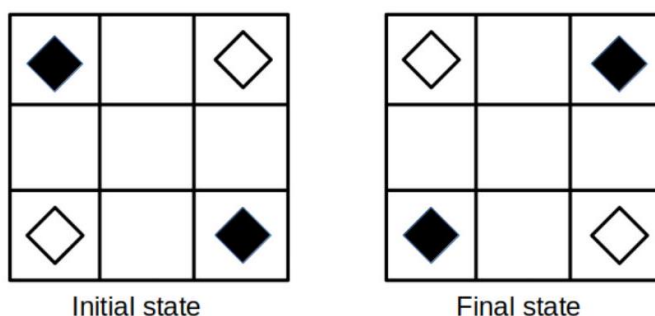


Figure 5.1: Example of knight moves on a 3×3 grid

First question to consider is, Is this final state possible? If yes, then what is the minimum number of steps required to reach final state? Solution to these questions can be hit and trial method but this situation can be tricky when we have a larger grid. So we use graph to make a understand the positions of the knights in the grid. Let us number these tiles from 1-9 on the grid as shown in the Fig. 5.2.

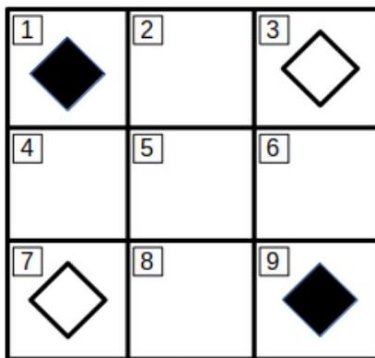


Figure 5.2: Numbering the tiles of the 3×3 grid

We make a graph of the moves that these 4 knights can move and understand the dynamics using the graph. This can be illustrated in the Fig. 5.3. We can clearly see the connections between these vertices that depicts the possible moves of the four knights on the grid.

To understand the possible moves and their minimum number of steps. We count the steps on the graph and that will be our required solution. Fig. 5.4. Depicts that minimum number of moves required to reach from initial to final state is 8 . We can model more such examples

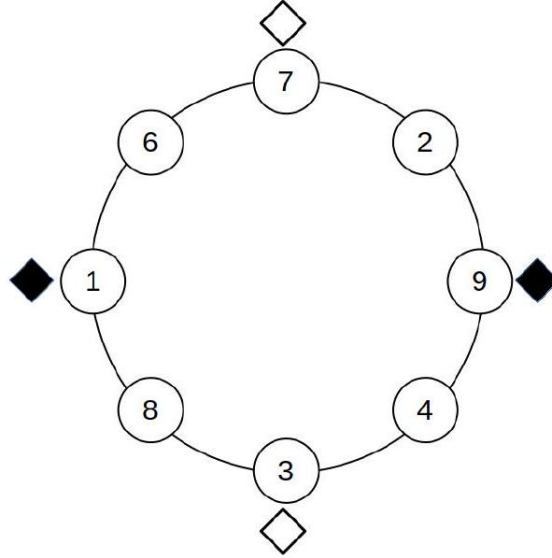


Figure 5.3: Graph of all the possible knights move on the grid

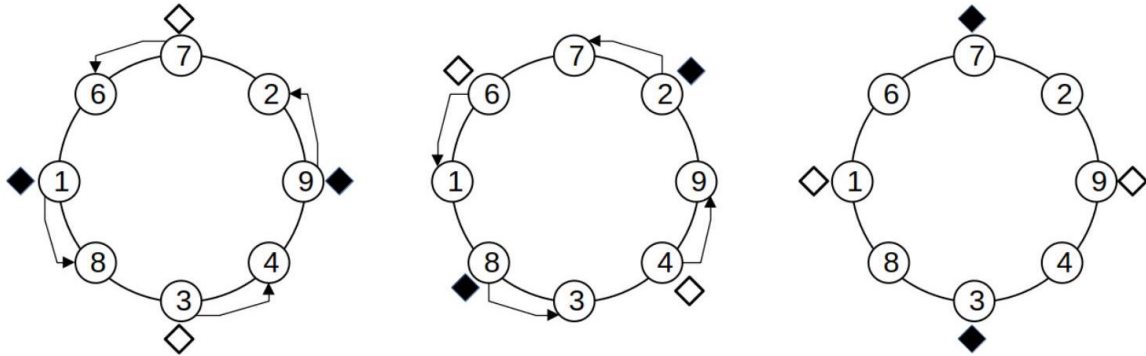


Figure 5.4: Knights move to reach from initial to final state

6 Conclusion and Discussion

Graph theory has gained a lot of attention due to its applicability and modeling power. It is vastly used in many areas of sciences. Types of graph like bipartite graph, complete graph, etc. have proven to be useful in different applications. Extension to graph theory have been studied extensively like Petri Nets [1, 2, 4, 5, 8, 9, 11, 13-19] and Signed Graphs. Petri Net is the modeling tool used to study the behavior of discrete event type dynamic systems. Signed graphs extends the modeling power of graphs by introducing the positive and negative sign to the edges of the graph. We have discussed the fundamentals of graph in this paper and readers are motivated to apply these concepts in their area of study.

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OPTIMIZATION INVENTORY MODEL FOR DETERIORATING ITEMS WITH STOCK DEPENDENT DEMAND AND CARBON TAX UNDER FUZZY ENVIRONMENTS

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Abstract

Inventory control models provide a structured framework for optimizing replenishment decisions under operational and economic constraints. To address the uncertainty commonly observed in real-world systems, this study proposes a fuzzy inventory control model that incorporates parameter imprecision within a fuzzy framework. The model considers deteriorating items with stock-dependent demand, allowable shortages, and complete backordering. Under the fuzzy environment, key parameters are represented as trapezoidal fuzzy numbers and defuzzified using the centroid method. A total cost function is developed to determine the optimal order quantity, the time at which inventory becomes zero, and the corresponding minimum cost. Numerical illustrations are presented for both fuzzy and crisp models, and the convexity of the cost function is verified graphically. Sensitivity analysis is conducted to examine the impact of variations in fuzzy parameters on optimal inventory decisions. The results indicate that the optimal total cost under the fuzzy environment is lower than that in the crisp case, demonstrating that incorporating fuzziness enables retailers to better capture uncertainty and improve inventory planning.

2020 Mathematical Sciences Classification: 90B05, 91B55, 03E72.

Keywords and Phrases: Inventory model; deteriorating product; carbon tax policy; fuzzy set theory.

1 Introduction

Inventory control models form the foundation of effective supply chain management by determining optimal policies for ordering and replenishment. Their importance lies in ensuring product availability, minimizing operational costs, and enhancing overall system efficiency. In traditional inventory models, the majority of solutions are developed in deterministic environments. However, market volatility prevents them from accurately reflecting real-world problems. But this problem could be resolved with the help of fuzzy set theory introduced by Zadeh [39, 1965]. A significant research has been made in the field of fuzzy inventory control models. Vujovesvic *et al.* [31] developed an inventory control model in which the various costs like holding cost, ordering cost or more, involved in economic order quantity formula are assumed to be fuzzy. A fuzzy inventory model with or without backorder for fuzzy order quantity with trapezoid fuzzy number was developed by Yao *et al.* [34] while Lee and Yao [15] examined an economic order quantity model in fuzzy sense for inventory without backordering model. Kao and Hsu [12] discussed a single-period inventory model with fuzzy demand. Wee [32] gave a replenishment policy for items with a price-dependent demand and a varying rate of deterioration. Fuzzy inventory control model with and without backordering along with various fuzzy parameters were developed by Yao *et al.* [34, 35, 36, 37]. Syed [23] discussed a fuzzy inventory control model without shortages using signed distance method. A periodic review of inventory control model involving fuzzy expected demand and fuzzy backorder rate was given by Lin [16]. Singh and Singh [24] discussed fuzzy inventory model for finite rate of replenishment using signed distance method while Ritha [22] discussed fuzzy inventory control model for imperfect quality items with shortages. De [4]

gave a fuzzy order quantity model with fuzzy shortage quantity and fuzzy promotional index. Samal [25] optimized variable demand fuzzy economic order quantity inventory models with and without backordering. Priyam [19] gave continuous review of inventory model with controllable lead time, lost sales rate and order processing cost when the received quantity is uncertain. An *EOQ* model with backlogging under fuzzy environment was developed by De and Sana [5]. Kazemi *et al.* [13] incorporated human learning into a fuzzy *EOQ* inventory model with backorders. An *EOQ* model under fuzzy reasoning was solved by De [6].

Deteriorating items are characterized by a gradual decline in usability or value during storage, driven by factors such as spoilage, evaporation, or obsolescence. Effective management of such items necessitates specialized inventory models that incorporate deterioration rates to ensure timely replenishment and cost efficiency. Traditional inventory models can find it difficult to effectively control the uncertainties associated with deteriorating items, such as fluctuating demand, undetermined lead times, and variable degradation rates. Here fuzziness allows a more versatile and accurate approach. Various scholars have conducted research in the field of fuzzy inventory control model for deteriorating items. A fuzzy *EOQ* model for time-dependent deteriorating items and time-dependent demand with shortages was discussed by Saha [26]. Weibull deteriorating items with a time-dependent exponential demand rate are the subject of this paper's infinite time horizon fuzzy Economic Order Quantity (*EOQ*) models, where the deterioration parameter is a triangular fuzzy number. Halim *et al.* [10] developed fuzzy economic order quantity model for deteriorating items with stochastic demand, partial backlogging and fuzzy deterioration rate. Additionally, the suggested model is expanded to include the scenario in which a fuzzy value is assumed for the partial backlog factor. By using the signed distance approach to defuzzify the total cost, the minimum total cost and ideal order quantity are determined. Kumar and Rajput [14] gave fuzzy inventory model for deteriorating items with time dependent demand and partial backlogging. It is expected that the backlog rate of unmet demand decreases exponentially with waiting time. It is assumed that the demand, degradation, and backlog rates are triangular fuzzy values and goal is to defuzzify the total profit function using the centroid and signed distance methods. A fuzzy inventory model for deteriorating items with exponentially decreasing demand under fuzzified cost and partial backlogging was given by Mohanty and Tripathy [17]. This model additionally takes degradation and fuzzy-based cost components (such as holding costs and shortage costs) into account. It is assumed that all associated expenses are trapezoidal fuzzy numbers and defuzzification is accomplished using the gradient mean integration approach and signed distance. Shekarian *et al.* [27] explored an economic order quantity model considering different holding costs for deteriorating items subject to fuzziness. This study develops a fuzzy model that uses triangular fuzzy numbers (*TFNs*) to describe the fuzzified parameters and decision variables. Fuzzy arithmetic operations are used to determine the total profit per unit of time, and the graded mean integration value (*GMIV*) method is used to defuzzify the results. Indrajit *et al.* [11] discussed a fuzzy inventory model for deteriorating items with stock dependent demand rate in which the triangular fuzzy numbers represent the demand, holding cost, deterioration cost, and deterioration rate and the total cost function is defuzzified using the centroid method, signed distance method, and graded mean representation. A fuzzy inventory model for a deteriorating item with variable demand, permissible delay in payments and partial backlogging with shortage that follows inventory (*SFI*) policy was given by Shaikh *et al.* [28]. Soni *et al.* [29] *EOQ* examined an *EOQ* model of deteriorating items for fuzzy demand and learning in fuzziness with finite horizon in which an algorithm is created to calculate the number of replenishment once the model is expanded by including learning in fuzziness.

With increasing awareness of environmental impacts, incorporating carbon emission policies into inventory control models has become essential. It allows firms to align inventory planning with modern environmental standards. Arora *et al.* [1] presented a re-manufacturing inventory model to control the carbon emission using cap-and-trade regulation with the hexagonal fuzzy number. This study explored the fuzzy model of economic production for both remanufacturing and recycling with uncertain cost parameters under the cap-and-trade rule to control the carbon emission due to different modes of transportation. Due to imprecise cost parameters, the hexagonal fuzzy numbers are set to fuzzify the overall cost, which leads to correct decisions in a more confident way. The result is defuzzified by using graded mean integration. De *et al.* [7] explored carbon emission issues over production manufacturing system with the context of joint inventory control and sustainable trade credit financing for deteriorating items in a supplier-retailer-customer model under volumetric fuzzy system. Mahato *et al.* [18] devised inventory models for deteriorating items with fixed lifetime, partial backordering and dynamic demand under two different scenarios (a) carbon cap and trade policy (b) carbon tax policy. Bhardwaj *et al.* [2] developed an innovative sustainable fuzzy

inventory approach that integrates green considerations, carbon emissions, and a bi-phase credit policy.

In this article, a fuzzy inventory control model for deteriorating items with stock-dependent demand and complete backlogging is presented, both fuzzy and crisp scenarios are examined through numerical analysis, and managerial insights are derived through sensitivity analysis.

1.1 Motivation and Contribution

Inventory control models generally involve uncertain or imprecise data for demand, deterioration, and costs. This study addresses such uncertainty by modeling all functions and parameters using fuzzy mathematics, offering a robust and realistic framework for decision making in inventory. Table 1.1 compares selected inventory control studies, outlining their features, assumptions, and methodologies, and highlighting existing gaps. Building on this, the present study introduces a model that simultaneously integrates several distinctive features not previously addressed together. The main contributions are as follows:

- A key distinction of the proposed model is its treatment of demand during stock-out. Unlike conventional models, which assume demand drops to zero or equals the linear demand intercept, the present model allows demand to transition to an independent constant level. This flexibility captures realistic market behavior where demand persists regardless of inventory, extending the applicability of stock-dependent frameworks.
- All model parameters are assumed to be fuzzy parameters, distinguishing the proposed framework from most existing studies that consider fuzziness only in selected parameters.
- Trapezoidal fuzzy numbers are employed to represent parameter uncertainty, ensuring analytical tractability while realistically capturing imprecision.
- The model integrates a carbon emission policy in the form of a carbon tax, enabling the simultaneous assessment of economic and environmental impacts of inventory decisions.

Table 1.1: Overview of some of the related study

Author	Year	Demand	Carbon emission policies	Fuzzy Logic
Xu <i>et al.</i> [33]	2019	Time dependent	Yes	No
Yadav <i>et al.</i> [38]	2021	Carbon emission dependent	Yes	No
Sahoo and Acharya [30]	2022	Time dependent	Yes	Yes
Parida <i>et al.</i> [20]	2023	Time dependent	Yes	Yes
Pakhira <i>et al.</i> [21]	2024	Constant	Yes	Yes
Dave <i>et al.</i> [8]	2025	Time dependent	No	Yes
Das <i>et al.</i> [9]	2025	Price dependent	No	Yes
This paper	2026	Stock dependent	Yes	Yes

1.2 Structural outline of paper

The paper is organized as follows. All the notations used in the paper are given in Section 2 followed by the assumptions made for the mathematical model given in Section 3. Section 4 provides the formulation of mathematical model. The solution of the mathematical model is presented in Section 5. Section 6 of this paper deals with the numerical computation part of the paper. Sensitivity analysis is done in Section 7. Finally the conclusion is discussed in Section 8.

2 Preliminaries:

Definition 2.1 ([3]). *Fuzzy Set:* Fuzzy set A on universe X is defined by function $\mu_A(x)$ called the membership function of set A . Membership function basically assigns a membership value to the elements from the universal set. It determines the degree of membership of each element in fuzzy set. The membership function is a real valued function which is defined as follows:- $\mu_A(x) : x \rightarrow [0, 1]$ Where,

$$\mu_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \\ (0, 1) & x \text{ is partly in } A. \end{cases}$$

Definition 2.2 ([3]). *Trapezoidal fuzzy numbers:* Let $\tilde{A}$ be a trapezoidal fuzzy number which is represented as (a, b, c, d) where a, b, c, d are real numbers and $a \leq b \leq c \leq d$. Then the membership function $\mu_A(x)$ will

be given as

$$\mu_A(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x \leq b \\ 1, & b < x \leq c \\ \frac{d-x}{d-c}, & c < x \leq d \\ 0, & x > d. \end{cases}$$

Definition 2.3 ([40]). *Centroid Method: The centroid (center-of-area) defuzzification of the trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ yields the crisp or defuzzified value*

$$A^* = \frac{a + 2b + 2c + d}{6}.$$

Defuzzification is the systematic procedure used to transform a fuzzy set, typically obtained from a fuzzy inference process, into a corresponding crisp numerical value. This step enables the interpretation of fuzzy conclusions in precise, actionable terms and is essential for implementing fuzzy models in real-world decision-making environments.

3 Notations and Assumptions:

3.1 Notations

The notations which are used for different variables are listed in Table 3.1.

Table 3.1: List of parameters with their description and symbols

Description of variables	Symbol of the variables
Inventory level at time t	$I(t)$
Fuzzy inventory level at time t	$\tilde{I}(t)$
Ordering quantity per order	q_d
The quantity of products that deteriorate during the interval $[0, t_0]$	P_d
Fuzzy deterioration rate	θ
The fuzzy integrated market demand	$\tilde{D}[t, \tilde{I}(t)]$
Fuzzy base demand rate when inventory is available	$\tilde{m}$
Fuzzy inventory dependent demand coefficient	$\tilde{n}$
Fuzzy demand rate	$\tilde{D}_0$
The point of time where the inventory becomes zero	t_0
Cycle length of inventory	T
Fuzzy total cost function	$\tilde{TC}$
Fuzzy unit buying cost	$\tilde{a}$
Fuzzy per unit inventory holding cost per unit time	$\tilde{b}$
Fuzzy per unit shortage cost	$\tilde{c}$
Fuzzy per unit handling cost of the deteriorated product	$\tilde{d}$
Fuzzy fixed cost per order	$\tilde{K}$
Fixed carbon emissions per order	A_e
Per unit carbon emission of orders	c_1^e
Per unit carbon emission for inventory per unit time	h_1^e
Fuzzy carbon tax under the carbon tax policy(rupees/per unit carbon emission)	$\tilde{\lambda}$

3.2 Assumptions

The assumptions for the proposed mathematical model are stated as follows:

- The lead time is assumed to be zero; that is, the time elapsed between placing an order and receiving the ordered items is negligible.
- The total length of the inventory cycle (or planning horizon) is denoted by T .
- The demand rate is assumed to be stock-dependent during the positive-inventory period and is given by $m + nI(t)$, where m represents the base demand rate when inventory is available, reflecting the fundamental level of customer demand independent of stock visibility, and n denotes the inventory-dependent demand coefficient, which measures the rate of change in demand with respect to inventory

level $I(t)$. Once the inventory level reaches zero, the demand becomes constant at the base rate D_0 . This assumption captures both the influence of available inventory on customer demand and the persistence of baseline demand during stock-out periods.

- Shortages are permitted and are completely backlogged.
- A single type of deteriorating item is considered in the inventory control model. The item is assumed to be in its optimal condition initially and begins to deteriorate over time until t_0 . The deterioration rate is assumed to be constant and is denoted by θ .
- Carbon emissions are assumed to arise from both inventory holding and transportation activities. To mitigate environmental impact, a carbon tax policy is imposed, whereby a cost proportional to the amount of carbon emitted is charged. This policy incentivizes firms to optimize inventory and transportation decisions so as to reduce both operational costs and carbon emissions.
- The uncertain parameters in the model are represented using fuzzy set theory. In particular, trapezoidal fuzzy numbers are employed to effectively capture the vagueness and imprecision inherent in real-world inventory systems.

4 Formulation of Mathematical Model:

In this model, during the initial interval $[0, t_0)$, the demand is assumed to be dependent on the inventory level and is expressed as $m + nI(t)$, where m denotes the base demand rate when inventory is available, n is the inventory-dependent demand coefficient, and $I(t)$ represents the inventory level at time t . Once the inventory is depleted, the demand becomes constant at the rate D_0 and is completely backlogged until the end of the cycle at time T . The integrated demand function is denoted by $D[t, I(t)]$ and is defined as follows:

$$D[t, I(t)] = \begin{cases} m + nI(t), & 0 \leq t < t_0 \\ D_0, & t_0 \leq t < T. \end{cases}$$

The items considered in this model are of a deteriorating nature and are assumed to deteriorate at a constant rate θ . The maximum inventory level at time $t = 0$ is denoted by S_0 , which gradually decreases and becomes zero at time t_0 due to the combined effects of demand and deterioration. Shortages are permitted and are completely backlogged during the interval $[t_0, T]$. At time T , the next replenishment occurs, restoring the inventory level to S_0 .

In real-world situations, it is often difficult to describe system parameters precisely using exact numerical values alone. Therefore, the above model is extended using fuzzy set theory, wherein the parameters m , n , D_0 , and θ are treated as imprecise. For instance, a decision-maker may be unable to specify an exact value for the deterioration rate; however, based on market studies, an approximate range within which the deterioration rate typically lies may be available. To capture such vagueness and uncertainty, the model represents these parameters using trapezoidal fuzzy numbers, thereby reflecting the imprecision inherent in their estimation from market surveys and empirical data.

The new demand function in fuzzy environment $\tilde{D}[t, \tilde{I}(t)]$ is taken as follows:

$$\tilde{D}[t, \tilde{I}(t)] = \begin{cases} \tilde{m} + \tilde{n}\tilde{I}(t), & 0 \leq t < t_0 \\ \tilde{D}_0, & t_0 \leq t < T. \end{cases} \quad (4.1)$$

Since the values of these parameters are specified in an imprecise manner, their corresponding defuzzified values are obtained using the centroid method. By substituting these defuzzified parameters into equation (4.1), the resulting crisp demand function is derived as follows:

$$D_d[t, I_d(t)] = \begin{cases} m_d + n_d I_d(t), & 0 \leq t < t_0 \\ D_{0d}, & t_0 \leq t < T. \end{cases}$$

Here x_d denotes the defuzzified value of parameter x .

In this model, the change in level of inventory at any point of time for an item following the demand is given in equation 4.2

$$\frac{dI_d(t)}{dt} = \begin{cases} -m_d - (n_d + \theta_d)I_d(t) & 0 \leq t < t_0 \\ -D_{0d} & t_0 \leq t < T. \end{cases} \quad (4.2)$$

Solving equation (4.2) with the condition $I_d(t_0) = 0$ gives $I_d(t)$ as follows

$$I_d(t) = \begin{cases} \frac{m_d}{n_d + \theta_d} [e^{(n_d + \theta_d)(t_0 - t)} - 1], & 0 \leq t < t_0 \\ D_{0d}(t_0 - t), & t_0 \leq t < T. \end{cases}$$

The initial inventory level can be computed by $I_d(0)$ in equation which gives

$$S_0 = I_d(0) = m_d t_0 \left\{ 1 + \frac{(n_d + \theta_d)t_0}{2} \right\}.$$

The total number of items deteriorated in the time interval 0 to t_0 can be computed as

$$P_d = S_0 - \int_0^{t_0} D_d(t, I_d(t))dt = \frac{m_d t_0^2 \theta_d}{2} - \frac{m_d n_d (n_d + \theta_d) t_0^3}{6}.$$

The order quantity in this cycle is calculated as:-

$$q = S_0 + \int_{t_0}^T D_d(t, I_d(t))dt = m_d t_0 + \frac{m_d t_0^2 (n_d + \theta_d)}{2} + D_{0d} (T - t_0).$$

To compute the components of the total cost function, the purchasing cost, holding cost, storage cost, handling cost of deteriorated items, carbon emission cost, and deterioration rate are treated as fuzzy parameters. All parameters are represented using trapezoidal fuzzy numbers, which are defuzzified to obtain crisp values for determining the optimal inventory decisions.

Let $\tilde{m} = (m_1, m_2, m_3, m_4)$, $\tilde{n} = (n_1, n_2, n_3, n_4)$, $\tilde{D}_0 = (D_{01}, D_{02}, D_{03}, D_{04})$, $\tilde{a} = (a_1, a_2, a_3, a_4)$, $\tilde{b} = (b_1, b_2, b_3, b_4)$, $\tilde{c} = (c_1, c_2, c_3, c_4)$, $\tilde{d} = (d_1, d_2, d_3, d_4)$, $\tilde{K} = (K_1, K_2, K_3, K_4)$, $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\tilde{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)$ are trapezoidal fuzzy numbers.

Here, centroid method is employed here to defuzzify the fuzzy parameters. If $\tilde{x} = (x_1, x_2, x_3, x_4)$ is the fuzzy trapezoidal number then the defuzzified value of $\tilde{x}$ using centroid method is $x_d = \frac{x_1 + 2x_2 + 2x_3 + x_4}{6}$.

Now the components of total cost function under crisp environment are computed as follows:-

Purchase cost:

$$c_1 = a_d q = a_d \left\{ m_d t_0 + \frac{m_d t_0^2 (n_d + \theta_d)}{2} + D_{0d} (T - t_0) \right\}.$$

Holding cost:

$$c_2 = b_d \int_0^{t_0} I(t)dt = \frac{b_d m_d t_0^2}{2} \left[1 + \frac{(n_d + \theta_d)t_0}{3} \right].$$

Shortage cost:

$$c_3 = c_d \int_{t_0}^T (-I(t))dt = c_d D_{0d} \left[\frac{t_0^2}{2} + \frac{T^2}{2} - T t_0 \right].$$

Handling cost of deteriorated products:

$$c_4 = d_d P_d = d_d \left\{ \frac{m_d t_0^2 \theta_d}{2} - \frac{m_d n_d (n_d + \theta_d) t_0^3}{6} \right\}.$$

Fixed ordering cost:

$$c_5 = K_d.$$

Carbon tax policy cost:

$$c_6 = \lambda_d \left\{ A_e + c_1^e \left\{ m_d t_0 + \frac{m_d t_0^2 (n_d + \theta_d)}{2} + D_{0d} (T - t_0) \right\} + h_1^e \left\{ \frac{m_d t_0^2}{2} \left[1 + \frac{(n_d + \theta_d)t_0}{3} \right] \right\} \right\}.$$

The retailers total cost function is as follows:

$$TC_d = a_d \left\{ m_d t_0 + \frac{m_d t_0^2 (n_d + \theta_d)}{2} + D_{0d} (T - t_0) \right\} + \frac{b_d m_d t_0^2}{2} \left[1 + \frac{(n_d + \theta_d)t_0}{3} \right] + c_d D_{0d} \left[\frac{t_0^2}{2} + \frac{T^2}{2} - T t_0 \right] \quad (4.3)$$

$$+ d_d \left\{ \frac{m_d t_0^2 \theta_d}{2} - \frac{m_d n_d (n_d + \theta_d) t_0^3}{6} \right\} + K_d + \lambda_d \left\{ A_e + c_1^e \left\{ m_d t_0 + \frac{m_d t_0^2 (n_d + \theta_d)}{2} + D_{0d} (T - t_0) \right\} \right. \\ \left. + h_1^e \left\{ \frac{m_d t_0^2}{2} \left[1 + \frac{(n_d + \theta_d)t_0}{3} \right] \right\} \right\}. \quad (4.1)$$

Then, this optimization problem is described as follows:

$$\text{Min } TC_d(t_0),$$

$$\text{Subject to } t_0 > 0.$$

This model can also be applied directly to crisp inputs by substituting the corresponding crisp values for the fuzzy parameters. In other words, instead of using defuzzified values from fuzzy inputs, the model can directly use crisp parameter values to determine the optimal inventory decisions.

5 Numerical Illustration and Convexity Validation:

To illustrate the model discussed above, two randomly generated data sets are presented and solved using *LINGO 18.0* under both fuzzy and crisp environments. In the first subsection 5.1, the inventory model is analyzed in a fuzzy environment that illustrates the optimal behavior of the total cost function under a fuzzy environment. In this case, the uncertain parameters are represented by trapezoidal fuzzy numbers to capture real-world imprecision and to assess the robustness of the proposed model in the presence of uncertainty. In the second subsection 5.2, the inventory model is analyzed in a crisp environment while maintaining the same demand structure and model assumptions. All parameters are treated as deterministic and are assigned precise or crisp numerical values, which are substituted into Equation (4.3). The model is then solved numerically to obtain the optimal time when inventory becomes zero, optimal order quantity, and the minimum total system cost. This crisp analysis serves as a computational benchmark for evaluating the performance of the proposed model.

5.1 Fuzzy Model

Let us consider $\tilde{a} = (6, 8, 12, 14)$, $\tilde{b} = (0.012, 0.016, 0.024, 0.028)$, $\tilde{c} = (0.6, 0.8, 1.2, 1.4)$, $\tilde{d} = (3, 4, 6, 7)$, $\tilde{K} = (60, 80, 120, 140)$, $\tilde{D}_0 = (16, 18, 22, 24)$, $\tilde{\theta} = (0.016, 0.018, 0.022, 0.024)$, $\tilde{n} = (0.006, 0.008, 0.012, 0.014)$, $\tilde{\lambda} = (0.06, 0.08, 0.12, 0.14)$ are given trapezoidal fuzzy inputs. Optimizing the above problem (Equation (4.3)) under fuzzy environment, we get the optimal value of TC_d i.e. $TC_d^* = 25890.32$, optimal value for t_0 i.e. $t_0^* = 30.84652$ and $q_d^* = 1495.6905$.

The convexity of the cost function TC_d with respect to t_0 for fuzzy case is shown graphically in the Figure 5.1.

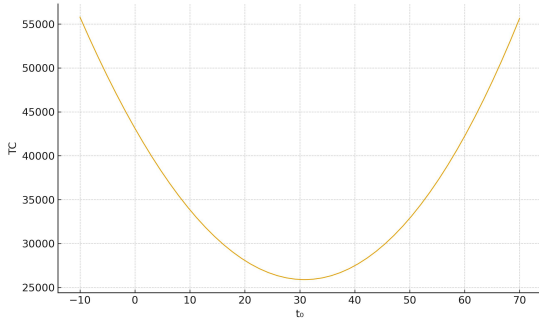


Figure 5.1: Total cost as function of t_0 (fuzzy model)

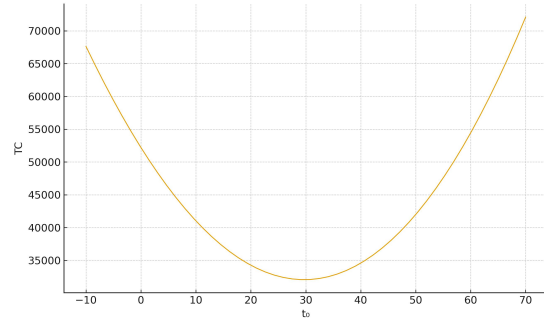


Figure 5.2: Total cost as function of t_0 (crisp model)

5.2 Crisp model

Instead of fuzzy inputs, if the crisp values are given as $T = 56$, $a = 11$, $b = 0.022$, $c = 1.1$, $d = 5.2$, $M = 105$, $D_0 = 22$, $\theta = 0.021$, $n = 0.013$, $m = 24$, $\lambda = 0.12$ then after optimizing Equation 4.3, we get the optimal total cost i.e. $TC_d^* = 32065.10$, optimal value for time when inventory becomes zero i.e. $t_0^* = 29.71537$ and optimal amount of quantity to be ordered i.e. $q_d^* = 1651.6951$.

The convexity of the cost function TC_d with respect to t_0 for crisp model is shown graphically in the Figure 5.2.

Upon comparing the optimal total costs obtained from the fuzzy and crisp models, it is observed that the total cost incurred under the fuzzy environment is lower than that of the crisp environment. Additionally, inventory depletion occurs later in the fuzzy environment.

6 Sensitivity Analysis

In this section, a sensitivity analysis is conducted to examine the impact of variations in key parameters on the optimal total cost of the system. Each parameter is independently decreased and increased by 50% to investigate its influence on the optimal depletion time (t_0^*), the economic order quantity (q^*), and the optimal total cost (TC_d^*) under both crisp and fuzzy environments. Tables 6.1 and 6.2 provide a systematic summary of the simulated results. Several important observations and managerial insights emerge from this analysis, which are discussed in detail in the subsequent section.

6.1 Observations

- The numerical results, along with the sensitivity analysis presented in Tables 6.1 and 6.2, indicate that the total cost under the fuzzy environment (25,890.32) is consistently lower than that under the crisp environment (32,065.10). This difference of 19.25% arises because the fuzzy approach incorporates parameter uncertainty through membership functions, allowing the model to distribute variability more flexibly across demand, deterioration rate, and other cost components. Consequently, the fuzzy total cost provides a more realistic and conservative estimate compared with the rigid, single-point assumptions of the crisp model.
- The results also indicate that incorporating a carbon tax policy increases the total cost of the inventory system in both crisp and fuzzy models. Specifically, increasing (decreasing) the carbon emission cost under the carbon tax policy (λ) by 50% leads to an increase (decrease) in total cost of 6.01% (7.24%) and 5.96% (7.09%) in the crisp and fuzzy environments, respectively. This increase occurs because the carbon tax directly penalizes emissions generated during ordering, transportation, storage, and deterioration processes, thereby adding an explicit cost component to the total system cost.
- Sensitivity analysis further reveals that the total cost is highly responsive to changes in both demand and purchasing cost in both environments. Even moderate fluctuations in these parameters result in significant variations in overall cost. For instance, a 50% increase (decrease) in demand (D_0) results in a total cost change of +18.2289% (−32.3350%) in the crisp environment and +17.1990% (−31.4086%) in the fuzzy environment. Similarly, a 50% increase (decrease) in purchasing cost (a) leads to a total cost change of +27.7002% (−29.0887%) in the crisp environment and +28.2901% (−29.5910%) in the fuzzy environment.
- The sensitivity analysis also shows that the total cost is relatively less sensitive to the inventory-dependent demand coefficient (n), shortage cost (c), carbon emission cost (λ), and deterioration rate (θ). The total cost increases moderately with changes in these parameters and is largely insensitive to holding cost and the handling cost of deteriorated products.

6.2 Insights for practitioners

- The numerical results emphasize that incorporating fuzziness into inventory parameters can reduce overall cost, particularly in systems with uncertain demand, deterioration, or other cost factors. By effectively capturing parameter vagueness, the fuzzy model provides a more flexible and economical framework for decision-making, enabling managers to establish reliable ordering policies under uncertainty. This suggests that companies dealing with imprecise data common in cases of deteriorating goods, fluctuating demand, or real-time supply chain disruptions can significantly lower total costs by adopting fuzzy-based inventory planning instead of traditional deterministic approaches.
- The incorporation of a carbon tax policy in the model indicates that firms operating under carbon taxation regimes must adjust their inventory strategies to account for the additional costs associated with environmental compliance. Managers are encouraged to adopt greener transportation modes, invest in low-emission preservation technologies, optimize order quantities and frequencies to minimize emission-related expenses, and redesign processes to reduce carbon footprints. Although carbon taxes may increase short-term operational costs, they incentivize firms to adopt more energy-efficient and sustainable practices, which can ultimately lower emissions and reduce long-term costs.
- Since the total cost in the model is highly sensitive to demand patterns and purchasing costs, managers must focus on accurate estimation and continuous monitoring of these parameters. Firms should enhance demand forecasting using advanced predictive tools or real-time data, negotiate stable supplier contracts to reduce purchasing cost volatility, consider bulk purchasing agreements where feasible, and review ordering policies more frequently under high demand uncertainty. Recognizing the significant impact of demand and purchasing costs on overall expenses helps managers implement strategies to stabilize costs, mitigate financial risk, and enhance the reliability of the inventory system.

Table 6.1: Sensitivity analysis of parameters for crisp model

Parameter	% change	t_0^*	TC_d^*	Change in TC_d^*	q^*
a	-50	33.0750	22737.76	-29.0887	1651.6958
a	+50	26.9098	40947.21	+27.7002	1651.6958
b	-50	29.9683	31907.79	-6.4905	1651.6958
b	+50	29.46783	32219.45	+0.4813	1651.6958
c	-50	20.45807	26423.64	-17.5937	1651.6958
c	+50	35.0416	35395.95	+10.3877	1651.6958
d	-50	30.3077	31583.11	-1.5031	1651.6958
d	+50	29.1729	32500.17	+1.3568	1651.6958
λ	-50	32.9865	29741.79	-7.2456	1651.6958
λ	+50	27.1587	33994.89	+6.0183	1651.6958
m	-50	41.9955	20849.67	-34.9770	1171.8238
m	+50	22.02416	38.840 .42	+21.1298	1837.1971
n	-50	30.96951	31203.74	-2.6862	1610.4425
n	+50	28.64823	32816.94	+2.3447	1688.1669
D_0	-50	16.6682	21696.84	-32.3350	946.0407
D_0	+50	37.13576	37910.23	+18.2289	2076.4365
θ	-50	33.2273	29737.37	-7.2593	1731.382
θ	+50	26.9274	33931.24	+5.8198	2009.5396

Table 6.2: Sensitivity analysis of parameters for fuzzy model

Parameter	% change	t_0^*	TC_d^*	Change in TC_d^*	q^*
a	-50	34.1107	18229.09	-29.5910	1495.6905
a	+50	28.0868	33214.72	+28.2901	1495.6905
b	-50	31.1151	25752.04	-0.5340	1495.6905
b	+50	30.5837	26025.97	+0.5239	1495.6905
c	-50	21.4746	21555.85	-16.7416	1495.6905
c	+50	36.12410	28388.69	+9.6498	1495.6905
d	-50	31.5548	25438.47	-1.7452	1495.6905
d	+50	30.1587	26323.97	+1.6749	1495.6905
λ	-50	33.9833	24053.85	-7.0932	1495.6905
λ	+50	28.3475	27434.59	+5.9646	11995.6905
m	-50	43.0953	16593.58	-35.9081	1038.5811
m	+50	23.0133	31615.29	+22.1123	1681.3308
n	-50	29.9353	26415.95	+2.03021	1426.3102
n	+50	31.8885	25302.22	-2.27150	1575.1814
D_0	-50	17.4883	17758.52	-31.4086	870.7870
D_0	+50	38.2872	30343.21	+17.1990	1857.4526
θ	-50	34.5568	23868.31	-7.8099	1451.83153
θ	+50	27.9127	27506.04	+6.2406	1512.6376

7 Conclusions

This study developed a fuzzy inventory control model for deteriorating items with dynamic, stock-dependent demand, allowing shortages and complete backlogging. Key parameters were modeled as trapezoidal fuzzy numbers and defuzzified using the centroid method. The numerical illustrations were provided to optimize the total cost functions in both (fuzzy and crisp) environments using *LINGO 18.0*, and the convexity of the cost function was validated graphically using *MATLAB*. The results demonstrate that incorporating fuzziness lowers total cost by improving the handling of parameter uncertainty and enabling more robust and flexible inventory planning, making it particularly valuable for retailers managing deteriorating items

with stock-dependent demand. Sensitivity analysis was performed on the parameters, $m, n, D_0, a, b, c, d, \lambda$ and θ for both crisp and fuzzy models in Table 6.1 and Table 6.2 respectively which reveals that total cost is highly responsive to changes in demand and purchasing cost, emphasizing the importance of monitoring these factors for effective inventory management. Additionally, the analysis of carbon tax policy highlights that regulatory constraints increase total costs, suggesting that firms should adopt energy efficient technologies or preservation strategies to maintain sustainability and cost efficiency. Overall, the proposed fuzzy framework offers practical advantages over traditional models, supporting firms in making informed, sustainable, and cost effective inventory decisions under uncertain and imprecise market conditions.

This study provides a flexible framework for understanding inventory systems under uncertainty, while leaving room for further refinement. The model could naturally accommodate more complex demand patterns, different deterioration behaviors, and alternative shortage scenarios, and the fuzzy parameter approach allows for richer representations of uncertainty. At the same time, the sustainability perspective considered here could be explored in broader inventory contexts, helping to make the model even more practical and relevant for decision making in the real world.

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EARLY DETECTION OF HUNTINGTONS DISEASE (HD) BY MACHINE LEARNING USING GENETIC DATA

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Abstract

A genetic condition known as Huntington's disease (*HD*) impairs voluntary movement by causing the death of brain cells called neurons in various brain regions. It is challenging to identify early-stage *HD* in clinical practice since there aren't many effective, simple-to-administer cognitive tests that are sensitive to organized data. Life expectancy has risen steadily as modern treatment has advanced. The prevalence of *HD* has also increased and is anticipated to increase over the next several decades as a result of escalating healthcare expenses. Early *HD* detection is therefore essential. Through this work, we used a clinical and genetic dataset of 10 *HD* patients. The present approach introduces a novel support vector machine (*SVM*)-based machine Learning (*ML*) algorithm for the detection of *HD* patients. The results of a classification study comparing various techniques indicate that *SVM*-based classification is more accurate at identifying subjects, even across varying stages of development. The detection capabilities of the suggested method were thoroughly examined using clinical and genetic data. With two-fold cross-validation, the maximum accuracies for early *HD* achieved by Gaussian *SVM*, Cubic *SVM*, Quadratic *SVM*, Naive Bayes, and Linear *SVM* are 70%, 100%, 100%, 80%, and 60%, respectively. These findings show that a useful and promising strategy for *HD* identification is the use of supervised classification techniques to genetic characteristics. The primary conclusions of this research are: 1) To detect substantial *HD*, a genetic test will be necessary. 2) The quadratic and cubic *SVMs* are the most effective predictors for *HD*. 3) According to experimental findings, quadratic and cubic *SVMs* outperform other classification algorithms for *HD* diagnosis as well. Even with a small dataset, the non-linear *SVM* approach improves diagnosis accuracy.

2020 Mathematical Sciences Classification: 00B99.

Keywords and Phrases: Huntington's disease (*HD*), symptoms, classification issues, and machine learning (*ML*) approaches.

1 Introduction

A neurological disorder that significantly affects people is *HD* [1]. It has a lasting impact on people's families in addition to themselves [20]. An uncontrollable progression of mental symptoms, cognitive decline, and physical dysfunction characterizes this complicated clinical picture, which ultimately leads to severe disability and a tragically reduced lifetime [12]. The seriousness of *HD* has led the medical and scientific communities worldwide to become more concerned, prompting a variety of initiatives not only to understand its etiology and pathophysiology but also to advance early detection and treatment [7]. The *HD* pathological

progression is still unclear, attracting interest from a variety of scientific fields. In order to better understand the motor engagement in *HD*, the work [17] investigates motor speech patterns. Paper [6] explores speech biomarkers at different phases of *HD*, highlighting the range from early manifestation to pre-symptomatic. The paper [19] provides information on Huntingtons complex role, which extends from neurodevelopment to neurodegeneration. In [21] and another study published in Nature Medicine [8] provide light on synaptic loss by highlighting the early roles of innate immune systems, complement activation, and microglia in the reduction of corticostriatal synapses. Views on the harmful consequences of mutant Huntington’s disease are provided by the study [8]. It is impossible to overestimate the significance of early *HD* diagnosis since it enables prompt treatment that could reduce the progression of this debilitating neurological condition. By combining existing knowledge and offering perceptive analysis, it seeks to encourage more study and innovation in this field in order to improve our capacity to identify and treat *HD* in its early stages and possibly enhance the lives of people impacted and their families [8]. Diseases can be diagnosed and tracked using biomarkers, which are measurable indicators of a certain biological process or state [22, 23]. The *HD* is a degenerative neurological condition that runs in families, is brought on by a mutation in the cytosine-adenine guanine (*CAG*) repeat of the *HTT* gene [4, 11]. Biomarkers can be utilized in the context of *HD* to offer more details regarding the patient’s condition, prognosis, and reaction to therapy. It provides a thorough analysis of *ML*-powered therapies for this illness and evaluates their efficacy. By showing how beneficial these technologies are in this situation, this study aims to close the gap between *ML* and *HD*. The important contributions of this study are:

- This study investigates the potential for machine learning (*ML*) to enhance *HD* diagnosis.
- This study presents a variety of *SVM*-based machine learning techniques that have been used for early diagnosis of *HD*.
- In this work, the cubic *SVM* outperformed other existing *SVM*-based algorithms.

This study presents a novel *SVM*-based machine learning method for *HD* classification. The suggested approach uses categorization techniques to accomplish this goal. In this investigation, data from ten *HD* patients were used. With the most important *SVM*-based method (Quadratic *SVM*), the highest accuracy is 100%, and the sensitivity and specificity values are almost 100%, respectively.

2 Problems and Related Work

A fatal autosomal dominant disorder, *HD* often manifests in early adulthood and develops over time to produce a range of motor, behavioral, and cognitive symptoms. Even though *HD* is difficult to forecast in medical science, researchers work hard to create the finest techniques and results. Because machine learning yields practical results, it is becoming more and more popular. In order to classify Huntington & 39;s illness, Miranda, A. *et al.* [13] used the support vector machine (*SVM*) technique on oculomotor variables that were aggregated from a four-task psychophysical experiment. With an accuracy of 73.47%, sensitivity of 74.31%, and specificity of 72.64%, it was possible to automatically separate control participants from presymptomatic *HD* (pre-*HD*) participants; 81.84%, sensitivity of 76.19%, and specificity of 87.48% were achieved when separating control participants from *HD* patients; and 83.54%, sensitivity of 92.62%, and specificity of 74.45% were achieved when separating pre-*HD* participants from *HD* patients. A Money Box Test (*MBT*) was created by Bennasar *et al.* [2] to measure involuntary movement. A novel method for automatically dividing the subjects into two groups, healthy and *HD*, is suggested using time series accelerometer data. After extracting a set of 90-time time-domain features from the accelerometer data, the dataset dimensionality is decreased, and the features; relevance is evaluated using a feature selection technique. Lastly, the participants are split into *HD* and healthy groups using an *SVM* classifier. Data from 15 *HD* patients and seven healthy controls are used in this study. The eight most important criteria yield the maximum accuracy of 86.36%, sensitivity of 87.50%, and specificity of 83.33%. According to Perez *et al.* [16], the proposed method can accurately distinguish between control and *HD* patients using non-linear forecasting techniques and *DNA CpG* methylation data as an input. It should be mentioned that the dataset being studied is not very large. Nonetheless, the findings appear to be very consistent, and if other data sets become available, the study might be conducted using them. The results will be evaluated using the neural network (*NN*) technique. The accuracy, specificity, and sensitivity of the *NN* are 92%, 80%, and 95%, respectively. Using a method of shallow machine learning, Maddury *et al.* [14] provided a prepared, validated dataset that includes biological signal data (*EEG, ECG, AND FNIRS*) from a clinic with 27 *HD*-positive patients, 36 controls, and 6 unknowns [14]. With a 91.35% accuracy rate and a 0.96 area under the receiver operator characteristic curve, the scaled-out Extremely Randomized Trees technique produced the best results. With

60.86% of the features having $p < 0.05$, the resulting feature analysis showed that the most significant features were those from the raw signal. The results imply that heart and brain signals may be useful in detecting abnormalities in *HD* and monitoring the course of the illness in individuals. Erda *et al.* [6] use convolutional neural networks (*CNNs*) to recognize gait data and convert it into *QR* codes to create a dependable and efficient system for detecting neurodegenerative diseases. By introducing a novel method to gait pattern analysis, this strategy aims to improve the accuracy of identifying neurodegenerative disorders such as *HD*, Parkinsons disease (*PD*), and amyotrophic lateral sclerosis (*ALS*). The dataset consists of recordings from 13 patients with *ALS*, 20 patients with *HD*, 15 patients with *PD*, and 16 healthy controls. Using a 10-fold *CV*, the accuracy rates for *ALS* versus control, *PD* versus *HD* versus *ALS* versus control, *PD* versus control, and *NDD* versus control were 95.81%, 93.86%, and 93.56%, respectively, and *HD* versus control, respectively. These results demonstrate how effectively the proposed method may distinguish between the control group and different neurodegenerative diseases.

Earlier research used data from many sources to solve the problem of categorizing neurodegenerative disorders by utilizing traditional machine learning. *ML* approaches like the Quadratic *SVM* can be used to generate more creative solutions for the classification challenge of neurodegenerative illnesses. In the current work, *HD* genomic data were categorized using the quadratic *SVM* machine learning technique. One of the innovative features of this paper is the reproduction of genetic data in the classification issue using Quadratic *SVM*.

3 Problems and Related Work

Standard protocol approval, ethical clearance, and consent: Following a Memorandum of Understanding between the two universities, protocol clearance was acquired from the institutional ethics committees of the two collaborating institutions (a teaching engineering institute and a tertiary care university hospital) (Ethics permission no. *IERB ID: COM-D-21- 001*). After gaining patients; or caregivers; written informed consent, participants were enlisted.

Study period and site: The study was conducted from August 2022 to September 2024. Data collection and nerve conduction velocity (*NCV*), and genetic tests were performed at a tertiary care university hospital. Data analysis and algorithm development were mainly performed at the Engineering Institute lab.

Study Design and Participants: The study was cross-sectional. For the study, ten *HD* patients were enlisted.

***NCV* and Genetic Features:** In our study, muscles were characterized on the basis of *NCV* parameters and genetic data. All patients were assessed for the diagnosis of *HD* clinically and on *NCV*. The *NCV* parameters were recorded with concentric needle electrodes. A region with many *CAG* repeats (Cytosine, Adenine, and Guanine) is found in the gene that causes *HD*. Although the gene encoding the Huntington protein has these *CAG* repeats in all of us, individuals with *HD* have an abnormally high amount of *CAG* repeats in one of their inherited genes. For the *NCV* of *HD* patients, we perform the Youden Index (*J*), which determines the maximal efficacy of a test and computes the performance of a single diagnostic test. The index's value falls between 0 and 1, where 0 denotes a test with no discriminatory power, and 1 denotes a perfect test.

Validation: In the study, clinical diagnosis (based on clinical and investigations by experts) was considered. Neurology experts (*AP* and *VNM*), from the Department of Neurology, Institute of Medical Sciences, Banaras Hindu University (*BHU*), Varanasi confirmed the clinical diagnosis of the disease. An *ML* Model was run to identify the diagnosis on the patients data.

Data Collection:

- **Clinical data:** The *HD* UHDRS-TMS=Unified Huntington's Disease Rating Scale-Total Motor Score was used to evaluate how *HD* affected day-to-day functioning. Among the demographic characteristics of the *HD* is the mean (*SE*).
- ***NCV* and Genetic data:** *NCV* and genetic data from 10 clinically diagnosed *HD* patients were taken. The demographic and clinical information of the *HD* patients was collected, which is reported in Table 3.1.
- **Data Analysis:** *NCV* and genetic data sets were used for an *ML*-based diagnosis detection system to identify that genetic data is a biomarker for *HD* diagnosis.
- ***HD* Detection with *SVM*-based *ML* Modeling** A separating hyperplane is constructed as a decision boundary using the *SVM* machine learning algorithm, which maximizes the margin of separation between the classes. More specifically, given training data that includes two distinct classes,

the *SVM* method creates a model using data from individuals in one class on one side of the hyperplane and those in the other class on the other side. The optimal separation hyperplane is found by reducing the amount of classification error and maximizing the distance between each class of data points and the decision boundary. Overall, this study has shown that *SVMs* are a useful tool for detecting *HD* and have led the way in applying machine-learning algorithms to genetic data in *HD*. The proposed model used an *SVM*-based *ML* algorithm, which is explained in Figure 3.1.

Table 3.1: Features of the *HD* patients' demographics.

Case	Gender (male: female)	Age	CAG Repetition	Duration of Disease (years)	Years Before the Start	UHDRS (Motor)
<i>HD</i> Patients	7:3	35.79 (2.63)	41.29 (0.52)	N/A	21.01 (2.92)	1.93 (0.46)

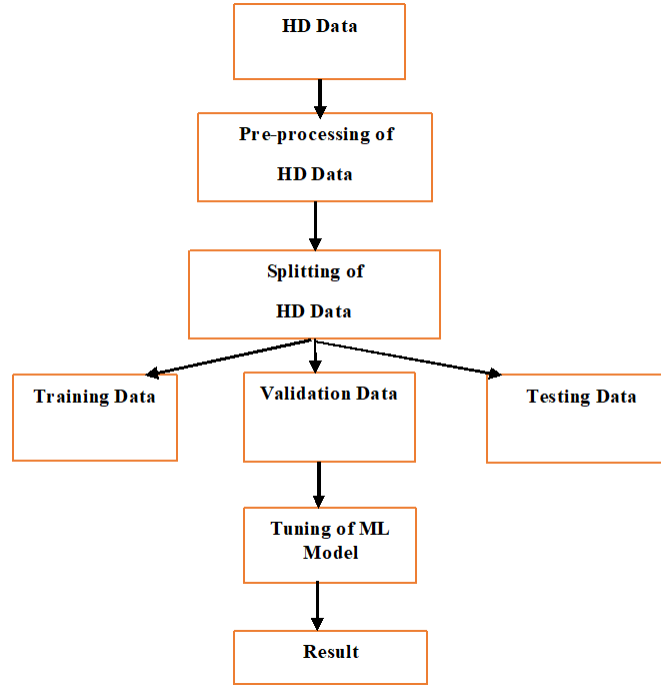


Figure 3.1: Proposed Model for *HD* Detection

4 Experimental Results

We evaluated the model's performance using an *SVM*-based *ML* algorithm using the confusion matrix for sensitivity, specificity, accuracy [3, 24], and Youden Index (J)[18], which is shown in equations (4.1), (4.2), (4.3), and (4.4).

- **Sensitivity :** Sensitivity is the only really positive statistic that considers every situation and can be calculated as follows:

$$Sensitivity = \frac{TP}{TP + FN}, \quad (4.1)$$

- **Specificity:** The number of true negatives that are accurately identified and calculated is ascertained in this manner:

$$Specificity = \frac{TN}{TN + FP}, \quad (4.2)$$

- **Accuracy:** The accuracy is used to determine the proportion of correctly classified values. It shows the frequency of accuracy of our classifier. The following formula can be used to calculate accuracy:

$$Accuracy = \frac{TP + TN}{TP + TN + FP + FN}, \quad (4.3)$$

- **Youden Index (J):** It is the most notable vertical difference between the receiver operating characteristic (*ROC*) curve and the diagonal line, which indicates how well a random guess performs. The *J* can be computed in this way:

$$YoudenIndex(J) = Sensitivity + Specificity - 100, \quad (4.4)$$

Here, *TN* stands for true negative, *FN* for false negative, *FP* for false positive, and *TP* for true positive.

The quadratic *SVM* model's confusion matrix in terms of *TPR* and *FNR* is displayed in Figure 4.1. The quadratic *SVM* model's confusion matrix is displayed in Figure 4.2 in terms of *PPV* and *FDR*. The *ROC* curve for the *SVM* quadratic model is displayed in Figure 4.3. For the cubic *SVM* model, Figure 4.4 displays the confusion matrix in terms of *TPR* and *FNR*. The confusion matrix for the cubic *SVM* model in terms of *PPV* and *FDR* is shown in Figure 4.5. The cubic *SVM* model's *ROC* curve is displayed in Figure 4.6. The training duration and prediction speed of the proposed model are shown graphically in Figure 4.7. The Youden index, sensitivity, specificity, and accuracy of the suggested models are shown graphically in Figure 4.8. The visual representation for contrasting the performance of the proposed model with earlier research is displayed in Figure 4.9. The total performance graphical depiction and comparison with the suggested model work are displayed in Figure 4.10.

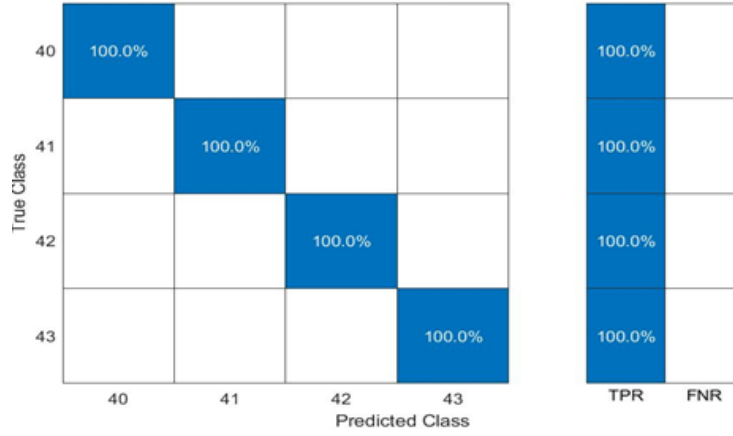


Figure 4.1: Confusion matrix in terms of *TPR* and *FNR* of the quadratic *SVM* model

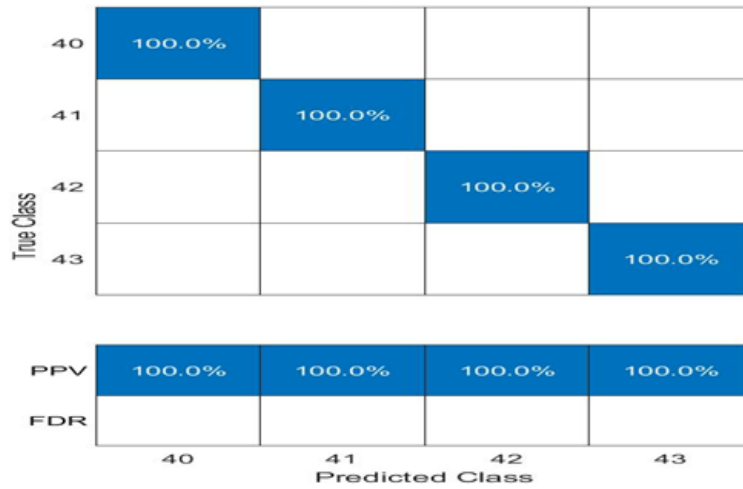


Figure 4.2: Confusion Matrix in terms of *PPV* and *FDR* for the quadratic *SVM* model.

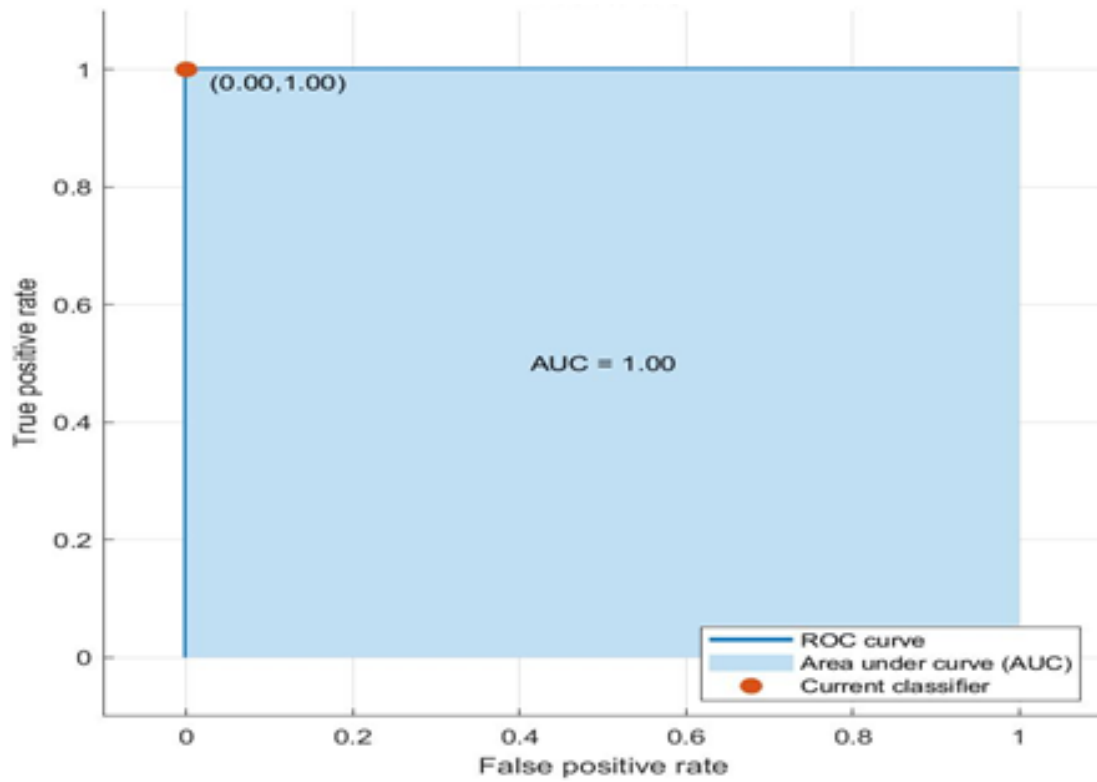


Figure 4.3: ROC curve for the SVM quadratic model

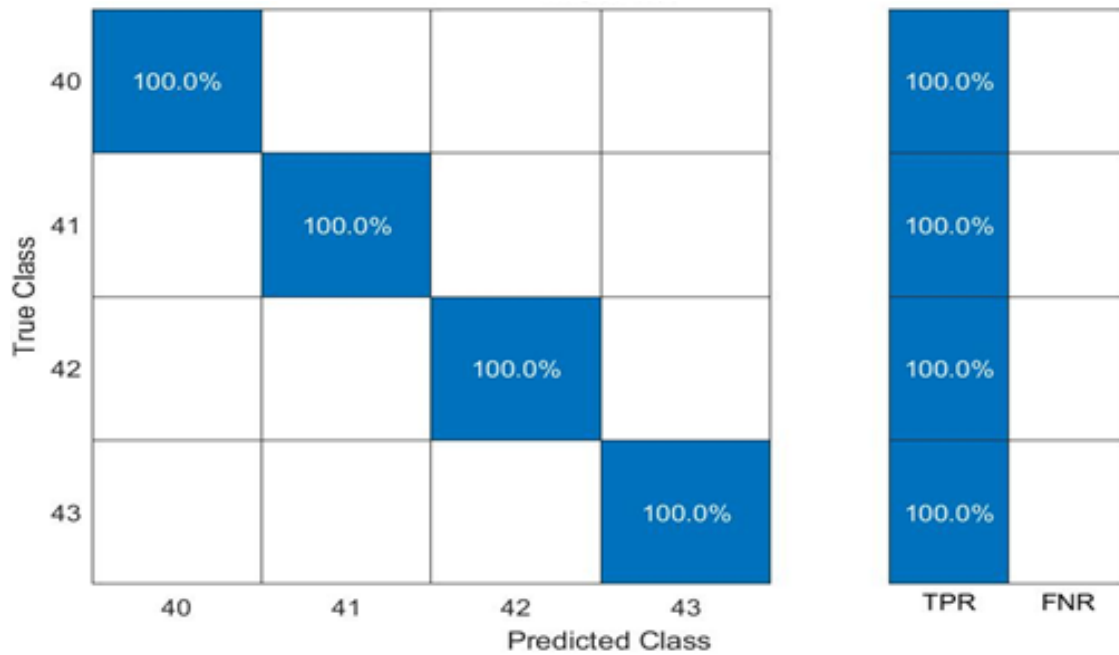


Figure 4.4: Confusion matrix in terms of TPR and FNR for the cubic SVM model

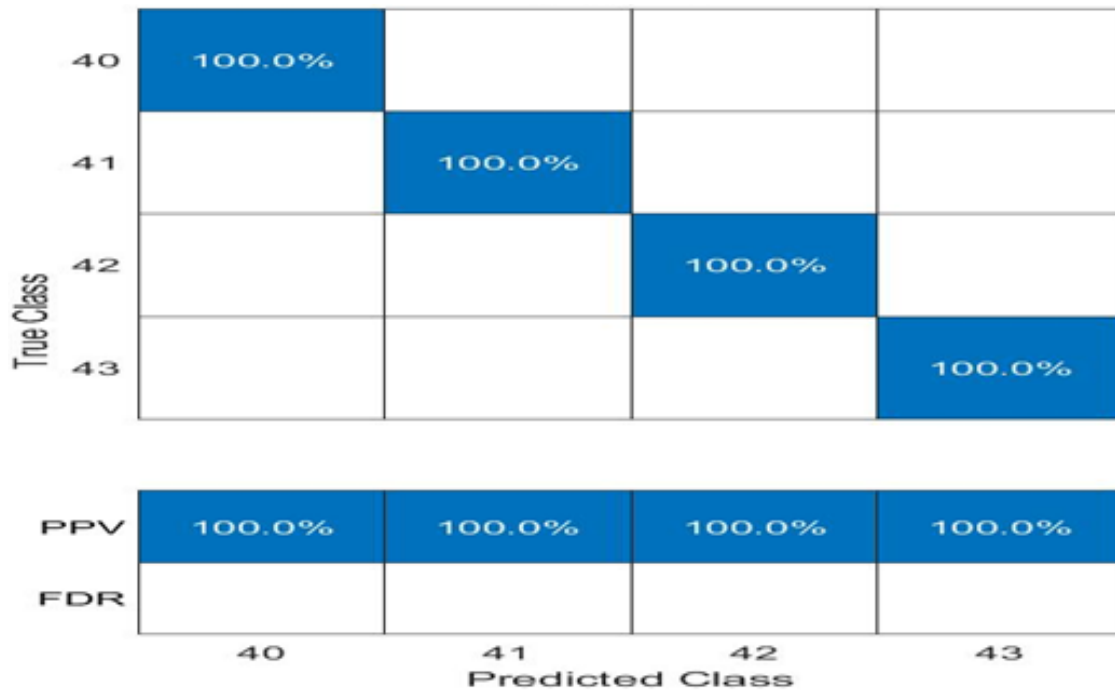


Figure 4.5: Confusion matrix in terms of PPV and FDR for the cubic SVM model

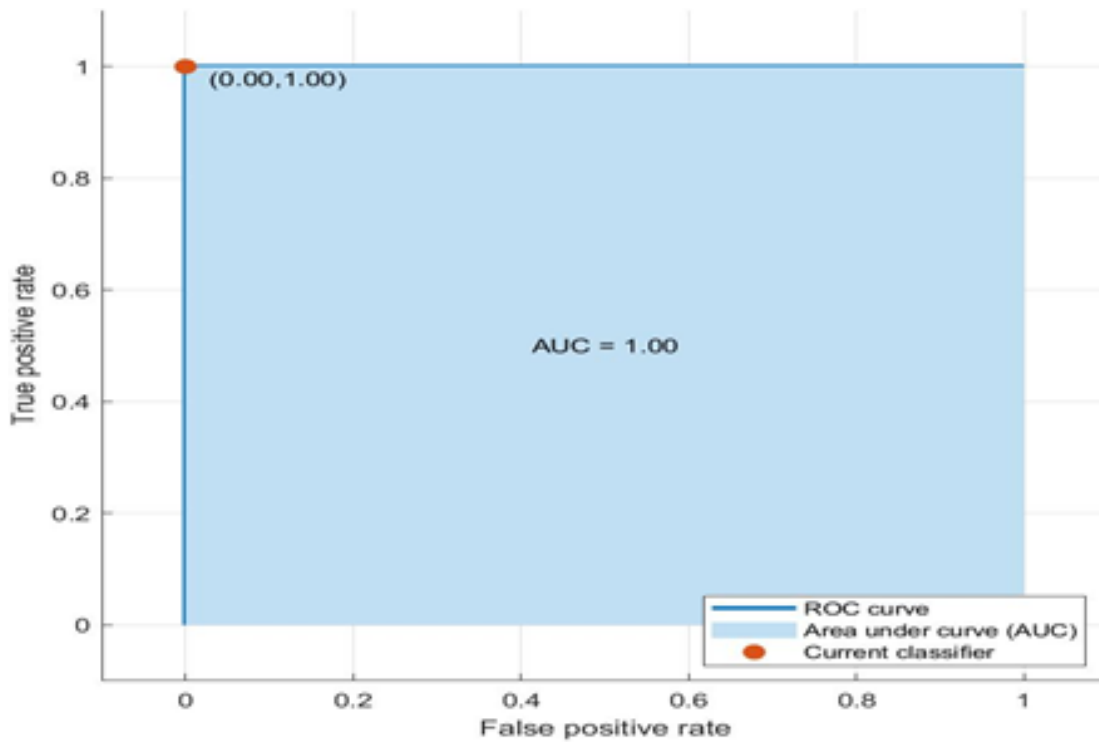


Figure 4.6: ROC curve for cubic SVM model

Table 4.1: The performance of the proposed model in terms of prediction speed and training time.

SVM-based Algorithms	Prediction speed (obs/sec)	Training time (sec.)
Gaussian <i>SVM</i>	43000	0.86
Cubic <i>SVM</i>	29000	0.91
Quadratic <i>SVM</i>	21000	0.73
Naive Bayes	46000	3.74
Linear <i>SVM</i>	41000	2

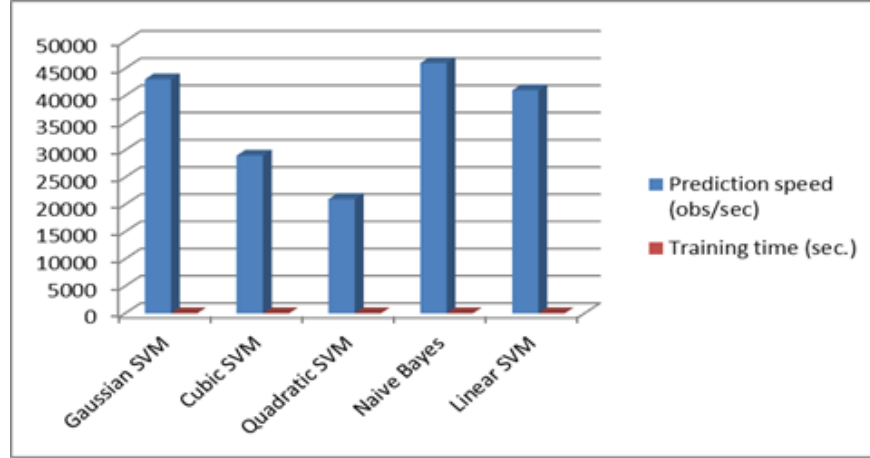


Figure 4.7: A visual representation of the suggested model's training time and prediction speed.

Table 4.2: Performance Matrices of the Proposed Model

Models	Sensitivity (%)	Specificity (%)	Accuracy (%)	J	AUC
Gaussian <i>SVM</i>	50	50	70	0	0.56
Cubic <i>SVM</i>	100	100	100	100	1
Quadratic <i>SVM</i>	100	100	100	100	1
Naive Bayes	85.42	99.99	80	85.41	1
Linear <i>SVM</i>	41.67	87.5	60	29.17	0

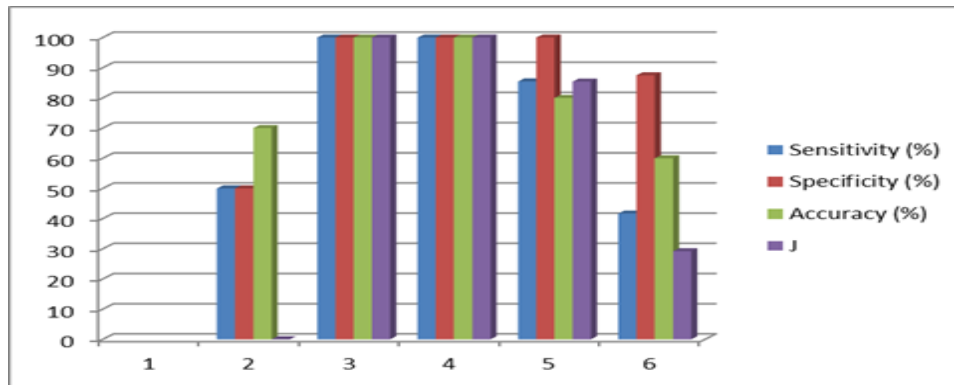


Figure 4.8: Visual illustration of the Youden index, sensitivity, specificity, and accuracy.

Table 4.3: Performance comparison with existing works and the proposed model.

Models	Sensitivity (%)	Specificity (%)	Accuracy (%)	J
Bennasar, M., <i>et al.</i> [2]	87.50	83.33	86.36	-
Erda, . B., <i>et al.</i> , [6]	93.86	93.56	95.81	
Miranda, A., <i>et al.</i> [13]	76.19	87.48	81.84	-
Maddury, S. <i>et al.</i> , [14]	-	-	91.35	-
Alfonso Perez, G., <i>et al.</i> [16]	95	80	92	-
Proposed Model	100	100	100	100

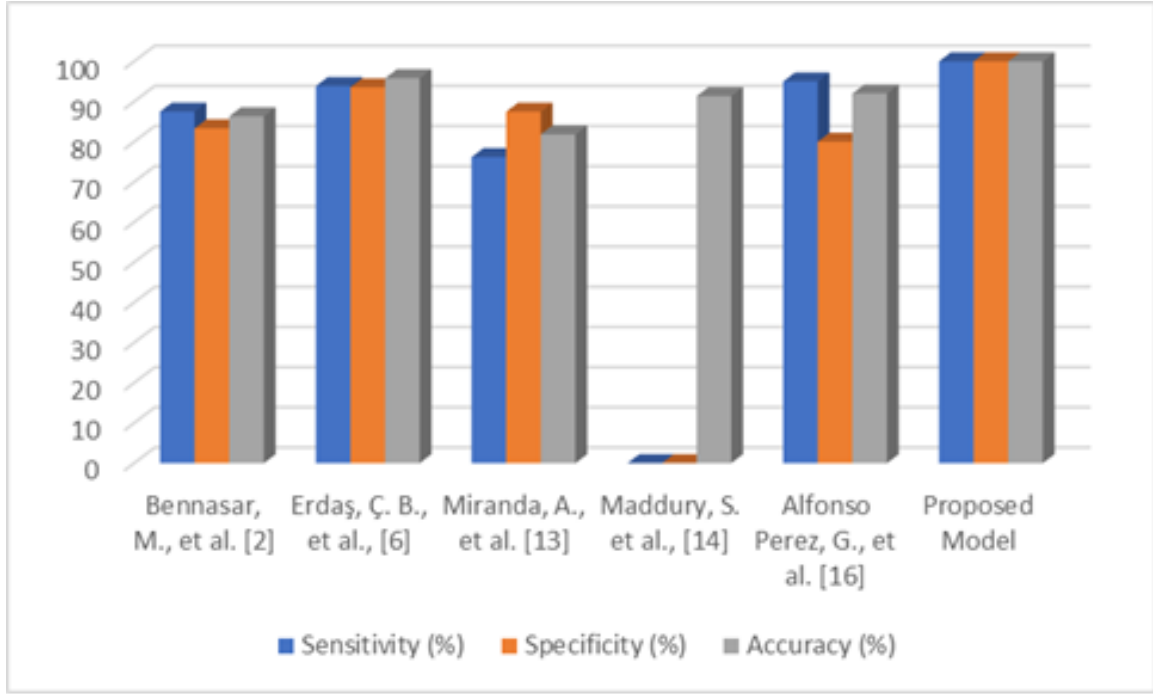


Figure 4.9: Visual representation for comparing the suggested model's performance to that of previous works

Table 4.4: Overall performance and comparison with the proposed model.

Model / Year	Modality	Algorithm Used	No. of features used	No. of <i>HD</i> Patients	Accuracy (%)	Research focus
Bennasar, M., <i>et al.</i> 2016 [2]	Accelerometer data	<i>SVM</i>	90	15	86.36	Classification
Cheng, J., <i>et al.</i> , 2020 [5]	Genetic data	RF	157	-	97.46	Identification
Erda, . B., <i>et al.</i> , 2024 [6]	Gait data	CNN	13	20	95.81	Classification
Miranda, A., <i>et al.</i> 2016 [13]	Oculomotor data	<i>SVM</i>	-	14	81.84	Classification
Maddury, S. 2023 [14]	Biological signal data (EEG, ECG, FNIRS)	Tree	-	27	91.35	Progression rate
Ouwerkerk, J., <i>et al.</i> , 2023 [15]	Genetic data	RNN	21000	486	85.2	Prediction
Alfonso Perez, G., <i>et al.</i> 2022 [16]	DNA CpG methylation data	NN	237	33	92	Classification
Proposed Model 2025	NCV and Genetic data	<i>SVM</i>	28	10	100	Detection

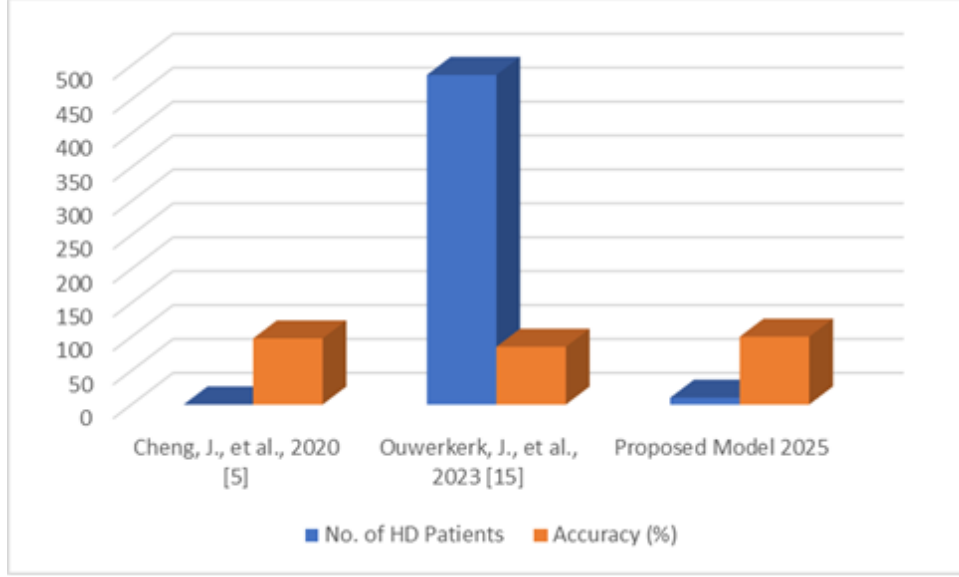


Figure 4.10: Graphical representation of the overall performance and comparison with the proposed model.

5 Discussion

Our findings demonstrate how genetic performance may be used to categorize people with Huntington's disease. We showed that genetic information directly affects the classifiers' performance. To achieve the highest accuracy, the *HD* selects which features to feed into the *SVM* algorithm. The best results were obtained for all classifiers. This study developed a novel representation strategy, using genetic data from *HD* patients. To aid early diagnosis, these data were then examined using a quadratic *SVM*-based machine learning technique. Notable categorization performance was achieved in this investigation. In particular, the classification accuracies of five distinct *SVM*-based models were 70%, 100%, 100%, 80%, and 60%, respectively, as shown in Table 4.2. The difficulty of differentiating between several neurodegenerative illnesses at once is the reason for the three *SVM*-based algorithms' comparatively poorer classification results when compared to other binary classification tasks. On the other hand, a more straightforward assessment based on the existence or lack of a disease is advantageous for binary classification sub-problems. This study's lack of comparable datasets makes it difficult to compare it to other studies. This distinction draws attention to an important contribution made to this research. Furthermore, the suggested method performs better than the best findings from previous research for additional sub-problems, as indicated in Table 4.3. Table 4.4 displays the total performance and comparison with the suggested model. These results imply that the suggested approach is not only generally effective but also has potential for use in treating additional neurological conditions. According to Table 4.2, the best predictor for *HD* in terms of training time and prediction speed is quadratic *SVM*. According to experimental findings, quadratic and cubic *SVMs* outperform other classification algorithms for *HD* diagnosis as well, because their underlying kernel function was better suited to capture the complex, non-linear patterns present in Huntingtons disease (*HD*). The cubic kernel implicitly maps the data into a higher-dimensional feature space, allowing for more sophisticated decision boundaries than simpler kernels like the linear one. The study's key conclusions are that a genetic test will be necessary to identify substantial *HD*.

6 Conclusion and future work

In this work, we showed that a genetic test can be a biomarker for the early detection of *HD* with reasonable accuracy and feasibility. Specifically, the genetic test tasks were the most accurate in detecting the *HD*. For the analysis of small real *HD* data, machine learning is a viable method. ML models maintained a lower prediction speed and training time while achieving a greater performance with various *SVM*-based ML algorithms and other previous research. Lastly, ML was used to develop a model that would help clinicians give advice and a future trajectory regarding *HD* patients' driving abilities. The findings also demonstrate that the quadratic *SVM*-based machine learning algorithm yields good sensitivity, specificity, and accuracy values. More *HD* participants and healthy controls that are available to patients and clinicians

will be improved in future studies to enable participation and adherence to the final clinical tool.

Authors' Contributions: Prof. Vijaya Nath Mishra participated in the study's design, data collection, and manuscript drafting. Data analysis, interpretation, and *NCV* were done by Prof. Abhishek Pathak. Mr. Arunesh Dutt, Ms. Komal Singh, and Dr. Pradeep Kumar Tiwari participated in the discussion of the findings and the interpretation of the data gathering. Prof. Ashish Khare participated in the study's design and result analysis, and Prof. Narendra Kumar Shukla oversaw the manuscript's composition.

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MODELING DENGUE TRANSMISSION WITH HUMAN AWARENESS AND STAGNANT WATER INDEX: ANALYSIS AND SIMULATION

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Abstract

Dengue fever remains a major vector-borne health threat, particularly in regions where climatic conditions and human activities such as improper waste management and water storage support mosquito breeding. This paper improves the classical nonlinear $SIR-SI$ host-vector framework by introducing an environmentally dependent mosquito recruitment function, $\lambda(\mathcal{A}) = \lambda_0 + \lambda_1 \mathcal{A}$, where $\mathcal{A}$ denotes a dimensionless Stagnant Water Index (SWI). This modification addresses a key limitation of earlier models, which assumed constant mosquito recruitment and therefore could not capture seasonal or ecological variability. By coupling the environmentally responsive recruitment term with human awareness (ϕ) and vector-control efforts (ψ), the proposed model captures dengue transmission dynamics more realistically under diverse environmental scenarios. We examine the existence and stability of both the disease-free equilibrium (DFE) and endemic equilibrium (EE), and compute the basic reproduction number $\mathcal{R}_0$ using the next-generation matrix approach. Numerical simulations establish that reducing stagnant-water accumulation (lower $\mathcal{A}$), together with enhanced community awareness, considerably curbs peak infection and reduces $\mathcal{R}_0$. The results highlight the combined importance of ecological management and human behavioral responses in controlling dengue transmission.

2020 Mathematical Sciences Classification: 34D20; 92D30; 37N25; 92B05

Keywords and Phrases: Mathematical modeling; Dengue dynamics; Basic reproduction number; Stability analysis; Next-generation matrix; Mosquito-human interaction; Endemic equilibrium; Stagnant water index

1 Introduction

Dengue fever is a viral disease transmitted by mosquitoes, primarily the female *Aedes aegypti*, and is a major public-health concern in tropical and subtropical regions worldwide. Environmental factors play a crucial role in the disease's transmission, particularly the availability of stagnant water that enables mosquito breeding, as well as social and behavioral factors within the human population [7, 11]. Mathematical models have long played an important role in understanding dengue transmission, with several classical host-vector frameworks developed to analyze epidemic patterns [4, 5, 8]. However, many existing formulations depend on fixed mosquito recruitment rates or purely biological assumptions, and therefore do not capture environmental variability or community-driven behavioral responses [1, 11, 16].

Recent research has incorporated human awareness as a factor influencing transmission dynamics. For instance, Phaijoo and Gurung [12] demonstrated that increased public awareness can substantially reduce infection levels. In parallel, several models have focused on clinical and patient-level predictions, aiming to assess disease severity and outcomes using statistical and machine-learning approaches [2]. These studies highlight the growing need to integrate behavioral, clinical, and environmental factors into dengue modeling. A number of works have explored advanced computational and numerical techniques for analyzing infectious disease systems [9, 14, 17, 18, 19], underscoring the relevance of improved modelling frameworks for complex biological dynamics.

Building on these contributions, the present study extends the standard $SIR-SI$ model by integrating three key components: human awareness (ϕ), vector-control interventions (ψ), and an environment-dependent

mosquito recruitment rate regulated by the stagnant-water index $\mathcal{A}$. The major contribution and motivation of this study involve incorporating both human behavior and environmental influences into the recruitment function,

$$\lambda(\mathcal{A}) = \lambda_0 + \lambda_1 \mathcal{A},$$

where λ_0 represents the baseline recruitment in low-risk conditions, and λ_1 captures the sensitivity of mosquito population growth to environmental change. Unlike earlier dengue models, including the widely used frameworks of Esteva and Vargas [4, 5] and the awareness-based model of Phaijoo and Gurung [12], the present formulation replaces the traditional constant mosquito recruitment rate with an explicit environmental term. This modification provides a direct and biologically meaningful mechanism for assessing how ecological variability—such as dry versus monsoon conditions—influences dengue transmission. The stagnant-water index $\mathcal{A}$ serves as a practical environmental indicator, enabling the model to assess how ecological interventions that reduce stagnant water can lower recruitment and limit transmission. By coupling $\mathcal{A}$ with ϕ and ψ , the model provides a unified framework for analyzing the combined effects of environmental conditions, human behavior, and control strategies.

Using this extended SIR – SI formulation, we determine the disease-free equilibrium (DFE), calculate the basic reproduction number $\mathcal{R}_0$ via the next-generation method, and examine the local stability of both the DFE and endemic equilibrium (EE). Numerical experiments further show how changes in $\mathcal{A}$, awareness levels, and vector-control efforts jointly influence the resulting epidemic dynamics.

2 Model Formulation

We develop a hostvector model to capture the spread of dengue fever between humans and mosquitoes. The human host population follows an SIR (Susceptible-Infected-Recovered) structure, reflecting the fact that individuals who recover from dengue get temporary immunity. For mosquitoes, an SI (Susceptible-Infected) formulation is employed, since infected mosquitoes do not recover; therefore, a recovered class is biologically unnecessary.

The human population is divided into three epidemiological classes: susceptible ($\mathcal{H}_S$), infected ($\mathcal{H}_I$), and recovered ($\mathcal{H}_R$). Similarly, the mosquito population is represented by susceptible ($\mathcal{M}_S$) and infected ($\mathcal{M}_I$) classes. The interaction between these compartments and the flow of infection in the system are illustrated in the following diagram.

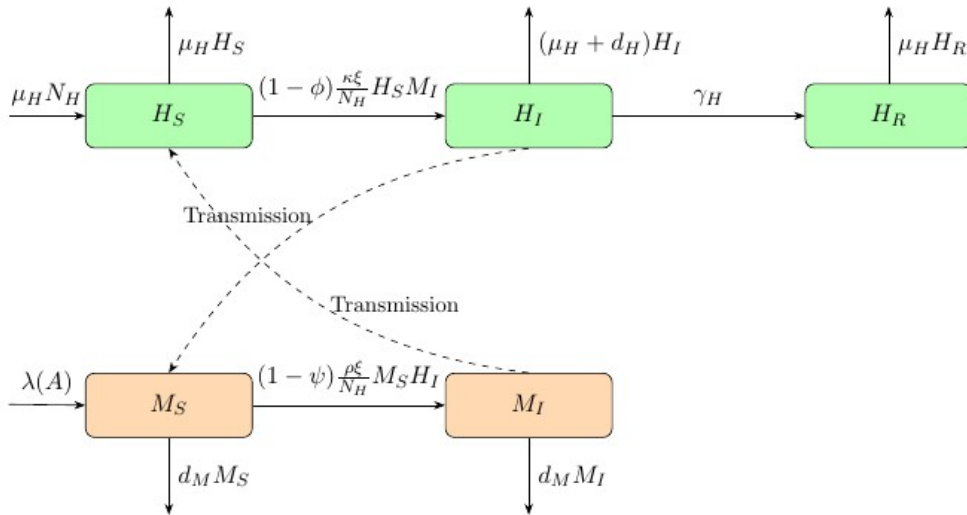


Figure 2.1: Flowchart representing the compartmental structure of the SIR – SI dengue model, showing the transitions between human hosts and mosquito classes and the corresponding epidemiological parameters.

Table 2.1: Description of the state variables used in the model.

Symbol	Description
$\mathcal{H}_S$	The population of susceptible humans
$\mathcal{H}_I$	The population of infected humans
$\mathcal{H}_R$	The population of recovered humans
$\mathcal{M}_S$	The population of susceptible mosquitoes
$\mathcal{M}_I$	The population of infected mosquitoes
N_H	Total human population ($N_H = \mathcal{H}_S + \mathcal{H}_I + \mathcal{H}_R$)

Table 2.2: Description of parameters used in the model.

Symbol	Description
μ_H	Human birth rate / natural death rate of humans
κ	Transmission probability from mosquito to human per bite
ρ	Transmission probability from human to mosquito per bite
ξ	Average mosquito biting rate per day
ϕ	Proportion of aware susceptible humans (reduces exposure by $(1 - \phi)$)
ψ	Proportion of mosquitoes under control (reduces biting by $(1 - \psi)$)
γ_H	Recovery rate of infected humans
d_H	Disease-induced mortality rate in humans
d_M	Natural death rate of mosquitoes
$\lambda(\mathcal{A})$	Mosquito recruitment rate depending on stagnant water index $\mathcal{A} \in [0, 1]$

Using the flow structure (Figure 2.1) and the above compartment descriptions (Tables 2.1-2.2), the *SIR-SI* dengue model equations are given by:

$$\begin{aligned}
\frac{d\mathcal{H}_S}{dt} &= \mu_H N_H - (1 - \phi) \frac{\kappa \xi}{N_H} \mathcal{H}_S \mathcal{M}_I - d_H \mathcal{H}_S = f_1(\mathcal{H}_S, \mathcal{H}_I, \mathcal{H}_R, \mathcal{M}_S, \mathcal{M}_I), \\
\frac{d\mathcal{H}_I}{dt} &= (1 - \phi) \frac{\kappa \xi}{N_H} \mathcal{H}_S \mathcal{M}_I - (\gamma_H + \mu_H + d_H) \mathcal{H}_I = f_2(\mathcal{H}_S, \mathcal{H}_I, \mathcal{H}_R, \mathcal{M}_S, \mathcal{M}_I), \\
\frac{d\mathcal{H}_R}{dt} &= \gamma_H \mathcal{H}_I - \mu_H \mathcal{H}_R = f_3(\mathcal{H}_S, \mathcal{H}_I, \mathcal{H}_R, \mathcal{M}_S, \mathcal{M}_I), \\
\frac{d\mathcal{M}_S}{dt} &= \lambda(\mathcal{A}) - (1 - \psi) \frac{\rho \xi}{N_H} \mathcal{M}_S \mathcal{H}_I - d_M \mathcal{M}_S = f_4(\mathcal{H}_S, \mathcal{H}_I, \mathcal{H}_R, \mathcal{M}_S, \mathcal{M}_I), \\
\frac{d\mathcal{M}_I}{dt} &= (1 - \psi) \frac{\rho \xi}{N_H} \mathcal{M}_S \mathcal{H}_I - d_M \mathcal{M}_I = f_5(\mathcal{H}_S, \mathcal{H}_I, \mathcal{H}_R, \mathcal{M}_S, \mathcal{M}_I).
\end{aligned} \tag{2.1}$$

The recruitment rate of mosquitoes, incorporating the environmental influence, is:

$$\lambda(\mathcal{A}) = \lambda_0 + \lambda_1 \mathcal{A}, \tag{2.2}$$

which increases linearly with the stagnant water index $\mathcal{A} \in [0, 1]$. Here, $\lambda_0 > 0$ denotes the baseline daily mosquito recruitment in the absence of environmental influence, while $\lambda_1 > 0$ accounts for the additional recruitment arising from human-generated stagnant-water breeding sites.

3 Model Analysis

3.1 Positivity and Well-Posedness of the Model

Theorem 3.1 (Positivity of Solutions). *Assume that all parameters in system (2.1) are non-negative and the initial conditions satisfy*

$$(\mathcal{H}_S(0), \mathcal{H}_I(0), \mathcal{H}_R(0), \mathcal{M}_S(0), \mathcal{M}_I(0)) \in \mathbf{R}_+^5.$$

Then the non-negative orthant $\mathbf{R}_+^5$ is positively invariant under the dynamics of system (2.1). That is, the solutions remain in $\mathbf{R}_+^5$ for all $t \geq 0$, ensuring the model is biologically valid.

Proof. Consider the *SIR-SI* dengue system given in (2.1) with initial conditions

$$(\mathcal{H}_S(0), \mathcal{H}_I(0), \mathcal{H}_R(0), \mathcal{M}_S(0), \mathcal{M}_I(0)) \in \mathbf{R}_+^5.$$

Step 1: Boundary Analysis. We examine each compartment at its zero boundary, assuming all other variables are non-negative:

- At $\mathcal{H}_S = 0$, we have:

$$\left. \frac{d\mathcal{H}_S}{dt} \right|_{\mathcal{H}_S=0} = \mu_H N_H \geq 0.$$

- At $\mathcal{H}_I = 0$, we have:

$$\left. \frac{d\mathcal{H}_I}{dt} \right|_{\mathcal{H}_I=0} = (1 - \phi) \frac{\kappa \xi}{N_H} \mathcal{H}_S \mathcal{M}_I \geq 0.$$

- At $\mathcal{H}_R = 0$, we have:

$$\left. \frac{d\mathcal{H}_R}{dt} \right|_{\mathcal{H}_R=0} = \gamma_H \mathcal{H}_I \geq 0.$$

- At $\mathcal{M}_S = 0$, we have:

$$\left. \frac{d\mathcal{M}_S}{dt} \right|_{\mathcal{M}_S=0} = \lambda(\mathcal{A}) \geq 0.$$

- At $\mathcal{M}_I = 0$, we have:

$$\left. \frac{d\mathcal{M}_I}{dt} \right|_{\mathcal{M}_I=0} = (1 - \psi) \frac{\rho \xi}{N_H} \mathcal{M}_S \mathcal{H}_I \geq 0.$$

Step 2: Invariance of the Non-negative Orthant. Since the derivative of each variable is non-negative at the corresponding zero boundary, no state variable can become negative. Hence, the non-negative orthant $\mathbf{R}_+^5$ is invariant, and

$$\mathcal{H}_S(t), \mathcal{H}_I(t), \mathcal{H}_R(t), \mathcal{M}_S(t), \mathcal{M}_I(t) \geq 0, \quad \forall t \geq 0.$$

The standard invariance arguments discussed in [6, 23] lead to this conclusion

Step 3: Biological Relevance. All compartments represent population counts. Positivity ensures that the model respects natural biological constraints, preventing negative populations.

Therefore, the solutions of system (2.1) remain non-negative for every $t \geq 0$. $\square$

3.2 Equilibria of the System

Equilibrium points of a system are determined by setting all time derivatives to zero, representing a state in which the populations in all compartments remain constant. For system (2.1), this condition implies:

$$\frac{d\mathcal{H}_S}{dt} = \frac{d\mathcal{H}_I}{dt} = \frac{d\mathcal{H}_R}{dt} = \frac{d\mathcal{M}_S}{dt} = \frac{d\mathcal{M}_I}{dt} = 0.$$

3.2.1 Disease-Free Equilibrium (DFE)

The disease-free equilibrium (DFE), denoted by E^0 , corresponds to an equilibrium state in which the disease is absent from the population. In this scenario, the number of infected humans ($\mathcal{H}_I^0$) and infected mosquitoes ($\mathcal{M}_I^0$) is zero:

$$\mathcal{H}_I^0 = 0, \quad \mathcal{M}_I^0 = 0.$$

By applying these conditions to the system's differential equations, (2.1) we can determine the steady-state values for the remaining variables.

- Susceptible humans: From the first equation of (2.1),

$$0 = \mu_H N_H - \mu_H \mathcal{H}_S^0 \Rightarrow \mathcal{H}_S^0 = N_H.$$

- Recovered humans: From the third equation of (2.1),

$$0 = \gamma_H \mathcal{H}_I^0 - \mu_H \mathcal{H}_R^0 \Rightarrow \mathcal{H}_R^0 = 0.$$

- Susceptible mosquitoes: From the fourth equation of (2.1),

$$0 = \lambda(\mathcal{A}) - d_M \mathcal{M}_S^0 \Rightarrow \mathcal{M}_S^0 = \frac{\lambda(\mathcal{A})}{d_M}.$$

Hence, the DFE is

$$E^0 = \left(N_H, 0, 0, \frac{\lambda(\mathcal{A})}{d_M}, 0 \right). \quad (3.1)$$

3.3 Basic Reproduction Number ($\mathcal{R}_0$)

The basic reproduction number, $\mathcal{R}_0$, measures the average number of new infections generated by a single infected individual in a population where all individuals are susceptible. To compute $\mathcal{R}_0$, we apply the next-generation matrix method, considering only the infected compartments $(\mathcal{H}_I, \mathcal{M}_I)$.

In this framework, we define $\mathcal{F}$ as the rate of appearance of new infections and $\mathcal{V}$ as the rate of transfer or removal from the infected compartments.

$$\begin{aligned}\mathcal{F}_1 &= (1 - \phi) \frac{\kappa \xi}{N_H} \mathcal{H}_S \mathcal{M}_I, & \mathcal{V}_1 &= (\gamma_H + \mu_H + d_H) \mathcal{H}_I \\ \mathcal{F}_2 &= (1 - \psi) \frac{\rho \xi}{N_H} \mathcal{M}_S \mathcal{H}_I, & \mathcal{V}_2 &= d_M \mathcal{M}_I\end{aligned}$$

The Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$, evaluated at the DFE E^0 given by (3.1), with respect to $(\mathcal{H}_I, \mathcal{M}_I)$ are:

$$\begin{aligned}F &= \begin{pmatrix} \frac{\partial \mathcal{F}_1}{\partial \mathcal{H}_I} & \frac{\partial \mathcal{F}_1}{\partial \mathcal{M}_I} \\ \frac{\partial \mathcal{F}_2}{\partial \mathcal{H}_I} & \frac{\partial \mathcal{F}_2}{\partial \mathcal{M}_I} \end{pmatrix}_{E^0} = \begin{pmatrix} 0 & \frac{(1 - \phi) \kappa \xi \mathcal{H}_S^0}{N_H} \\ \frac{(1 - \psi) \rho \xi \mathcal{M}_S^0}{N_H} & 0 \end{pmatrix}, \\ V &= \begin{pmatrix} \frac{\partial \mathcal{V}_1}{\partial \mathcal{H}_I} & \frac{\partial \mathcal{V}_1}{\partial \mathcal{M}_I} \\ \frac{\partial \mathcal{V}_2}{\partial \mathcal{H}_I} & \frac{\partial \mathcal{V}_2}{\partial \mathcal{M}_I} \end{pmatrix}_{E^0} = \begin{pmatrix} \delta_H & 0 \\ 0 & d_M \end{pmatrix}, \quad \text{where } \delta_H := \gamma_H + \mu_H + d_H.\end{aligned}$$

The next-generation matrix is $K = FV^{-1}$, giving

$$K = \begin{pmatrix} 0 & \frac{(1 - \phi) \kappa \xi \mathcal{H}_S^0}{N_H d_M} \\ \frac{(1 - \psi) \rho \xi \mathcal{M}_S^0}{N_H \delta_H} & 0 \end{pmatrix}.$$

The basic reproduction number $\mathcal{R}_0$ is the spectral radius of K , $\mathcal{R}_0 = \rho(K)$.

$$\mathcal{R}_0 = \sqrt{\left(\frac{(1 - \phi) \kappa \xi \mathcal{H}_S^0}{N_H d_M} \right) \cdot \left(\frac{(1 - \psi) \rho \xi \mathcal{M}_S^0}{N_H \delta_H} \right)}$$

Substituting the DFE values for $\mathcal{H}_S^0$ and $\mathcal{M}_S^0$:

$$\mathcal{H}_S^0 = N_H, \quad \mathcal{M}_S^0 = \frac{\lambda(\mathcal{A})}{d_M}.$$

Thus, the basic reproduction number is:

$$\boxed{\mathcal{R}_0 = \sqrt{\frac{(1 - \phi)(1 - \psi) \kappa \rho \xi^2 \lambda(\mathcal{A})}{N_H d_M^2 \delta_H}}}. \quad (3.2)$$

Letting

$$C = (1 - \phi) \kappa \xi, \quad B = \frac{(1 - \psi) \rho \xi \lambda(\mathcal{A})}{N_H d_M},$$

we can write

$$\mathcal{R}_0 = \sqrt{\frac{CB}{\delta_H d_M}}. \quad (3.3)$$

3.4 Stability Analysis at Disease Free Equilibrium point

Local stability of the disease-free equilibrium E^0 is ensured when the Jacobian J at E^0 has eigenvalues with strictly negative real components, under classical stability theory [15] and the Routh-Hurwitz criterion.

Theorem 3.2. *The disease-free equilibrium point E^0 of the system given in (2.1) is:*

- (i) **Locally asymptotically stable** if $\mathcal{R}_0 < 1$,
- (ii) **Unstable** if $\mathcal{R}_0 > 1$.

Proof. The Jacobian matrix of system (2.1) is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial \mathcal{H}_S} & \frac{\partial f_1}{\partial \mathcal{H}_I} & \frac{\partial f_1}{\partial \mathcal{H}_R} & \frac{\partial f_1}{\partial \mathcal{M}_S} & \frac{\partial f_1}{\partial \mathcal{M}_I} \\ \frac{\partial f_2}{\partial \mathcal{H}_S} & \frac{\partial f_2}{\partial \mathcal{H}_I} & \frac{\partial f_2}{\partial \mathcal{H}_R} & \frac{\partial f_2}{\partial \mathcal{M}_S} & \frac{\partial f_2}{\partial \mathcal{M}_I} \\ \frac{\partial f_3}{\partial \mathcal{H}_S} & \frac{\partial f_3}{\partial \mathcal{H}_I} & \frac{\partial f_3}{\partial \mathcal{H}_R} & \frac{\partial f_3}{\partial \mathcal{M}_S} & \frac{\partial f_3}{\partial \mathcal{M}_I} \\ \frac{\partial f_4}{\partial \mathcal{H}_S} & \frac{\partial f_4}{\partial \mathcal{H}_I} & \frac{\partial f_4}{\partial \mathcal{H}_R} & \frac{\partial f_4}{\partial \mathcal{M}_S} & \frac{\partial f_4}{\partial \mathcal{M}_I} \\ \frac{\partial f_5}{\partial \mathcal{H}_S} & \frac{\partial f_5}{\partial \mathcal{H}_I} & \frac{\partial f_5}{\partial \mathcal{H}_R} & \frac{\partial f_5}{\partial \mathcal{M}_S} & \frac{\partial f_5}{\partial \mathcal{M}_I} \end{pmatrix}.$$

The Jacobian matrix at DFE point E^0 is:

$$J(E^0) = \begin{pmatrix} -\mu_H & 0 & 0 & 0 & -C \\ 0 & -\delta_H & 0 & 0 & C \\ 0 & \gamma_H & -\mu_H & 0 & 0 \\ 0 & -B & 0 & -d_M & 0 \\ 0 & B & 0 & 0 & -d_M \end{pmatrix}.$$

The eigenvalues are obtained from the characteristic equation $\det(J(E^0) - \lambda I) = 0$. This yields:

$$\lambda_1 = -\mu_H, \quad \lambda_2 = -\mu_H, \quad \lambda_3 = -d_M,$$

and the remaining two satisfy the quadratic equation derived from the lower right 2×2 block of the matrix:

$$\lambda^2 + (\delta_H + d_M)\lambda + \delta_H d_M - CB = 0.$$

Using $CB = \mathcal{R}_0^2 \delta_H d_M$ from (3.3), the quadratic can be written as

$$\lambda^2 + (\delta_H + d_M)\lambda + \delta_H d_M(1 - \mathcal{R}_0^2) = 0.$$

Let $c_1 = \delta_H + d_M$ and $c_2 = \delta_H d_M(1 - \mathcal{R}_0^2)$. Since $c_1 > 0$:

- If $\mathcal{R}_0 < 1$, then $c_2 > 0$. By the Routh-Hurwitz criterion, both roots have negative real parts. Together with $\lambda_1, \lambda_2, \lambda_3 < 0$, all eigenvalues are negative, and E^0 is *locally asymptotically stable*.
- If $\mathcal{R}_0 > 1$, then $c_2 < 0$. Since the product of the roots is negative, the quadratic has one positive and one negative real root. Thus E^0 is *unstable*.

□

3.5 Endemic Equilibrium (E^*): Existence and Stability

An endemic equilibrium, denoted E^* , represents a state where the disease persists in both human and mosquito populations, with the number of infected individuals remaining positive and constant, i.e., $\mathcal{H}_I^* > 0$ and $\mathcal{M}_I^* > 0$. The existence and stability of E^* are governed by the basic reproduction number $\mathcal{R}_0$.

Theorem 3.3 (Existence of Endemic Equilibrium). *The system (2.1) possesses a unique endemic equilibrium*

$$E^* = (\mathcal{H}_S^*, \mathcal{H}_I^*, \mathcal{H}_R^*, \mathcal{M}_S^*, \mathcal{M}_I^*)$$

with all components positive if and only if $\mathcal{R}_0 > 1$. Here, let

$$\alpha = 1 + \frac{\gamma_H}{\mu_H}, \quad K = \frac{(1 - \phi)\kappa\xi}{\delta_H N_H}, \quad L = \frac{(1 - \psi)\rho\xi}{d_M N_H}, \quad N_M = \frac{\lambda(\mathcal{A})}{d_M}.$$

Note that $\mathcal{R}_0^2 = KLN_H N_M$. The equilibrium components are:

$$\mathcal{H}_I^* = \frac{N_H(\mathcal{R}_0^2 - 1)}{LN_H + \alpha\mathcal{R}_0^2}, \tag{3.4}$$

$$\mathcal{H}_S^* = \frac{N_H(LN_H + \alpha)}{LN_H + \alpha\mathcal{R}_0^2}, \quad (3.5)$$

$$\mathcal{H}_R^* = \frac{\gamma_H N_H (\mathcal{R}_0^2 - 1)}{\mu_H (LN_H + \alpha\mathcal{R}_0^2)}, \quad (3.6)$$

$$\mathcal{M}_I^* = \frac{LN_M N_H (\mathcal{R}_0^2 - 1)}{\mathcal{R}_0^2 (LN_H + \alpha)}, \quad (3.7)$$

$$\mathcal{M}_S^* = \frac{N_M (\alpha\mathcal{R}_0^2 + LN_H)}{\mathcal{R}_0^2 (LN_H + \alpha)}. \quad (3.8)$$

Proof. At equilibrium, setting the derivatives in (2.1) to zero and solving for the infected classes gives:

$$(1 - \phi) \frac{\kappa_S^{\xi}}{N_H} \mathcal{H}_S^* \mathcal{M}_I^* = \delta_H \mathcal{H}_I^*, \quad \text{or } \mathcal{H}_I^* = K \mathcal{H}_S^* \mathcal{M}_I^*$$

$$(1 - \psi) \frac{\rho_S^{\xi}}{N_H} \mathcal{M}_S^* \mathcal{H}_I^* = d_M \mathcal{M}_I^*, \quad \text{or } \mathcal{M}_I^* = L \mathcal{M}_S^* \mathcal{H}_I^*.$$

The total human population N_H remains constant at equilibrium. Solving $\frac{d\mathcal{H}_S}{dt} = \frac{d\mathcal{H}_I}{dt} = \frac{d\mathcal{H}_R}{dt} = 0$ along with $N_H = \mathcal{H}_S^* + \mathcal{H}_I^* + \mathcal{H}_R^*$ implies:

$$\mathcal{H}_S^* = N_H - \alpha\mathcal{H}_I^*, \quad \text{where } \alpha = 1 + \frac{\gamma_H}{\mu_H}.$$

Similarly, the total number of mosquitoes N_M is $N_M^* = \mathcal{M}_S^* + \mathcal{M}_I^* = N_M$, implying:

$$\mathcal{M}_S^* = N_M - \mathcal{M}_I^*.$$

Substituting $\mathcal{M}_S^*$ into the mosquito infection balance and eliminating $\mathcal{M}_I^*$:

$$\mathcal{M}_I^* = L(N_M - \mathcal{M}_I^*)\mathcal{H}_I^* \Rightarrow \mathcal{M}_I^* = \frac{LN_M \mathcal{H}_I^*}{1 + L\mathcal{H}_I^*}.$$

Substituting this into the human infection balance ($\mathcal{H}_I^* = K \mathcal{H}_S^* \mathcal{M}_I^*$) and dividing both sides by $\mathcal{H}_I^* > 0$ yields:

$$1 = K \cdot (N_H - \alpha\mathcal{H}_I^*) \cdot \left(\frac{LN_M}{1 + L\mathcal{H}_I^*} \right).$$

Rearranging to collect terms in $\mathcal{H}_I^*$:

$$1 + L\mathcal{H}_I^* = KLN_M N_H - KLN_M \alpha\mathcal{H}_I^*$$

$$\mathcal{H}_I^* (L + KLN_M \alpha) = KLN_M N_H - 1.$$

Using the relation $\mathcal{R}_0^2 = KLN_H N_M$, we simplify the expression for $\mathcal{H}_I^*$:

$$\mathcal{H}_I^* = \frac{\mathcal{R}_0^2 - 1}{L + \alpha KLN_M} = \frac{N_H(\mathcal{R}_0^2 - 1)}{LN_H + \alpha\mathcal{R}_0^2}.$$

This proves (3.4). The remaining components, (3.5)-(3.8), follow from the conservation and relation equations:

$$\mathcal{H}_S^* = N_H - \alpha\mathcal{H}_I^*, \quad \mathcal{H}_R^* = \frac{\gamma_H}{\mu_H} \mathcal{H}_I^*, \quad \mathcal{M}_I^* = \frac{LN_M^0 \mathcal{H}_I^*}{1 + L\mathcal{H}_I^*}, \quad \mathcal{M}_S^* = N_M - \mathcal{M}_I^*.$$

□

Positivity: For the endemic equilibrium E^* is biologically relevant, all state variables must be strictly positive. From (3.4), the numerator of $\mathcal{H}_I^*$, $N_H(\mathcal{R}_0^2 - 1)$, is positive if and only if $\mathcal{R}_0 > 1$, while the denominator $LN_H + \alpha\mathcal{R}_0^2$ is always positive. Therefore, $\mathcal{H}_I^* > 0$ precisely when $\mathcal{R}_0 > 1$. Since $\mathcal{H}_R^*$ is proportional to $\mathcal{H}_I^*$, and both $\mathcal{M}_I^*$ and $\mathcal{M}_S^*$ remain positive under the same condition, a unique positive endemic equilibrium exists if and only if $\mathcal{R}_0 > 1$.

Theorem 3.4 (Stability of Endemic Equilibrium). *If $\mathcal{R}_0 > 1$, the endemic equilibrium E^* is locally asymptotically stable.*

To simplify interpretation, the system can also be expressed in terms of normalized population proportions. In this normalized form, the basic reproduction number takes the compact expression

$$\mathcal{R}_0 = \sqrt{\frac{(1-\phi)(1-\psi)\kappa\rho\xi^2 m_S}{\delta_H d_M}}, \quad (3.9)$$

where

$$m_S = \frac{N_M}{N_H} = \frac{\lambda(\mathcal{A})}{d_M N_H}, \quad \delta_H = \gamma_H + \mu_H + d_H.$$

This normalized form reduces to the reproduction number of Phaijoo and Gurung [12] when the mosquito recruitment rate $\lambda(\mathcal{A})$ is constant and the disease-induced human death rate d_H is assumed negligible. In contrast, by allowing λ to depend on the stagnant water index $\mathcal{A}$, our model explicitly incorporates environmental effects.

4 Numerical Simulation and Interpretation

4.1 Numerical Methodology

The compartmental model (2.1) is formulated as a system of nonlinear ordinary differential equations. This system, given in (2.1), was solved numerically using the classical fourth-order Runge–Kutta (RK4) method implemented in Python.

4.2 Parameter Justification and Initial Conditions

To ensure the biological relevance of our numerical experiments, we adopt parameter values drawn from the literature wherever available, complemented by biologically reasonable assumed values.

Table 4.1: *Parameter values, definitions, and sources/assumptions.*

Parameter	Value	Unit	Definition	Source/Assumption
κ	0.75	(unitless)	Human infection rate parameter	[13]
ρ	0.75	(unitless)	Mosquito infection rate parameter	[13]
ξ	1/3	day ⁻¹	Human biting/contact rate	[10]
γ_H	0.14286	day ⁻¹	Human recovery rate (7 days)	[10]
μ_H	0.0000457	day ⁻¹	Human natural death rate	[10]
d_M	0.14286	day ⁻¹	Mosquito natural death rate (7 days lifespan)	[10]
d_H	0.00143	day ⁻¹	Disease-induced death rate	Assumed
λ_0	5000	day ⁻¹	Baseline mosquito recruitment	Assumed
λ_1	5000	day ⁻¹	SWI-dependent recruitment coefficient	Assumed
$\mathcal{A}$	0.1, 0.7	(dimensionless)	Stagnant Water Index (dry/monsoon)	Assumed

The disease-induced mortality rate for humans (d_H) is taken as 0.00143 day⁻¹. This value corresponds to an assumed case fatality proportion of approximately 1% relative to the human recovery rate (γ_H), which is consistent with typical clinical outcomes for dengue patients receiving adequate medical care. The baseline mosquito recruitment rate (λ_0) and the environmentally driven recruitment coefficient (λ_1) are both assigned the value 5000 day⁻¹. These choices ensure that the mosquito population attains a stable, high endemic level while also exhibiting a substantial increase in recruitment as environmental conditions become more favorable (i.e., as the stagnant water index $\mathcal{A}$ increases). This setup effectively captures strong ecological sensitivity in mosquito proliferation.

For environmental comparison, the Stagnant Water Index ($\mathcal{A}$) is evaluated at two representative levels: 0.1 and 0.7. The value $\mathcal{A} = 0.1$ corresponds to a dry, low-risk setting with minimal stagnant water, whereas $\mathcal{A} = 0.7$ represents a monsoon-driven, high-risk environment characterized by abundant breeding sites and favorable conditions for vector growth.

Initial Conditions:

1. $H_S(0) = 9900$, $H_I(0) = 100$, $H_R(0) = 0$,
2. $M_S(0) = 34750$, $M_I(0) = 350$.

These conditions correspond to a human population of $N_H = 10,000$ with 1% initial infection, and a mosquito population of 35,000 with 1% infected vectors.

4.3 Comparison with the Phaijoo and Gurung (2015) Model

4.3.1 Full Model of Phaijoo and Gurung (2015):

Phaijoo and Gurung ([12]) proposed an *SIR-SI* dengue model incorporating human awareness (ϕ) and vector control (ψ). Denoting total human population by

$$N_H = H_S + H_I + H_R,$$

their system is given by:

Human Population Equations

$$\begin{aligned} \frac{dH_S}{dt} &= \mu_H N_H - (1 - \phi) \frac{\xi \kappa}{N_H} H_S M_I - \mu_H H_S, \\ \frac{dH_I}{dt} &= (1 - \phi) \frac{\xi \kappa}{N_H} H_S M_I - (\mu_H + \gamma_H) H_I, \\ \frac{dH_R}{dt} &= \gamma_H H_I - \mu_H H_R. \end{aligned} \quad (4.1)$$

Vector Population Equations

$$\begin{aligned} \frac{dM_S}{dt} &= \lambda_0 - (1 - \psi) \frac{\rho \xi}{N_H} M_S H_I - d_M M_S, \\ \frac{dM_I}{dt} &= (1 - \psi) \frac{\rho \xi}{N_H} M_S H_I - d_M M_I. \end{aligned} \quad (4.2)$$

Basic Reproduction Number:

$$\mathcal{R}_0^{PG} = \sqrt{\frac{(1 - \phi)(1 - \psi) \kappa \rho \xi^2 \lambda_0}{(\mu_H + \gamma_H) d_M^2 N_H}}. \quad (4.3)$$

4.3.2 Extended Model with Environment-Dependent Recruitment

In our extended framework, mosquito recruitment incorporates stagnant-water effects:

$$\lambda(\mathcal{A}) = \lambda_0 + \lambda_1 \mathcal{A}.$$

This relationship reflects that greater availability of stagnant water (represented by $\mathcal{A}$) increases the mosquitoes recruitment rate. From the normalized reproduction number (3.9), the extended reproduction number is:

$$R_0^{\text{ext}}(\mathcal{A}) = \mathcal{R}_0^{PG} \sqrt{\left(1 + \frac{\lambda_1 \mathcal{A}}{\lambda_0}\right) \cdot \left(\frac{\mu_H + \gamma_H}{\delta_H}\right)}. \quad (4.4)$$

If we assume no disease-induced death (i.e., $d_H = 0$, implying $\delta_H = \mu_H + \gamma_H$) and equate the recruitment parameter ($\lambda_0 = \lambda_1$), the expression simplifies to a direct relationship with the environmental factor:

$$R_0^{\text{ext}}(\mathcal{A}) = \mathcal{R}_0^{PG} \sqrt{1 + \mathcal{A}}. \quad (4.5)$$

This shows that increasing stagnant water ($\mathcal{A}$) directly increases the basic reproduction number by the factor:

$$\sqrt{1 + \mathcal{A}}.$$

4.4 Interpretation and Validation of $\mathcal{R}_0$

Table 4.2: Comparison of $\mathcal{R}_0$ under varying stagnant-water conditions.

Scenario	Awareness-only Model	Extended Model ($\mathcal{A} = 0.1$)	Extended Model ($\mathcal{A} = 0.7$)	Outcome
No control ($\phi = 0, \psi = 0$)	3.2733	3.4160	4.2467	Epidemic
High control ($\phi = 0.8, \psi = 0.8$)	0.6547	0.6833	0.8495	Disease Eradication

Environmental influence. For both control regimes, $\mathcal{R}_0$ increases as $\mathcal{A}$ rises from 0.1 (dry) to 0.7 (monsoon). For example, under no control, $\mathcal{R}_0$ grows from 3.4160 to 4.2467, showing that favorable breeding conditions sharply enhance transmission potential.

Comparison with the baseline model. The awareness-only model assumes fixed mosquito recruitment, yielding a constant $\mathcal{R}_0$. The extended model provides a more realistic seasonal range:

- $\mathcal{A} = 0.1$: $\mathcal{R}_0 = 3.4160$ (dry season risk lower than baseline),
- $\mathcal{A} = 0.7$: $\mathcal{R}_0 = 4.2467$ (approaches baseline during monsoon).

Effectiveness of control measures. High awareness ($\phi = 0.8$) and strong vector control ($\psi = 0.8$) push all values of $\mathcal{R}_0$ below one, even for high $\mathcal{A}$. This demonstrates that public health interventions can effectively counteract environmental risk.

Overall, these results confirm that incorporating the environmental index $\mathcal{A}$ is essential to capture seasonal variation in dengue transmission, and that the combination of awareness and vector control remains effective across all environmental conditions.

4.5 Comparative Evaluation of Control Methods in High Transmission Risk Environments

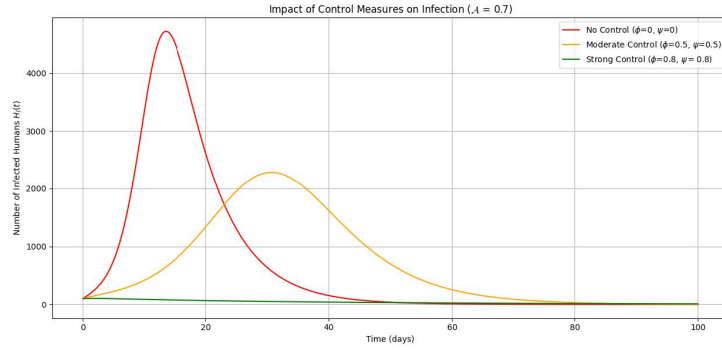


Figure 4.1: Time series of $\mathcal{H}_I$ (Infected Humans) under high stagnant-water index ($\mathcal{A} = 0.7$) for different levels of awareness (ϕ) and vector control (ψ). The curves correspond to: $(\phi = 0, \psi = 0)$, $(\phi = 0.5, \psi = 0.5)$, and $(\phi = 0.8, \psi = 0.8)$.

Interpretation: Figure 4.1 demonstrates that in the absence of awareness or vector control ($\phi = \psi = 0$), the number of infected humans rises sharply and reaches its maximum peak. Even under high-risk environmental conditions ($\mathcal{A} = 0.7$), the combined application of awareness and vector-control interventions markedly suppresses transmission. When both measures are strong ($\phi = \psi = 0.8$), infection levels remain low and decline rapidly, highlighting the effectiveness of integrated public-health strategies.

4.6 Influence of Stagnant Water Index on Infection Peaks

To investigate the role of environmental conditions, the model is simulated for two values of the stagnant-water index $\mathcal{A}$.

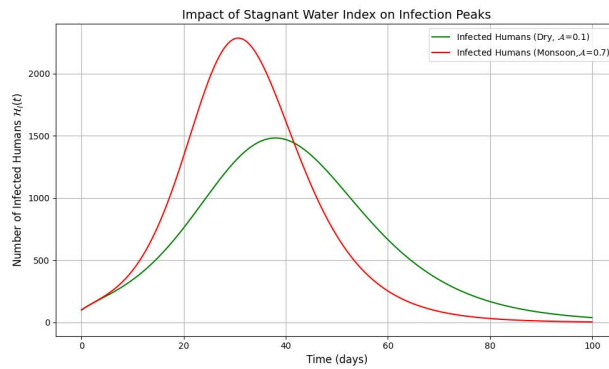


Figure 4.2: Time series of $\mathcal{H}_I$ (Infected Humans) for dry-season conditions ($\mathcal{A} = 0.1$) and monsoon-like conditions ($\mathcal{A} = 0.7$), with moderate awareness ($\phi = 0.5$) and vector control ($\psi = 0.5$).

Interpretation: Figure 4.2 compares infection dynamics during dry-season ($\mathcal{A} = 0.1$) and monsoon-season ($\mathcal{A} = 0.7$) scenarios. A higher stagnant-water index substantially enhances mosquito recruitment, resulting in larger and more persistent infection peaks. The comparison underscores that environmental factors—particularly water accumulation—play a critical role in shaping outbreak intensity, even when awareness and vector control are moderately applied.

4.7 Contour and Threshold Plots with Respect to ϕ and $\mathcal{A}$

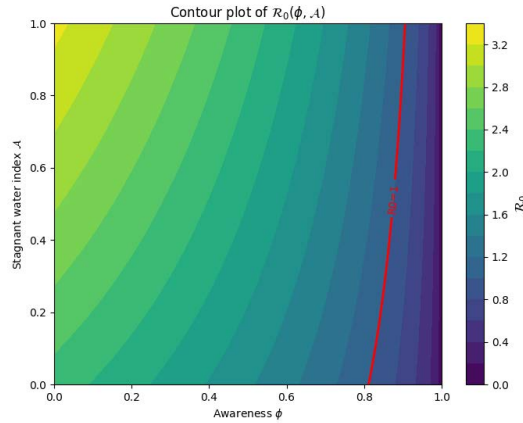


Figure 4.3: Contour plot of $\mathcal{R}_0(\phi, \mathcal{A})$, illustrating level curves of constant basic reproduction number. Values increase toward high $\mathcal{A}$ and low ϕ .

Interpretation: The contour map in Figure 4.3 illustrates how awareness (ϕ) and stagnant water index ($\mathcal{A}$) jointly affect $\mathcal{R}_0$. The highest transmission risk occurs in regions of high $\mathcal{A}$ and low awareness. The $\mathcal{R}_0 = 1$ boundary (pink line) shows the combined level of ϕ and $\mathcal{A}$ required to eliminate the disease. This confirms that environmental sanitation and behavioral interventions act synergistically in reducing dengue transmission.

4.8 Peak Infection vs. Awareness and Stagnant Water

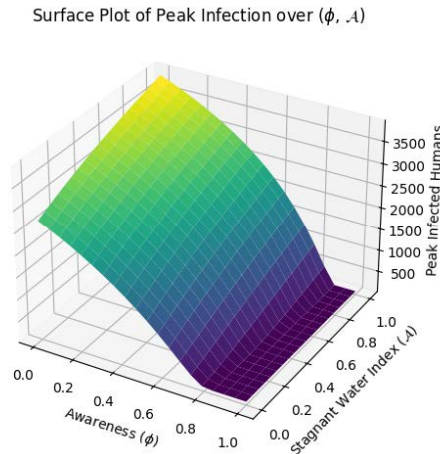


Figure 4.4: Surface plot of peak infection (maximum $\mathcal{H}_I$) as a function of awareness (ϕ) and stagnant-water index ($\mathcal{A}$). Higher awareness and lower $\mathcal{A}$ sharply decrease peak infections.

Interpretation: The surface plot in Figure 4.4 shows that peak infection levels decline sharply as awareness (ϕ) increases and stagnant water ($\mathcal{A}$) decreases. This reinforces the importance of combining public awareness campaigns with environmental sanitation.

4.9 Sensitivity Analysis

The influence of key parameters on $\mathcal{R}_0$ is examined using the normalized forward sensitivity index:

$$\Upsilon_p^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial p} \cdot \frac{p}{\mathcal{R}_0}, \quad (4.6)$$

which measures the proportional change in $\mathcal{R}_0$ caused by a proportional change in the parameter p .

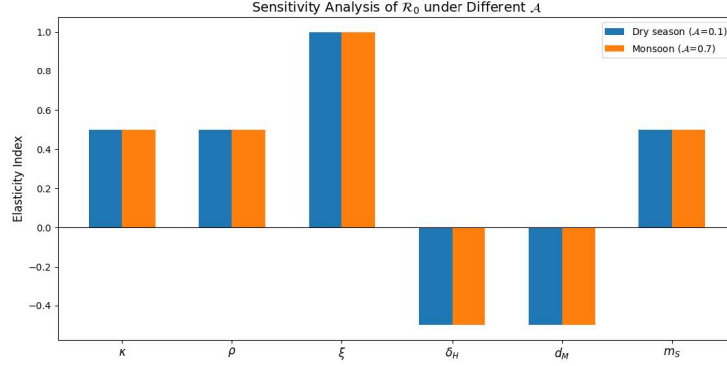


Figure 4.5: Sensitivity of $\mathcal{R}_0$ to key parameters ($\xi, \kappa, \rho, \delta_H, d_M, m_S$) under dry-season ($\mathcal{A} = 0.1$) and monsoon-season ($\mathcal{A} = 0.7$) conditions.

Interpretation: The sensitivity analysis (Figure 4.5) shows that the mosquito biting rate ξ has the strongest positive effect on $\mathcal{R}_0$, followed by m_S , κ , and ρ . Conversely, human removal δ_H and mosquito death d_M exert strong negative effects. These results highlight that interventions targeting mosquito survival and biting behavior are the most effective for controlling transmission.

Table 4.3: Sensitivity indices for $\mathcal{R}_0$ (see Figure 4.5).

Parameter	Sign	Interpretation
ξ	Positive (Strong)	An increase in the mosquito biting rate ξ substantially raises $\mathcal{R}_0$ by exposing more humans per unit time.
κ	Positive (Moderate)	The mosquito-to-human transmission probability moderately increases $\mathcal{R}_0$.
ρ	Positive (Moderate)	Increased human-to-mosquito transmission probability ρ per bite sustains the infection cycle and elevates $\mathcal{R}_0$.
δ_H	Negative (Strong)	Higher human removal rate δ_H reduces $\mathcal{R}_0$ by shortening the infectious period.
d_M	Negative (Strong)	Increased mosquito mortality d_M strongly limits vector lifespan and reduces $\mathcal{R}_0$.
m_S	Positive (Moderate)	Higher equilibrium susceptible mosquito-to-human ratio m_S increases $\mathcal{R}_0$, reflecting the role of environmental recruitment.

5 Conclusion

This study presents an extended mathematical model (2.1) to evaluate the combined effects of stagnant water accumulation ($\mathcal{A}$), public awareness (ϕ), and vector-control (ψ) on vector-borne disease transmission. By incorporating $\mathcal{A}$ linearly into mosquito recruitment $\lambda(\mathcal{A})$, the model extends the awareness-only framework of Phaijoo and Gurung [12]. The unique contribution of this model lies in its ability to quantify the synergistic

interaction between socio-behavioral responses and ecological conditions. Specifically, by replacing the standard constant mosquito recruitment with dynamic recruitment function based on the Stagnant Water Index (SWI), we demonstrate that environmental management acts as a crucial force-multiplier for human awareness, directly determining the baseline effort required for effective disease suppression.

Key findings include:

1. Higher stagnant water levels ($\mathcal{A} = 0.7$) increase mosquito breeding and intensify outbreaks (Figure 4.2).
2. Public awareness (ϕ) mitigates infections by promoting preventive behavior, lowering peak incidence and shortening outbreak duration (Figure 4.1).
3. The basic reproduction number ($\mathcal{R}_0$) rises with stagnant water $\mathcal{A}$ but decreases with increasing awareness ϕ (Figure 4.3).
4. Sensitivity analysis (Figure 4.5) identifies the mosquito biting rate ξ and mosquito death rate d_M as the most influential parameters for $\mathcal{R}_0$, confirming that vector-targeted interventions are paramount.

Effective disease control requires combined strategies addressing both environmental management (reducing $\mathcal{A}$) and community awareness (increasing ϕ), especially during high-risk periods. Future work in our research direction could incorporate spatial heterogeneity, demographic structure, or stochastic elements to further improve predictive accuracy and intervention planning. Furthermore, solving these complex extensions, which often involve time-delay or fractional-order dynamics, will require highly advanced numerical techniques. For instance, methods employing wavelet operational matrices are promising computational directions for future research, including those used to solve complex Fredholm integro-differential equations [20], singular value problems [21], and advanced biological models like the *HIV*-infected cells model [22].

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RELIABILITY AND AVAILABILITY ANALYSIS OF DEPENDENT COMPONENTS SYSTEMS

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Abstract

In this study we will discuss the method of availability and reliability of repairable apparatus with s -dependent failures among components. The foundation of system reliability is the dependability of its parts or components. Additionally dependent on component interaction, or s -dependence, is system reliability. There are s -dependent and s -independent in the system. As a result, numerous researchers have examined the accessibility and reliability of the s -dependencies among components; the failure and repair rates were regarded as constants. When we study time-related system reliability with varying failure and rates of repair, the s -dependencies among components are not considered. However, the system reliability and availability, which involve the s -dependent failures and failures varying with time and repair rates, are best suited for simulating actual systems.

This paper tells us about one type of system that resists environmental shocks and the solutions of system availability and reliability with time-varying failure and repair rates. Solutions for mechanical system availability and reliability are exemplified by a particular type of special vehicle.

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Keywords and Phrases: System reliability, Availability, Failure rates, Repair rates.

1 Introduction

A huge intricate system that makes many different functional subsystems. This subsystem can further be separated into various subsystems, components, or elements. The foundation of system reliability is the dependability of its parts or constituents [22]. Reliability of the system is also based upon interactions among the elements and the s -dependence among components. The widely accepted assumption of s -dependence among components is, to some extent, an approximation on some systems, and certain conditions-such as share-load redundant systems can not be represented with the s -independence assumption among components. Hence, in a system we can therefore conclude that not only can there be s -independent failures among components, but also s -dependent failures among them. As a result, numerous studies have considered the s -dependencies among components in system reliability and availability analysis, but they have also assumed constant failure and repair rates.

However, mechanisms of s -dependence among components are so intricate that it is exceedingly challenging to quantify the degree of this s -dependence without making certain assumptions. Studies on s -dependent failure theories for system safety have proliferated recently and have emerged in a variety of engineering domains, including the nuclear, chemical, and oil industries, as well as the mechanical and rocket launch industries. Research mostly focuses on common-cause failures in redundant systems. Consequently, they developed a technique for studying common-cause failures [6].

In this paper we have made an effort to study the system reliability as well availability with interdependent components where failure and repair rate that varies over time. The system dependability has been explored with fluctuating failure and repair rate have been taken in account rather than considering constant thus making the problem more realistic and practically applied in real world situation.

2 Brief Survey of Literature

Many researchers for examples Sharma *et al.* [17, 2007], Walter *et al.* [23, 2008], and Peyghami *et al.* [14, 2020], knowledge-based approximate reasoning methodologies which are required to analyze the reliability because of presented the growing complexity of technological systems. Zhang and Horigome [26, 2001] have obtained reliability and availability of a system with interdependent components and failure that varies over time and rates of repair. Trivedi [21, 2002] derived availability bounds for systems with limited information about the distributions. Ananda [2, 2003] discussed the long-run availability of a system by assuming lognormal repair times and exponential failure times. For system safety, s -dependent failure models have been investigated by El-Damcese and Temraz [5, 2012], which are extensively developed in a variety of engineering domains, including the nuclear, chemical, and oil industries, as well as the mechanical and rocket-launching industries. Vaurio [22, 1995] created a data bank of common cause failures in addition to developing a strategy for assessing them.

Redundant systems in parallel have been described by Shooman [18, 1968], Lin *et al.* [10, 1993], Huang and Xu [7, 2010], and Yang *et al.* [24, 2024], and analysis for parallel repairable systems with degradation facility was done by El-Damcese and Temraz [5, 2012]. Singh [19, 1980] presented the *MCS* technique for expanding the cut-set approach to systems with s -dependencies. The preventative replacement of the system using s -dependent components had been obtained by Albin and Chao [1, 1992]. Okumoto [12, 1981] studied the accessibility of a 2-component dependent system where a shock-damage process has the potential to cause both components to fail at the same time, that is, Poisson with parameter $\lambda_{1,2}$. Temraz [20, 2018] did an analysis of a system consisting of two dependent components with degradation facility and load sharing. Yin *et al.* [25, 2022] and Lesanovsky [9, 1988] presented the multistate Markov models for systems that have s -dependent units, wherein the framework is a homogenous continuous-time Markov process with discrete states, and investigated the system's transition-rate matrix, which is represented by a modified Kronecker total of its units' transition-rate matrices, and introduced an algorithm that determined the Kronecker algebra-based transition-rate matrix. Dorre [4, 1992] explained a stochastic multiple-state explanation on the s -dependent failures and studied the relationship between variability of objective and uncertainty of subjective and the impact of both characteristics on the created result of s -dependent failures. That shows a cohesive approach has both the state of knowledge correlation technique as a framework for explaining subjective parameter uncertainty and the 'generalized model of the binomial distribution function of real-world variability integrated into a multiple-state stochastic quantitative description of systems. The Clayton's framework for s -dependent elements that make up the system reliability was considered by Lyderson [11, 1992]. The main point, a general representation of system reliability in this model, is explained in accordance with the component reliabilities and the s -dependency parameter. The advantage of a model is that it explains the s -dependency regarding a single parameter and also explains the practical interpretation. For use in this model, estimation of s -dependency works as the key.

Humphreys and Jenkins [8, 1991] developed a technique for dealing with s -dependent failures. The method for s -dependent failure problems in real systems has been discussed by Parkinson [13, 1988] in which the failure and repair rates are assumed constant and which all concern s -dependent failures. There are other studies that handled the system reliability of time-varying failure rates. Atwood [3, 1992] has obtained the parametric estimation of time-dependent failure rates for probabilistic risk assessment, in which failures of components are assumed to follow the Poisson process, and it explains many parametric models for the time-dependent failure rate $\lambda(t)$. A concise review of key methodologies for availability, reliability, and maintainability in industrial systems was done by Saini and Kumar [16, 2025]. Rausand and Hoyland [15, 2004] considered the role of mathematical models, statistical methods, and their applications in the practical analysis of maintenance failure data.

3 The Model

The main contents of this paper are divided into three parts.

- i Description of state for repairable system with s -dependent failures.
- ii One type of special vehicle with such mechanical systems illustrates system reliability and availability solutions.
- iii System reliability and availability solutions involve time-varying failure and repair rates.

Assumptions

- i There are two discrete states of system components: one failed and the other operational

- iii Transition of states occurs instantaneously; the probability of two or more states coming into being state transitions in an interval of time dt is $0(dt)$.
- iii Each component has a failure and repair rate that are functions of time.
- iv There are n components of the system, which has a total of 2^n states.
- v A set with k components failed is one failure state of the system regardless of which one is active or passive in the failure mechanism.

Assumptions (iv) and (v) describe s-dependencies among components and also consider splitting component failures into single failure and joint failure.

Let us consider the s-dependencies among components. There can be an occurrence of $1, 2, \dots, k, \dots, n$ components failing simultaneously. Then the system fails. We put the system into repair after failure. After repair, it is put into work again. As we know, that system is always in the failure or operational state $\{ \text{system state } i = 0, 1, \dots, 2^n - 1. \text{ If } i = 0 \text{ such that it is a working state and } i > 0 : \text{ failure states} \}$. System states are defined as

0: The system has n components, which are all operational; thus, the system is operational.

I: The system is in 1 of the following 4 failure types, according to the amount of sdependence, which are:

- i If one component fails, there are ${}^nC_1 = n$ failure states. Then these failure states are written as: $1, 2, \dots, n$.
- ii Two out of n components fail at the same time; there are a total of nC_2 failure states. These failure states are written as follows: $(n+1), (n+2), \dots, (n+{}^nC_2)$, respectively.
- iii Similarly, k out of n components fail simultaneously; there is a total of nC_k failure states. Then these failure states are arranged in this form: $({}^nC_1 + {}^nC_2 + \dots + {}^nC_{k-1} + 1), ({}^nC_1 + {}^nC_2 + \dots + {}^nC_{k-1} + 2), \dots, ({}^nC_1 + {}^nC_2 + \dots + {}^nC_k)$, respectively, continuing in this manner that $(k-1)$ out of the n components are failed.
- iv If all components have failed, then in this case there is only one failure state, $i = \sum_{i=1}^n C_i = 2^n - 1$.

Then $E = \{0, 1, 2, \dots, 2^n - 1\} = \{WUF\}$. If $X(t) = 0$, the components of the system are in operation state. Where E is the state-space, W is the operation - state set $\{0\}$, F is the failure state set: $\{1, 2, \dots, 2^n - 1\}$ and $X(t)$ is the variate of the system state at t .

If $X(t) = I$, the system is in state i , which means that at least one component has failed and $\{X(t), t \geq 0\}$ is a stochastic process whose state space is E .

When failure and repair rates are constant, then this stochastic process becomes a homogeneous Markov process, and this homogeneous Markov process is used to solve such types of problems.

4 Reliability and Availability of Time-Varying Failure and Repair Rates Systems.

Assumptions

1. There are 4 different components of the system.
2. When 1 component fails, then the system fails. When the system has failed, no other components can fail. The system operates under a random environment.
3. At least 2 components can fail simultaneously due to a cumulative shock - damage process caused by environment with $\lambda_i(t)$.
4. At least 3 components can fail simultaneously due to a cumulative shock - damage process caused by environment with $\lambda_i(t)$.
5. When all components fail simultaneously, they are all replaced by new ones so that there is a state transition from this state to the system operative state with $\mu_r(t)$: the system state transition diagram is given in Fig. 4.1.
6. The system has 4 repair facilities, and they all have the same operational proficiency. The failure and repair rates are the function of time, which is shown in Fig. 4.1.

From Fig. 4.1 the equation is:

$$\begin{aligned}
 P'(t) &= T'_R(t)P(t), \\
 P'(t) &=> (P'_0(t), P'_1(t), \dots, P'_{15}(t))^T, \\
 P(t) &=> (P_0(t), P_1(t), \dots, P_{15}(t))^T, \\
 \lambda &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_{1,2} + \lambda_{1,3} + \lambda_{1,4} + \lambda_{2,3} + \lambda_{2,4} + \lambda_{3,4} + \lambda_{1,2,3} + \lambda_{1,2,4} + \lambda_{1,3,4} + \lambda_{2,3,4} + \lambda_{1,2,3,4}, \quad (4.1)
 \end{aligned}$$

where, $(P_0(0), P_1(0), \dots, P_{15}(0)) = (1, 0, \dots, 0)$ is the initial condition of (1) and $T'_R(t)$ is the state-transition rate matrix.

In the same way, the above procedure can be solved easily. For n components and the diagram for n components, like as Fig. 4.1, then

$$P'(t) = T_R(t)XP(t), \quad (4.2)$$

where $T_R(t)$ is the transition rate matrix. The initial condition of (4.2) is

$$\begin{aligned} (P_0(0), P_1(0), \dots, P_{2^n-1}(0)) &= (1, 0, \dots, 0), \\ P'(t) &= (P'_0(t), P'_1(t), \dots, P'_{2^n-1}(t))^\top, \\ P(t) &= (P_0(t), P_1(t), \dots, P_{2^n-1}(t))^\top. \end{aligned}$$

$T_R(t)$ and $T'_R(t)$ have the same characteristic. Therefore, the numerical solution for equation (4.2) is possible. We know that there exists a coupling term in this group, which is related to the state-transition rate matrix of differential equation (4.2), and it contains a time-varying factor.

Let us suppose that the factors of equation (4.2) are constant, then on solving equation (4.2) by Laplace transform, we get

$$P_0^*(s) = \frac{\prod_{i=1}^{2^n-1} (s + \mu_i)}{s.H(s)}, s > 0 \text{ and where, } H(s) = \prod_{i=1}^{2^n-1} (s + \mu_i) + \sum_{i=1}^{2^n-1} \lambda_i \prod_{j \neq i}^{2^n-1} (s + \mu_j). \quad (4.3)$$

$P_0^*(s)$ is Laplace transform of $P_0(t)$, and $s > 0$. $H(s)$ is a polynomial of degree $(2^n - 1)$. It has $-S_j$ real roots for all $j = 1, 2, 3, \dots, (2^n - 1)$, such that $H(-S_j) = 0 \forall j = 1, 2, 3, \dots, (2^n - 1)$.

Now taking inverse Laplace transform of (4.3), we find

$$P_0(t) = A_0 + \sum_{j=1}^{2^n-1} A_j e^{-S_j t}, A_0 = \frac{1}{1 + \sum_{i=1}^{2^n-1} \frac{\lambda_i}{\mu_i}}, A_j = \frac{1}{-S_j} \cdot \left[1 + \sum_{i=1}^{2^n-1} \frac{\lambda_i \mu_i}{(-S_j + \mu_i)^2} \right]^{-1}. \quad (4.4)$$

From (4.4), the system-transitory availability is

$$A(t) = P_0(t), \quad (4.5)$$

where $A(t)$ is the steady-state availability, which shows that,

$$A = \frac{1}{1 + \sum_{j=1}^{2^n-1} \frac{\lambda_j}{\mu_j}}. \quad (4.6)$$

Now from Fig. 4.1 the failure state, which is absorbing state, then,

$$R(t) = \exp \left[- \sum_{i=1}^{2^n-1} \lambda_i \right] t, \quad (4.7)$$

$$R_k(t) = \exp \left[- \sum_{i \in G} \lambda_i \right] t, \quad k = 1, 2, \dots, n; \quad (4.8)$$

$G = \{i \mid \text{all figure states containing component } k\}$. If failure and repair rates are not constant in (4.2), then equation (4.2) is known as linear time-varying homogeneous equations; the solution of (4.2) is

$$X(t) = \Phi(t, t_0) \times X(t_0), t \geq t_0;$$

where $\Phi(t, t_0)$ is a non-singular matrix, and $\Phi(t, t_0)$ is the state-transition matrix of a linear time-varying system that depends only on $T_R(t)$. $X(t)$ is the solution vector of (4.2).

Then,

$$\frac{\partial}{\partial t} \Phi(t, t_0) = T_R(t) \times \Phi(t, t_0), \quad (4.9)$$

with initial condition $\Phi(t, t_0) = I$.

The above equation is known as the unique solution of the matrix differential equation. From (4.9) and its initial condition $\Phi(t, t_0) = I$, the result is

$$\frac{\partial}{\partial t} \Phi(t, t_0) \times \Phi^{-1}(t, t_0) = T_R(t). \quad (4.10)$$

Let $\Phi(t, t_0)$ have the simplistic form. Now, integrating equation (4.10) in $[t_0, t]$

$$\begin{aligned}
\int_{t_0}^t \frac{\partial}{\partial t} \Phi(t, t_0) dt &= \int_{t_0}^t T_R(\tau) d\tau, \\
[\log \Phi(t, t_0)]_{t_0}^t &= \int_{t_0}^t T_R(\tau) d\tau, \\
\log \Phi(t, t_0) - \log \Phi(t_0, t_0) &= \int_{t_0}^t T_R(\tau) d\tau, \\
\log \left[\frac{\Phi(t, t_0)}{\Phi(t_0, t_0)} \right] &= \int_{t_0}^t T_R(\tau) d\tau, \\
\frac{\Phi(t, t_0)}{\Phi(t_0, t_0)} &= \exp \int_{t_0}^t T_R(\tau) d\tau, \\
\Phi(t, t_0) &= \Phi(t_0, t_0) \times \exp \int_{t_0}^t T_R(\tau) d\tau.
\end{aligned} \tag{4.11}$$

Function $\Phi(t, t_0)$ and $\Phi(t - t_0)$ are different from each other because $\Phi(t, t_0)$ is a function of t and t_0 and $\Phi(t - t_0)$ is a function of $(t - t_0)$. $\Phi(t - t_0)$ and $\Phi(t - t_0)$ are similar in their characteristic where $\Phi(t - t_0)$ is the state-transition matrix of a linear constant-factor system, and $\Phi(t, t_0)$ is the state-transition matrix of a linear time-varying system. We can write $\Phi(t, t_0)$ in exponential form.

$$\Phi(t - t_0) = \exp \left[\int_{t_0}^t T_R d\tau \right] = \exp [T_R(t - t_0)]. \tag{4.12}$$

$T_R(t)$ is time variant. Therefore $\Phi(t, t_0)$ cannot be found in the form of equation (4.12); if $T_R(t)$ satisfies the condition of result (4.1), then $\Phi(t, t_0)$ can be written as a matrix exponential:

$$\begin{aligned}
\Phi(t, t_0) &= \exp \int_{t_0}^t T_R(\tau) d\tau \\
&= 1 + \int_{t_0}^t T_R(\tau) d\tau + \frac{1}{2!} \left[\int_{t_0}^t T_R(\tau) d\tau \right]^2 + \frac{1}{3!} \left[\int_{t_0}^t T_R(\tau) d\tau \right]^3.
\end{aligned} \tag{4.13}$$

Result 1: $\Phi(t, t_0)$ is the solution of the matrix differential equation of linear time-varying systems from (4.9) with initial condition $\Phi(t_0, t_0) = 1$ that $\Phi(t, t_0) = \exp \left[\int_{t_0}^t T_R(\tau) d\tau \right]$. It will be satisfied when inserting (4.11) into it. Then, the L.H.S. and R.H.S. of (9) are, respectively,

$$\frac{\partial}{\partial t} \Phi(t, t_0) = \exp \left[\int_{t_0}^t T_R(\tau) d\tau \right] \times T_R(t).$$

Therefore, if

$$\exp \left[\int_{t_0}^t T_R(\tau) d\tau \right] \times T_R(t) = T_R(t) \times \exp \left[\int_{t_0}^t T_R(\tau) d\tau \right]. \tag{4.14}$$

If $T_R(t)$ is a matrix with constant elements or a diagonal matrix, then we proved that equation (4.14) is satisfied; otherwise, the solution of $\Phi(t, t_0)$ can be obtained from (4.9), with initial condition $\Phi(t_0, t_0) = I$. That is, when integrating (4.9) in $[t_0, t]$, then

$$\begin{aligned}
\int_{t_0}^t \frac{\partial}{\partial \tau} \Phi(\tau, t_0) d\tau &= \int_{t_0}^t T_R(\tau) \times \Phi(\tau, t_0) d\tau \\
[\Phi(\tau, t_0)]_{t_0}^t &= \int_{t_0}^t T_R(\tau) \times \Phi(\tau, t_0) d\tau.
\end{aligned}$$

Hence,

$$\Phi(t, t_0) = 1 + \int_{t_0}^t T_R(\tau) \times \Phi(\tau, t_0) d\tau. \tag{4.15}$$

Equation (4.15) is extended into a series:

$$\begin{aligned}
\Phi(t, t_0) &= 1 + \int_{t_0}^t T_R(\tau) \times \Phi(\tau, t_0) d\tau \\
&= 1 + \int_{t_0}^t T_R(\tau) \times \left[1 + \int_{t_0}^{\tau} T_R(\tau_1) \times \Phi(\tau_1, t_0) d\tau_1 \right] d\tau \\
&= 1 + \int_{t_0}^t T_R(\tau) d\tau + \int_{t_0}^t T_R(\tau) \times \int_{t_0}^{\tau} T_R(\tau_1) \times \Phi(\tau_1, t_0) d\tau_1 d\tau
\end{aligned}$$

The final solution is:

$$\begin{aligned}
\Phi(t, t_0) &= 1 + \int_{t_0}^t T_R(\tau) d\tau + \int_{t_0}^t T_R(\tau) \times \int_{t_0}^{\tau} T_R(\tau_1) d\tau_1 d\tau \\
&\quad + \int_{t_0}^t T_R(\tau) \times \int_{t_0}^{\tau} T_R(\tau_1) \times \int_{t_0}^{\tau_1} T_R(\tau_2) d\tau_2 d\tau_1 d\tau + \dots
\end{aligned} \tag{4.16}$$

5 Discussion and Conclusion

In this paper we have been discussing the method for analyzing availability and reliability of repairable systems with s -dependent failures among components; failure and repair rates of components are functions of time. Thus, there are many studies where the s -dependencies among components are taken into account in system reliability and availability analysis, but in which failure and repair rates were assumed constant. On the other hand, when we studied system reliability involving time-varying failure and repair rates, in this case the s -dependencies among components were not considered. However, to study system reliability by combining s -dependencies among components and time-varying failure and repair rates is the most appropriate for real systems. But it is very difficult to handle them mathematically; the closed-form solution of system reliability and availability is very difficult to obtain from an analytical solution. Here numerical or simulation methods must be used.

Whereas, with the increase in the number of components in the system, the manipulator time in calculation is enormous, and the calculation requires very large computer memory. The preferable method is used for solving such a problem and simplifying its structure in detail.

This paper analyzes one kind of system that endures environmental shocks and where one or more components can fail simultaneously due to a cumulative shock-damage process. Thus, in all studies concerning s -dependent failures, the failure and repair rates are constant, which becomes unacceptable to knowledgeable practitioners (in recent years) in some situations, especially for mechanical systems.

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INTEGRATED MODELING OF COLD STANDBY, MAINTENANCE STRATEGY, AND REPAIR DELAY IN MACHINING SYSTEMS

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Abstract

This paper presents a comprehensive mathematical framework for analyzing the performance of machining systems incorporating cold standby units, preventive maintenance (*PM*), and repair delays. The model captures the real-world complexity of manufacturing environments, where machines may fail unpredictably, repairs are subject to resource limitations and delays, and preventive interventions are scheduled to minimize unexpected downtimes. Using a Markovian approach with state-space representation, the system is modeled through balance equations that account for various operational states, including delay queues and active maintenance. Key performance indicators such as system availability, expected number of operational machines, and server utilization are derived to quantify system efficiency under different failure, repair, and maintenance regimes. A cost function integrating repair, delay, maintenance, and unavailability penalties is formulated and optimized under system constraints. Numerical illustrations and *MATLAB*-based optimization reveal that strategic tuning of preventive maintenance frequency and repair responsiveness can significantly reduce operational costs and improve system reliability. Graphical results further demonstrate the impact of failure rate and maintenance parameters on performance metrics. This study provides valuable insights for decision-makers aiming to enhance productivity, reduce downtime, and optimize maintenance policies in complex industrial systems.

2020 Mathematical Sciences Classification: Primary: 62N05; Secondary: 90B25, 60K10, 60K05

Keywords and Phrases: Cold standby machining system, Preventive maintenance, Repair delay, System availability, System performance analysis, Optimization of repair policy.

1 Introduction

Mathematical modeling of machining systems with cold standby, preventive maintenance, and repair delay plays a crucial role in enhancing the reliability, availability, and overall efficiency of industrial production environments. In such systems, cold standby units are inactive yet available to replace failed primary machines, thus improving fault tolerance without incurring wear during standby. Preventive maintenance (*PM*) is a proactive strategy designed to service machines before failure occurs, minimizing unexpected downtime and extending equipment life. Meanwhile, repair delay accounts for real-world constraints like technician unavailability, spare part procurement, or administrative lags, which can significantly affect recovery times and system performance. By integrating these elements into a comprehensive mathematical framework often using tools like Markov processes, queueing theory, and matrix-analytic methods researchers can evaluate key performance metrics such as system availability, mean time to failure (*MTTF*), and expected repair load. These models support optimal decision-making for maintenance scheduling, resource allocation, and system design under uncertainty, contributing to cost-effective and resilient manufacturing operations.

Yang *et al.* [14] proposed a preventive maintenance policy that accounts for two-stage dependent deterioration and external shocks. Their model highlights the importance of coupling degradation processes with stochastic shock environments to develop more robust maintenance schedules. The study successfully integrates condition-based and time-based maintenance principles, setting the groundwork for

more sophisticated models in multi-state systems. Ruiz-Castro [6] addressed the reliability behavior of complex multi-state systems under the influence of repair vacations. The study introduced a flexible framework that integrates vacation policies into the repair process, providing key insights into the behavior of degraded systems that experience service interruptions. This work laid a foundational approach for later developments in cold standby modeling and repair delays. Ruiz-Castro and Dawabsha [9] advanced the modeling of multi-state warm standby systems by incorporating preventive maintenance, unit loss, and multiple repairpersons. The model acknowledged real-world challenges like the variability in repair capacity and failures during standby, offering a more comprehensive picture of availability and reliability under constrained service resources. Shekhar *et al.* [11] explored the significance of integrating vacation policies within reliability models. Their work emphasized the critical impact of repairman availability and idle time on system performance, particularly under realistic operational conditions. By reviewing various vacation models, the study contributed to broader applications of queuing theory in reliability analysis. Cha *et al.* [1] introduced the concept of delayed worse-than-minimal repair models, offering a novel perspective on preventive replacement policies. Their study provided analytical tools for comparing different repair strategies based on their long-term effects on system failure probability, proving especially useful for aging systems with repair inefficiencies. Ruiz-Castro [7] focused on optimizing cold standby systems under multiple repair vacations and unit losses. His model evaluated the interplay between standby unit availability and service interruptions, shedding light on how repair delays and multiple vacations degrade system reliability. The results emphasized the importance of synchronizing repair schedules and standby allocation. Yang *et al.* [13] presented a joint optimization framework that integrated preventive maintenance with production scheduling in multi-state systems. Leveraging reinforcement learning, the study offered a dynamic approach for adjusting maintenance and production decisions in real-time, based on system performance and resource availability. It demonstrated the feasibility of *AI* techniques in maintenance planning. Ruiz-Castro and Acal [8] employed Marked Markovian Arrival Processes (*MMAPs*) to model complex multi-state systems with embedded vacation policies. Their analytical approach captured the bursty and correlated nature of failures and repair demands, improving the realism and precision of modeling repair delays and system transitions. Gao [3] investigated a redundant system with dependent failures and variable repair rates, using advanced stochastic methods to explore system dynamics. His work underlined the impact of repair rate heterogeneity on system performance and provided optimization strategies for repair resource allocation in redundant architectures. Gao *et al.* [4] extended the analysis to series systems experiencing common-cause failures and delayed vacations. Their approach offered reliability bounds and cost-effective repair strategies in the presence of correlated failures and structured repair interruptions. This study highlighted how common-cause effects compound the risks introduced by repair delays. Dong and Bai [2] developed a preventive maintenance policy for consecutive k -out-of- n : F balanced systems subjected to shock environments. Their methodology focused on reliability optimization under high-failure-stress conditions, providing analytical tools for balancing system redundancy with scheduled maintenance to minimize downtime. Wu *et al.* [12] explored preventive maintenance in performance-sharing multi-state systems. By modeling how performance levels influence the maintenance needs of interconnected subsystems, their study provided a nuanced understanding of how degradation propagates through shared-resource systems, helping to prevent cascading failures. Ning *et al.* [5] proposed a joint optimization framework combining preventive maintenance scheduling and protective triggering mechanisms in k -out-of- n : F systems. Their approach emphasized periodic inspections and dynamic adaptation to environmental and operational conditions, offering efficient risk-reduction strategies under uncertain failure behavior. Ruiz-Castro and Zapata-Ceballos [10] presented an algorithmic modeling approach for redundant multi-state systems subject to multiple stochastic events, including preventive maintenance, unit losses, and vacation policies. Their *MMAP*-based model allowed for detailed state transitions and service dynamics, capturing the complex interdependencies among system components. This study represents one of the most comprehensive efforts in modeling modern manufacturing systems with cold standby, delayed repair, and maintenance strategies.

2 Assumption and Notations

Assuming the system operates with:

- (i) N active machines (can fail),
- (ii) S cold standby machines (not operating but ready to be activated),
- (iii) A single repair server with repair delay δ ,

- (iv) A preventive maintenance rate θ ,
- (v) A failure rate λ per machine,
- (vi) A repair rate μ ,

and states defined as a tuple (i, j, k) , where:

- i : number of operating machines,
- j : number of failed machines under repair delay,
- k : number of machines under active repair or PM.

3 Balance Equations (Steady Equations)

Let P_{ijk} denote the steady-state probability of the system being in state (i, j, k) .

State (N, 0, 0): All machines are operational; no failure, no repair yet

$$-N\lambda P_{N,0,0} + \mu P_{N-1,0,1} = 0. \quad (3.1)$$

Machines can fail at rate $N\lambda$, or a repair from state $(N-1, 0, 1)$ brings the system back to full operation.

State (N-1, 1, 0): One failure under delay; no active repair yet

$$N\lambda P_{N,0,0} - \delta P_{N-1,1,0} = 0. \quad (3.2)$$

From full operation, one machine fails, enters delay queue; it leaves delay at rate δ to go into repair.

State (N-1, 0, 1): One machine under repair, none under delay

$$\delta P_{N-1,1,0} - \mu P_{N-1,0,1} = 0. \quad (3.3)$$

Machine moves from delay to active repair (rate δ), then completes repair (rate μ).

State (N-2, 2, 0): Two failed machines under delay

$$(N-1)\lambda P_{N-1,1,0} - 2\delta P_{N-2,2,0} = 0. \quad (3.4)$$

A second failure occurs when one is already delayed. Both leave delay at rate 2δ .

State (N-1, 0, 1), triggered by PM:

$$\theta P_{N,0,0} - \mu P_{N-1,0,1} = 0. \quad (3.5)$$

Preventive maintenance starts at rate θ , treated similarly to repair; completes at μ .

State (N-2, 1, 1): One failed machine in delay, one under PM

$$\lambda P_{N-1,0,1} + \theta P_{N-1,1,0} - (\delta + \mu) P_{N-2,1,1} = 0. \quad (3.6)$$

Failure or PM leads here; exit due to delay completion or PM ending.

4 Performance Measures

- (i) **System Availability A:** Probability that at least one machine is operational.

$$A = \sum_{i>0, j, k} P_{ijk}. \quad (4.1)$$

This excludes all states with $i = 0$, i.e., when no machine is working.

- (ii) **Expected Number of Operational Machines E_{op} :**

$$E_{op} = \sum_{i, j, k} i \cdot P_{ijk}. \quad (4.2)$$

Represents the mean number of machines currently working.

- (iii) **Expected Number of Failed Machines in Delay Queue E_{delay} :**

$$E_{delay} = \sum_{i, j, k} j \cdot P_{ijk}. \quad (4.3)$$

Captures the average number of machines in the repair delay buffer.

- (iv) **Expected Number of Machines under Repair or PM E_{repair} :**

$$E_{repair} = \sum_{i, j, k} k \cdot P_{ijk}. \quad (4.4)$$

Indicates how many machines are, on average, being actively repaired or maintained.

- (v) **Probability of Complete System Failure P_{fail} :**

$$P_{fail} = \sum_{j, k} P_{0jk}. \quad (4.5)$$

This is the probability that all machines are down, either failed or under PM.

- (vi) **Server Utilization U :** Assuming a single repair/PM server, the utilization is the probability that at least one machine is under repair or PM:

$$U = \sum_{i,j,k>0} P_{ijk}. \quad (4.6)$$

- (vii) **Preventive Maintenance Rate (PM frequency):**

$$\text{PM Rate} = \theta \cdot \sum_{i,j,k} P_{ijk}. \quad (4.7)$$

- (viii) **Failure Frequency:**

$$\text{Failure Rate} = \sum_{i,j,k} i \cdot \lambda \cdot P_{ijk}. \quad (4.8)$$

5 Cost Analysis

We introduce cost components, formulate the total cost function, and then pose a constrained optimization problem.

$$\text{Total Cost (TC)} = C_r \cdot \lambda E_{op} + C_d \cdot E_{delay} + C_p \cdot \theta + C_f \cdot E_{down} + C_s \cdot U. \quad (5.1)$$

where $E_{down} = N - E_{op}$ (Expected downtime).

C_r : cost due to failures and repairs,

C_d : cost due to failed machines waiting in delay,

C_p : cost of performing preventive maintenance,

C_f : penalty due to unavailability,

C_s : operational cost of repair server.

Objective:

$$\min_{\theta, \delta, \mu} TC(\theta, \delta, \mu). \quad (5.2)$$

Subject to Constraints:

$$\text{Repairservercapacity : } U \leq 1, \quad (5.3)$$

$$\text{Availabilityrequirement : } A \geq A_{min}, \quad (5.4)$$

$$\text{Preventivemaintenancefrequencylimits : } \theta_{min} \leq \theta \leq \theta_{max}, \quad (5.5)$$

$$\text{Delayandrepairservicelimits : } \delta_{min} \leq \delta \leq \delta_{max}, \quad \mu_{min} \leq \mu \leq \mu_{max}. \quad (5.6)$$

6 Numerical Illustration

Sample Size: 2 Machines ($N = 2$), 1 cold standby ($S = 1$), 1 repair server.

State (i, j, k) : i = working machines, j = in delay, k = in repair or preventive maintenance.

Table 6.1: Numerical values of system parameters

Parameter	Symbol	Value
Failure rate per active machine	λ	0.1
Repair rate	μ	0.5
Delay rate	δ	0.2
Preventive Maintenance rate	θ	0.05

$$-2\lambda P_{2,0,0} - \theta P_{2,0,0} + \mu P_{1,0,1} = 0, \quad (6.1)$$

$$2\lambda P_{2,0,0} - \delta P_{1,1,0} = 0, \quad (6.2)$$

$$\delta P_{1,1,0} + \theta P_{2,0,0} - \mu P_{1,0,1} = 0, \quad (6.3)$$

$$\lambda P_{1,1,0} - 2\delta P_{0,2,0} = 0, \quad (6.4)$$

$$2\delta P_{0,2,0} - (\delta + \mu) P_{0,1,1} = 0, \quad (6.5)$$

$$\delta P_{0,1,1} - 2\mu P_{0,0,2} = 0. \quad (6.6)$$

Table 6.2: Possible States

Label	State (i,j,k)	Description
S_0	(2,0,0)	Both machines working
S_1	(1,1,0)	One fails, in delay
S_2	(1,0,1)	One under repair/PM
S_3	(0,2,0)	Two in delay
S_4	(0,1,1)	One in delay, one in repair
S_5	(0,0,2)	Both in repair or under PM (system failed)

Normalization Condition:

$$P_{2,0,0} + P_{1,1,0} + P_{1,0,1} + P_{0,2,0} + P_{0,1,1} + P_{0,0,2} = 1. \quad (6.7)$$

$$\text{From (6.2): } P_{1,1,0} = \frac{2\lambda}{\delta} P_{2,0,0} = \frac{2 \times 0.1}{0.2} P_{2,0,0} = P_{2,0,0}.$$

$$\text{From (6.3): } P_{1,0,1} = \frac{\delta P_{1,1,0} + \theta P_{2,0,0}}{\mu} = \frac{0.2 P_{1,1,0} + 0.05 P_{2,0,0}}{0.5} = 0.5 P_{2,0,0}.$$

$$\text{From (6.4): } P_{0,2,0} = \frac{\lambda}{2\delta} P_{1,1,0} = \frac{0.1}{2 \times 0.2} P_{2,0,0} = 0.25 P_{2,0,0}.$$

$$\text{From (6.5): } P_{0,1,1} = \frac{2\delta}{\delta + \mu} P_{0,2,0} = \frac{2 \times 0.2}{0.2 + 0.5} (0.25 P_{2,0,0}) \approx 0.142857 P_{2,0,0}.$$

$$\text{From (6.6): } P_{0,0,2} = \frac{\delta}{2\mu} P_{0,1,1} = \frac{0.2}{2 \times 0.5} (0.142857 P_{2,0,0}) \approx 0.028571 P_{2,0,0}.$$

$$\text{Using (6.7): } P_{2,0,0} + P_{2,0,0} + 0.5 P_{2,0,0} + 0.25 P_{2,0,0} + 0.142857 P_{2,0,0} + 0.028571 P_{2,0,0} = 1. \\ 2.921428 P_{2,0,0} = 1 \implies P_{2,0,0} \approx 0.3423.$$

Results:

$$P_{2,0,0} \approx 0.3423, P_{1,1,0} \approx 0.3423, P_{1,0,1} \approx 0.1712, P_{0,2,0} \approx 0.0856, P_{0,1,1} \approx 0.0489, P_{0,0,2} \approx 0.0097.$$

$$\text{Availability} \approx 1 - P_{0,0,2} \approx 0.9903.$$

$$E_{op} = 2P_{2,0,0} + 1P_{1,1,0} + 1P_{1,0,1} \approx 1.1981.$$

$$E_{delay} = 1P_{1,1,0} + 2P_{0,2,0} + 1P_{0,1,1} \approx 0.5624.$$

$$E_{repair} = 1P_{1,0,1} + 1P_{0,1,1} + 2P_{0,0,2} \approx 0.2395.$$

$$U = E_{repair} \approx 0.2395.$$

Table 6.3: Numerical values for cost analysis

Symbol	Value	Units
C_r	100	\$/repair
C_d	50	\$/unit delay
C_p	80	\$/PM
C_f	200	\$/unit downtime
C_s	30	\$/time
$\theta_{min}, \theta_{max}$	0.01, 0.1	PM rate bounds
μ_{min}, μ_{max}	0.2, 1.0	Repair rate bounds
$\delta_{min}, \delta_{max}$	0.1, 1.0	Delay rate bounds
A_{min}	0.95	availability

MATLAB Optimization Results:

Optimal PM rate (theta): 0.0100,

Optimal Delay rate (delta): 1.0000,

Optimal Repair rate (mu): 1.0000,

Minimum Total Cost: 94.53.

7 Results and Discussion

The Figure 7.1 illustrates how the performance of a machining system degrades as the failure rate (λ) increases, under various levels of repair delay (δ). The vertical axis represents the expected number of operational machines, while the horizontal axis shows the failure rate λ ranging from 0 to 0.5. Each curve corresponds to a different value of repair delay δ (from 0.1 to 1.0). As observed, for all δ values, the expected

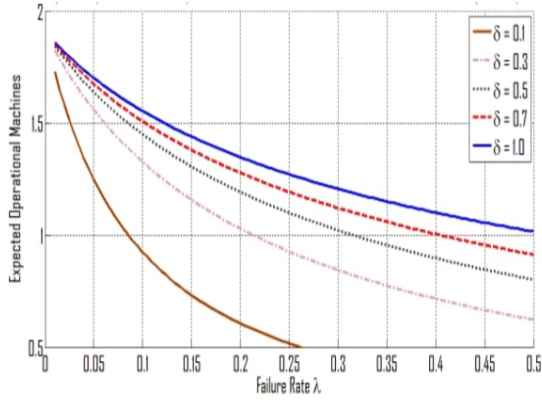


Figure 7.1: Expected number of operational machines vs. failure rate λ for different values of repair delay (δ).

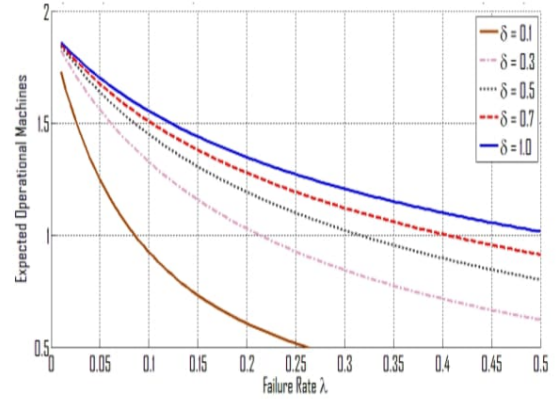


Figure 7.2: Expected number of operational machines vs. failure rate λ for different values of repair delay (δ).

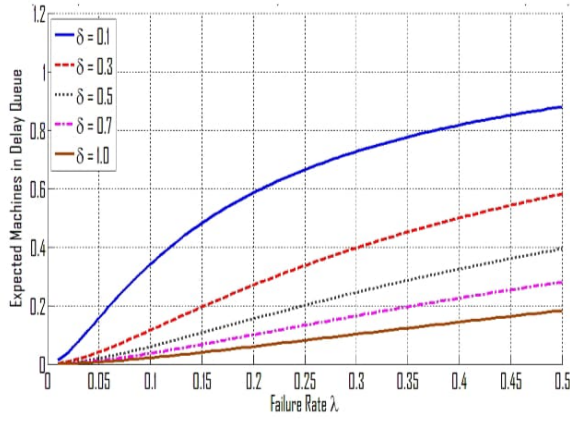


Figure 7.3: Expected number of failed machines in delay queue vs. failure rate λ for different values of repair delay (δ).

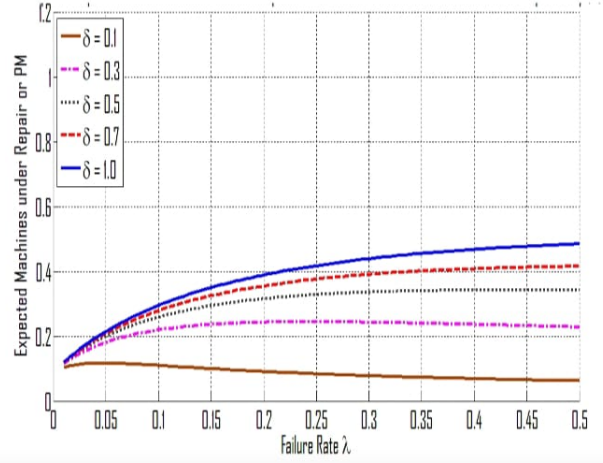


Figure 7.4: Expected number of machines under repair or PM vs. failure rate λ for different values of repair delay (δ).

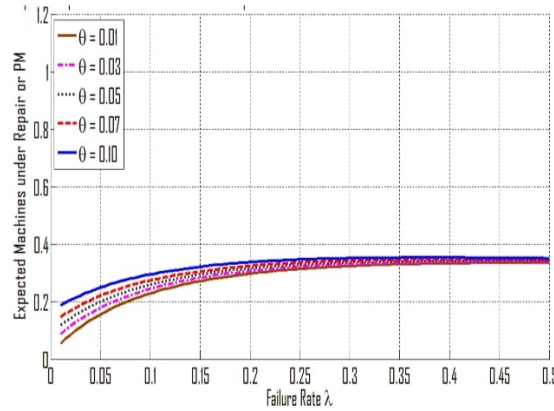


Figure 7.5: Expected number of machines under repair or PM vs. failure rate λ for different preventive maintenance rates (θ).

number of operational machines decreases with increasing λ , indicating that more frequent failures lead to fewer machines being available for operation. Furthermore, curves with higher δ values (i.e., longer repair delays) consistently lie above those with lower δ , showing that systems with faster repair (lower δ) experience a sharper drop in operational availability when failure rates increase. This suggests that repair delay plays a crucial role in sustaining machine operability, particularly under conditions of high failure risk.

The Figure 7.2 depicts the influence of varying preventive maintenance rates (θ) on the operational performance of machines as failure rate (λ) increases. The vertical axis shows the expected number of operational machines, while the horizontal axis represents the failure rate λ (ranging from 0 to 0.5). Each curve corresponds to a different value of θ (from 0.01 to 0.10). As λ increases, all curves show a decline in the expected number of operational machines, indicating that higher failure rates reduce system availability. However, the curves for different θ values are closely clustered, especially beyond $\lambda > 0.1$, suggesting that preventive maintenance rate has only a marginal impact on improving system availability under increasing failure rates. Minor differences appear in the low- λ region, where higher θ values (more frequent maintenance) slightly improve availability. This implies that while preventive maintenance does enhance reliability, its effectiveness diminishes as failure rates grow more severe.

The Figure 7.3 illustrates how the accumulation of failed machines waiting for repair (i.e., in the delay queue) varies with increasing failure rate λ for various repair delay parameters δ . The x-axis represents the failure rate (λ), while the y-axis shows the expected number of failed machines in the delay queue. Each curve corresponds to a different value of δ (from 0.1 to 1.0). The graph clearly shows that as the failure rate λ increases, the number of failed machines in the queue also increases. However, lower values of δ (i.e., higher repair delay) lead to a steeper and higher curve, meaning more machines accumulate in the queue. Conversely, higher δ values (faster repair initiation) result in flatter curves, indicating better system responsiveness and fewer queued failures. This emphasizes the critical role of minimizing repair delays to reduce system congestion and improve repair efficiency in machining systems.

The Figure 7.4 shows how the number of machines undergoing either repair or preventive maintenance (PM) changes with increasing failure rate λ , across various levels of the repair delay parameter δ . The x-axis represents the failure rate λ , while the y-axis shows the expected number of machines currently in the repair or PM phase. Each curve corresponds to a specific value of δ , ranging from 0.1 to 1.0. The trend demonstrates that as λ increases, more machines enter the repair or PM state. Importantly, higher values of δ (i.e., faster response to repair initiation) lead to higher values of the expected number under repair or PM, while lower δ values (i.e., longer delays before initiating repair) result in fewer machines actively being repaired or maintained. This is because at low δ , failures accumulate in the delay queue instead of transitioning to active repair or PM promptly. Thus, the graph emphasizes how improving the responsiveness of repair initiation (increasing δ) allows more machines to be processed for recovery, enhancing system efficiency.

Figure 7.5 illustrates the relationship between the Expected Number of Machines under Repair or Preventive Maintenance (PM) and the Failure Rate (λ) for different values of the Preventive Maintenance Rate (θ). As shown on the graph, the x-axis represents the failure rate (λ), while the y-axis shows the number of machines engaged in repair or PM activities. Each curve corresponds to a different value of θ , ranging from 0.01 to 0.10. The graph reveals that, in general, as λ increases, more machines are expected to be under repair or PM, though the increase becomes gradual at higher values of λ . It also shows that higher preventive maintenance rates (e.g., $\theta = 0.10$, blue line) result in a slightly greater number of machines in the repair or PM state. This is intuitive, as more frequent preventive maintenance increases the likelihood that machines are temporarily unavailable for operation due to ongoing maintenance. However, the differences between the curves are relatively small, indicating that θ has a subtler influence on this metric compared to other system parameters like repair delay. The graph highlights how preventive strategies impact the load on repair and maintenance capacity, especially as system stress increases due to rising failure rates.

8 Concluding Remarks

The study presents a robust mathematical framework that effectively integrates cold standby configuration, preventive maintenance strategies, and repair delay dynamics into the reliability modeling of machining systems. Through a Markovian state-space approach, it evaluates critical performance metrics such as system availability, server utilization, and the expected number of machines in repair or delay. The numerical analysis, supported by MATLAB-based optimization, reveals that minimizing preventive maintenance rate while maximizing repair and delay response rates significantly reduces operational costs and improves system efficiency. Graphical illustrations reinforce the finding that while preventive maintenance has a marginal

effect, reducing repair delays is paramount for sustaining operational readiness, especially under rising failure rates. Overall, the model provides valuable insights for industrial decision-makers to optimize maintenance schedules, manage repair resources effectively, and enhance the overall reliability of complex machining environments.

8.1 Declaration of Competing Interest

The authors declares that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

8.2 Funding

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8.3 Data availability

No data was used for the research described in the article.

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**A COMPREHENSIVE STUDY OF SOME PETRI NETS WITH THEIR TYPES,
PROPERTIES AND APPLICATIONS**

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Abstract

Modeling a discrete event dynamic system with the help of Petri nets is the main task in the application area. These models must satisfy each and every system design and property in order to be applicable in the real world. What if a system is associated with time? Or what if a system is continuously flowing in nature? What if the resources available are to be utilized minimum for the greatest profit? What if there is a random state change in the system? To answer such difficulties, new functions were introduced with the ordinary Petri nets and this gave rise to many types of Petri nets. Now, the Petri nets convey the time delays associated with the system in Time-dependent Petri nets and the continuous flow of nature has been explained by the continuous and hybrid Petri nets. In this article, we try to show the basics of two different types of Petri nets that have been used in different research areas using their examples. Also, the applicability of Petri nets in diverse areas has been described with a few examples.

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Keywords and Phrases: Petri nets; Types of Petri nets; Properties of Petri nets; Place/transition nets; Coloured Petri nets; Timed Petri nets; Reachability; Invariants.

1 Introduction

Processes in nature work in a systematic order. Some are complex while some are not. There are many mathematical models designed to solve the complexity of a system and to study the system's behavior theoretically. This helped in saving the resources available and designing a system correctly. Similar thought came into practice when Dr. Carl Adam Petri introduced the concept of Petri net (PN) during his Ph.D. dissertation in 1962. Petri nets (PNs) [11] further proved to be of greater use as it gives precise modeling of the system and its behavioral analysis. They are commonly used in biological networks, communication protocols, chemical systems, and many other areas of computer science [5-8, 14-21].

As per my study, fewer works introducing the basics of types of Petri nets have mostly resulted in ignorance about the wider applications of Petri nets. Many researchers and scholars have worked on the types of Petri nets but have described them by combining two or more two types. It definitely is a good way to approach the application process yet this kind of work doesn't really help a beginner to understand the insights used for the practical

approach. The objective of this work is to understand the basics of two different types of Petri nets so that all the researchers do not have to wander around in search of different scholarly works to find an alternative to their motives. In "Petri nets for Dynamic Event-Driven System Modeling", though J. Wang [22] has explained only time-dependent Petri nets, i.e. Time Petri nets and Stochastic Petri nets along with colored Petri nets, his work motivates one to read about time-dependent and high-level Petri nets in more detail. Though colored and time Petri nets are also discussed by David and Alla [1], they have focused more on hybrid, discrete, and continuous Petri nets in their work "Discrete, continuous, and hybrid Petri nets". Similarly, Time and Petri nets due to [12] is a whole work entirely dedicated to time and Petri nets. The authors have explained the different ways of introducing time with transitions whether it is interval or duration. You'll find glimpses of these works in our work.

We have commenced this work with basic definitions of Petri nets and their modeling example followed by their behavioral properties. Then we dug into the general methods used for analyzing the Petri nets using the coverability graph, reachability tree, etc. Further, we went for defining the basics of types of Petri nets, explaining them with a simple example followed by an application area in the chemical systems in order to make the readers understand the process better. We have also discussed the modeling tools and software we used to model the systems and examples presented in the paper, which may also help the readers model their own applications.

2 Fundamentals of Petri nets

PN is particularly a directed bipartite multigraph [11], that contains two disjoint sets of points/nodes, called place (O) and transition ($\square$), and these points/nodes are connected by directed arc ($\rightarrow$). Arcs are used to connect place node and transition node but in a manner that joins no two places or transitions simultaneously. An arc that is directed from place to transition is the input place of that transition and an arc from the transition to a place is the output place of the transition. We assign tokens ($\bullet$) to the places, which we call marking. These tokens are further used for the execution of Petri nets. Theoretically, we define marking as n -vector $= (M_1, M_2, M_3, \dots, M_n)$, where n is the number of places in the Petri net structure and each $M_i \in \{0, 1, 2, 3, \dots, n\}$. In many cases, we assign some nonnegative integer to the arcs in a Petri net, which we call the weight of the arc. And if weight is not assigned, we consider it 1 by default.

Formally we define Petri net [10] as a 5-tuple $C = (P, T, F, W, M_0)$ where:

$P = \{p_1, p_2, p_3, \dots, p_m\}$: A finite set of places ,

$T = \{t_1, t_2, t_3, \dots, t_n\}$: A finite set of transitions,

$F \subseteq (P \times T) \cup (T \times P)$: A set of arcs,

$W = F \rightarrow \{1, 2, 3, \dots\}$: Weight function assigned with arcs,

$M_0 = P \rightarrow \{0, 1, 2, 3, \dots\}$: The initial marking.

Here $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

2.1 Petri net firing rule

Removing $w(p, t)$ tokens from the input place of a transition and depositing $w(t, p)$ tokens into the output place of the transition causes the state change in a Petri net, which we call firing of that transition. Here, $w(p, t)$ denotes the weight of the arc from place to transition and similarly $w(t, p)$ denotes the weight of the arc from transition to a place. In Fig. 2.1, we have $w(p, t) = w(t, p) = 1$ for all the transitions.

2.2 Execution Rule in a Petri net

The execution of a Petri net is by the firing of transitions and any transition fires if it is enabled. Also, it is important to notice that not every enabled transition fires.

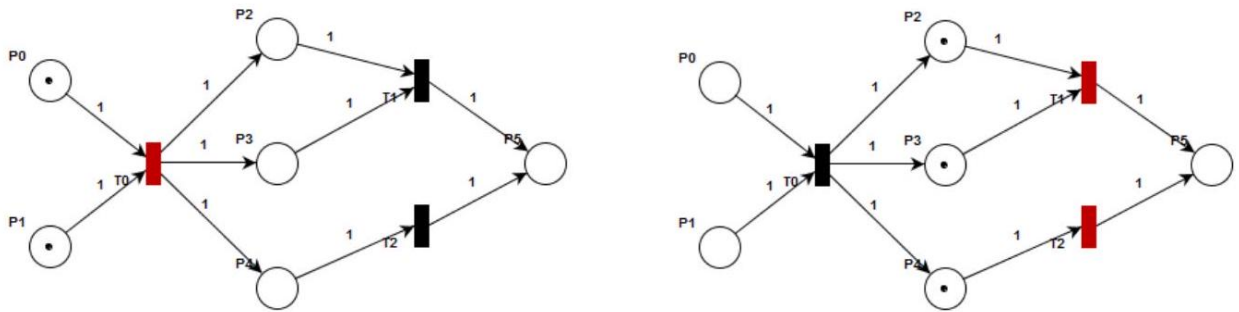


Figure 2.1: (a) Petri net showing enabled transition $T0$ with initial marking $(1,1,0,0,0,0)$
(b) Illustration of the firing of transition $T0$ in the marked Petri net

Here, we have $P = \{P0, P1, P2, P3, P4, P5\}$ is the set of places and $T = \{T0, T1, T2\}$ is the set of transitions. Fig. 2.1. shows marking reached after firing of transition $T0$ and let us suppose now transition $T1$ fires.

Hence, we have a firing sequence $\sigma = T0 \rightarrow T1$ which gives us the markings: $M_0 = (1, 1, 0, 0, 0, 0) \rightarrow M_1 = (0, 0, 1, 1, 1, 0) \rightarrow M_2 = (0, 0, 0, 0, 1, 1)$. Here, M_0 is the initial marking, M_1 is the marking reachable from firing $T0$ and M_2 is the marking reachable from firing $T1$. Hence, we have reachable markings and a random firing sequence that we study for analyzing Petri nets.

2.3. Analysis of a Petri net

2.3.1. Reachability Graph

We analyze a Petri net by constructing a reachability tree [13, 15, 18] from all the possible reachable states of the marked Petri net. In the example above, given is the initial marking $M_0 = (1, 1, 0, 0, 0, 0)$. Now for enabled transitions in the Petri net, only transition $T0$ is enabled and we have the state $M_1 = (0, 0, 1, 1, 1, 0)$ after firing $T0$. After this, we have two enabled transitions. So we consider both the conditions and write both the markings reached from the different transitions, i.e $M_2 = (0, 0, 0, 0, 1, 1)$ after firing of $T1$ while $M_3 = (0, 0, 1, 1, 0, 1)$ after firing of $T2$. And lastly M_2 reaches $M_4 = (0, 0, 0, 0, 0, 2)$ after firing of $T2$ and M_3 reaches $M_4 = (0, 0, 0, 0, 0, 2)$ after firing of $T1$. Hence, we draw the reachability tree as:

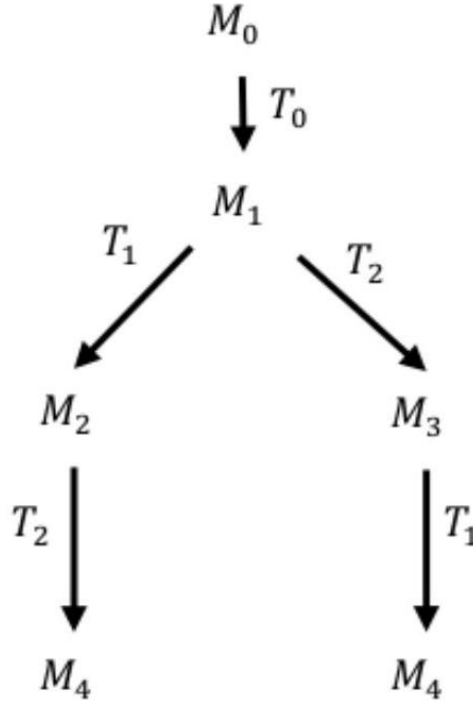


Figure 2.2: The Reachability Tree

3 Types of Petri nets

A. Time-dependent Petri nets

Introducing time variables in the form of interval or duration with Petri nets gives us a variety of possibilities that can be applied to model systems. Somewhere, time was introduced with places, transitions, tokens, or even with the arcs according to the need. Again, this will definitely lead to many different firing rules. So, in order to define a Time Petri net, these are the conditions that need to be fixed.

3.1 Time Petri nets

The most commonly discussed are the Time Petri nets and Timed Petri nets where transitions are connected with time variables. The time variable introduced here is deterministic in nature. Merlin [8] described Time Petri nets by introducing time intervals $[a_t, b_t]$ attached with the transition t such that no transition can fire before a_t time and after b_t time. We call a_t and b_t , earliest firing time of transition t and latest firing time of transition t respectively. Whereas, Ramchandani introduced Timed Petri nets by assigning a time duration d_t to the transition t . The difference, in this case, is that, transition has to fire immediately after being enabled. We will discuss Time Petri nets in this paper. One can refer to [9] for a detailed explanation.

Formally, Time Petri nets are defined as 6-tuple $TPN = (P, T, F, W, M_0, I)$ where

1. $C = (P, T, F, W, M_0)$ is a Petri net and
2. I denotes the Interval function of TPN such that $I : T \rightarrow N \times (N \cup \{\infty\})$ and $\forall t \in T$, with $I(t) = (e(t), l(t))$ such that $e(t) \leq l(t)$, where $t \in T$.

Here, $e(t)$ denotes earliest firing time of t and $l(t)$ denotes latest firing time of t .

Definition 3.1 (p-marking and t-marking). *The p-marking in Time Petri net is defined as the function mapping from set of places to the natural numbers including $\{0\}$. i.e. $M : P \rightarrow \{0, 1, 2, 3, \dots\}$*

And t-marking in a Time Petri net is defined as the function mapping from the set of transitions to $\{R_0^+ \cup \#\}$. i.e. $H : T \rightarrow \{R_0^+ \cup \#\}$.

Definition 3.2 (State). *A state in a Time Petri net is defined as $z = (M, H)$, $z \in TPN$ where M and H represents p-marking and t-marking respectively such that-*

- i. *For all those transitions that are disabled with p-marking, we will take $H(t) = \#$.*
- ii. *For all those transitions that are enabled with p-marking, and the time $H(t)$ is less than or equal to $l(t)$, we have $H(t) \in R_0^+$.*

Also, the initial state is defined as $z_0 = (M_0, H_0)$ where

$$H_0(t) = \begin{cases} 0, & \text{if } t \text{ is enabled in } M_0 \\ \#, & \text{else} \end{cases}.$$

For the state change in a Time Petri net, the first reason is due to the firing of transition.

3.1.1 Execution rule in Time Petri net.

A transition t in a Time Petri net is ready to fire in state (M, H) if and only if it is enabled in the marking $M(t)$ and $H(t) \geq e(t)$. Then the firing of transition t will take place if it is ready to fire.

This will convert state $z = (M, H) \rightarrow z' = (M', H')$ where $M'(t) = M(t) + \Delta t$ and

$$H'(t') = \begin{cases} \#, & \text{if } t \text{ is not enabled in } M' \\ \frac{H(t')}{H(t')}, & \text{if } t \text{ is enabled in } M \text{ and } M' \text{ along with pre-places of } t \text{ and } t' \text{ are disjoint and } t \neq t' \end{cases},$$

for all $t' \in T$.

Hence we say if, after firing of one of the two transitions that are enabled and having at least one common pre-place, another transition is still enabled, then the time is set to zero with that enabled transition.

Another case we need to take care of is t-marking also changes with elapsing of time even when the transition doesn't fire. Consider τ , a positive real number and state $z = (M, H)$ then elapsing of time τ is possible if $\forall t$, when $H(t) \neq \#, H(t) + \tau \leq l(t)$. And this elapsing of time convert the state $z = (M, H) \rightarrow z' = (M', H')$ where $M' = M$ and

$$H'(t) = \begin{cases} H(t) + \tau, & \text{if } t \text{ is enabled in } M' \\ \#, & \text{if } t \text{ is disabled in } M' \end{cases} \forall t \in T.$$

Definition 3.3 (Run). *The sequence $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots \tau_n$ of state transitions, where $\sigma = t_1 t_2 \dots t_n$ is the transition sequence and $\tau = \tau_0 \tau_1 \dots \tau_n$ is the sequence of times having $\tau_i \in R_0^+$ is known as run of.*

Definition 3.4 (Feasible run). *In a time Petri net, if a state $z' = (M', H')$ reachable from state $z = (M, H)$ with the run $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots \tau_n$, then the run is called feasible run from the state z .*

Definition 3.5 (Feasible transition sequence). *The transition sequence $\sigma = t_1 t_2 \dots t_n$ in a time Petri net is feasible if there exists a feasible run $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots \tau_n$ from the state z .*

Definition 3.6 (Reachable state). *The state we reached $z = (M, H)$ from z_0 , whenever there exists a feasible run $\sigma(\tau) = \tau_0 t_1 \tau_1 \dots \tau_n$ is the reachable state from initial state $z_0 = (M_0, H_0)$.*

Let us understand the rules of the firing of Time Petri nets by considering the Petri net-

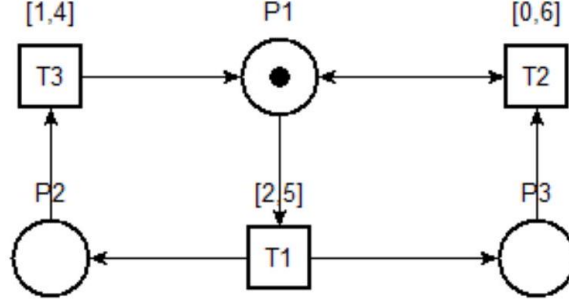


Figure 3.1: A simple example of TPN

In the Petri net, time intervals are associated with the transitions. We have-

$P = \{P_1, P_2, P_3\}$, the set of places and

$T = \{T_1, T_2, T_3\}$, the set of transitions with

$I(T_1) = (2, 5)$, $I(T_2) = (0, 6)$ and $I(T_3) = (1, 4)$.

Initial state/marking, $z_0 = (M_0, H_0) : z_0 = ((1, 0, 0), (0, \#, \#))$.

Now, let us suppose we have attached a clock with every transition that is enabled in the initial marking.

Here, only transition T_1 is enabled. Let $\tau_0 = 1.3$ unit of time passes, which changes the state of Petri net into another state, ie $z_0^* = ((1, 0, 0), (1.3, \#, \#))$. We know that the $(T_1) = 2$, hence transition can't fire even though it is enabled. After some more time say $\tau_1 = 1$ time units, the transition is now allowed to fire since $2 \leq 1.3 + 1 = 2.3 \leq 5$. Suppose T_1 fires now. The state that we reach after firing will be $z_1 = ((0, 1, 1), (\#, \#, 0))$. $(\#, \#, 0)$ in z_1 because only transition T_3 is enabled in $(0, 1, 1)$. Proceeding in the same way, we shall have

Transition sequence: $\sigma = T_1 T_3 T_1$

Run: $\sigma(\tau) = 1.3 T_1 2 T_3 2 T_1 1.5$

Feasible run: Run $\sigma(\tau)$ is feasible as we reach z_3 from z_0 by $-z_0 = ((1, 0, 0), (0, \#, \#)) \rightarrow 1.3 \rightarrow z_0^* = ((1, 0, 0), (1.3, \#, \#)) \rightarrow 1, T_1 \rightarrow z_1 = ((0, 1, 1), (\#, \#, 0)) \rightarrow 2 \rightarrow z_1^* = ((0, 1, 1), (\#, \#, 2)) \rightarrow T_3 \rightarrow z_2 = ((1, 0, 1), (0, 0, \#)) \rightarrow 2 \rightarrow z_2^* = ((1, 0, 1), (2, 2, \#)) T_1 \rightarrow z_3 = ((0, 1, 2), (\#, \#, 0))$ and so on.

Hence, $\sigma = T_1 T_3 T_1$ is a feasible transition sequence. And z_3 in this case, is the reachable state because $z_0 \sigma(\tau) \rightarrow z_3$.

Definition 3.7 (State Space). Set of all reachable marking is known as the State Space of TPN.

The properties that can be discussed in this type of Petri net are mainly- liveness, boundedness, and reachability which requires the knowledge of State Space of the Petri net. So, in order to find out the state space, we need to learn some more concepts. Not only the state space, we sometimes are required to find out the minimum or maximum time length that the system requires in reaching a particular state.

Replacing the real positive values τ_i with a variable x_i such that all x_i 's are bounded by some conditions, gives us the Parametric run $(\sigma(x), B_\sigma)$ of σ in TPN with $\sigma(x) = x_0 T_1 x_1 T_2 \dots x_{n-1} T_n x_n$ where B_σ is the set of conditions for all x_i . In continuation,

Definition 3.8 (Parametric State). is defined as (z_σ, B_σ) in TPN where, $z_\sigma = (M_\sigma, H_\sigma)$ such that $M_0 \sigma \rightarrow M_\sigma$ and $H_\sigma(t)$ is the variables' sum.

So, we can conclude: $z_0 \sigma(x) \rightarrow (z_\sigma, B_\sigma)$ and State Space of TPN = $\cup_{\sigma(x)} \{z_{\sigma(x)} \mid x \text{ satisfies } B_\sigma\}$.

Consider the above-discussed example, consider $\sigma = T_1 T_3 T_1$ sequence. Here, $\sigma(x) = x_1 T_1 x_2 T_3 x_3 T_1 x_4$. So, the parametric state will be $(z_\sigma, B_\sigma) = (((0, 1, 2), (\#, \#, 0)), \{2 \leq x_1 \leq 5, 1 \leq x_2 \leq 4, x_1 + x_2 \leq 6, 2 \leq x_3 \leq 5, x_1 + x_2 + x_3 \leq 6, 0 \leq x_3 \leq 4\})$.

Definition 3.9 (Integer Run). Run in which $\tau_i \in \mathbb{N}, \forall i = 0, 1, \dots, n$ is known as Integer Run.

As in the above example, one can easily see that the state z_3 reachable from the Run $\sigma(\tau) = 1.3 T_1 2 T_3 2 T_1 1.5$ is also reachable from $\sigma'(\tau) = 1 T_1 2 T_3 2 T_1 1$. This run $\sigma'(\tau) = 1 T_1 2 T_3 2 T_1 1$ is the Integer run and is solving the purpose that the run $\sigma(\tau) = 1.3 T_1 2 T_3 2 T_1 1.5$ did. Hence, it gives a good result that every feasible transition sequence can be realized by an Integer run.

3.1.2. Reachability Analysis for finite TPN

The reachability graph will be finite if all the reachable p-markings in the net are finite. In order to obtain a reachability graph, it will be difficult to mention all the reachable states. The integer states that we obtained after integer run, where $h(t) \in N, \forall m$ in $t \in T$, carry with themselves all the useful information not only regarding reachability but for other properties as well. Hence, in order to obtain a finite reachability graph, we reduce it by using the idea of "fusing together" all the consecutive edges that represent the elapsing of a single time unit and firing of a single transition. The states obtained in between are removed from the reachability graph. For example, consider an arbitrary path $z_0 t_1 \rightarrow z_1 1 \rightarrow z_1^* 1 \rightarrow z_1^{**} t_2 \rightarrow z_2 \dots$. We will reduce the path to $z_0 t_1 \rightarrow z_1 2, t_2 \rightarrow z_2$. Continuing in the same way, the reachability graph we obtained is the reduced reachability graph for finite TPN.

These are some results and properties that we discussed for finite TPN. One can easily now understand the concepts in case of Infinite TPN from the reference [9].

B. High-level Petri nets

Why High-level Petri nets? Because the tokens in this kind of Petri net carry some information with them, while tokens in Petri nets that we discussed above were all of the same types and represented the state of place by an integer. The aim is just to explain a modeled system in a proper and easier way. The most commonly used in application areas are colored Petri nets.

3.2. Colored Petri nets

Sometimes, there are different systems in the real world that use the same resources. So, it becomes difficult to model such systems practically. So, the need was felt to develop more powerful Net types. Coloured Petri nets [4] solve the purpose. The token assigned in this Petri net with some data values that they carry with themselves, and are called token color. Given a place, all tokens in that place have token colors which belong to a specific type,

which we call a color set of that place. Same color sets can be labelled to the Different places. Marking in these Petri nets is the tokens that are distributed in places. The underlined markings besides the places are the initial marking in these Petri nets. Practically, we can't just put tokens inside the places to show the current marking of colored Petri nets as we did in previous Petri nets because we also need to show the token colors in this case. As a result of introducing token colors and color sets, we will be using a lesser no. of places in comparison to Normal Petri nets. Now, Type of token colors are responsible for the enabling of transitions that are in the input places of the transitions, they can affect the output tokens as well. In order to remove the complexity, now Arc expressions are introduced. These arc expressions specify the collection of the tokens and contain no. of variables that are associated with the transitions. In order to combine many different systems in a system, along with the places one needs to find a way to combine transitions as well, i.e. different actions can be performed by a transition. The variables introduced with the arc expressions will solve our purpose by evaluating different values.

Let's see the formal definition of colored Petri nets, which first requires us to learn what a multi-set is. A set that allows repetition of its elements, different from a set is informally defined as a multi-set. Now,

A colored Petri net [4] is defined as $CPN = (P, T, F, \Sigma, C, N, E, G, I)$ where

$P = \{p_1, p_2, p_3, \dots, p_m\}$: A finite set of places.

$T = \{t_1, t_2, t_3, \dots, t_n\}$: A finite set of transitions.

$F \subseteq (P \times T) \cup (T \times P)$: A set of arcs.

Σ : Set of color sets containing every possible operation and function used in CPN.

C : It is the colour function that is a mapping from places into colours in Σ .

N : It is the node function which is a mapping from F into $(P \times T) \cup (T \times P)$.

E : It is the arc expression function which maps every arc into an expression.

G : It is called the Guard function which is a mapping from every transition to an expression of a particularly Boolean type and,

I : It is the Initialization function that maps every place into an expression type that is of particularly a multiset over a color set, $C(p)$.

Let us understand CPN more clearly with the help of an example.

We have three courses offered in an institution and the examination sheets of the students collected after the examination are randomly collected. We need to arrange these mark sheets course-wise. The following are the roll numbers of the students according to the course:

Table 3.1: Example of modeling using Coloured Petri nets

S. No.	Course	Roll no. of students
1	B.Sc.(H) Math	2001-2042
2	B.Sc.(H) Statistics	2043-2056
3	B.Sc.(H) Computer Science	2057-2091

We have tried to model the problem using colored Petri nets where we have four places and one transition. Also, we have $\Sigma = \{ \text{Green, Red, Yellow} \}$, arc expression associated with every arc, and color function which distinguishes colors of the tokens by allowing firing in respective transitions.

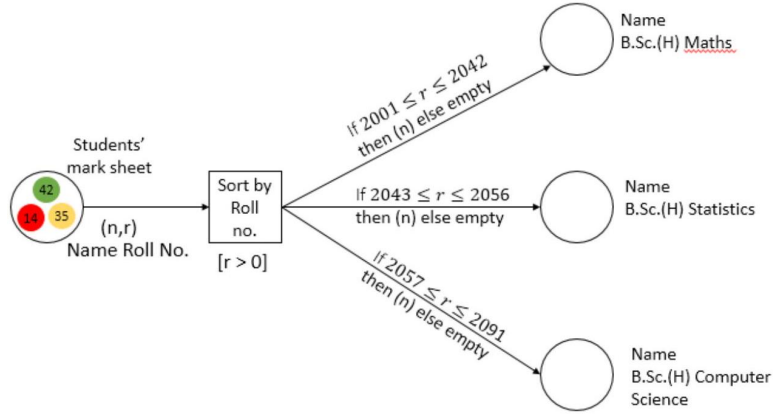


Figure 3.2: An example of a Coloured Petri net graph

4 Application of Petri nets in diverse areas

4.1 Chemical Systems - Lime Cycle

Limestone is a type of Sedimentary rock containing a high percentage of CaCO_3 in it. It has many uses in real-life applications like in constructing houses and buildings. We are going to learn here the Lime Cycle modeling through Petri nets. Consider the Lime Cycle as shown in the figure.

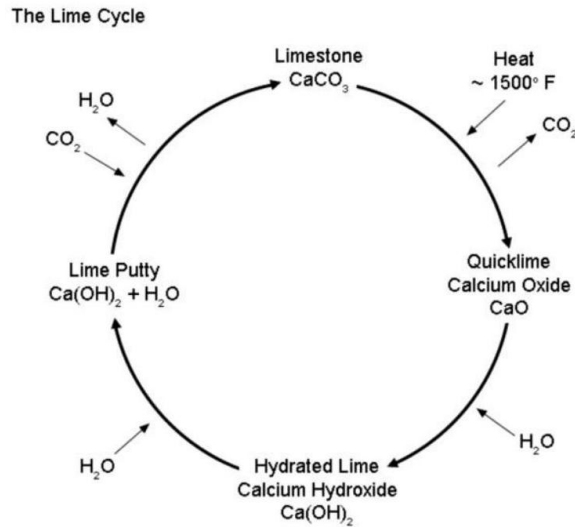


Figure 4.1: Lime cycle in nature

We are trying to show here a simple modeling example in chemical systems that one can use for learning of Petri net applications. Here is the PN modeled in the WoPeD tool for Petri nets:

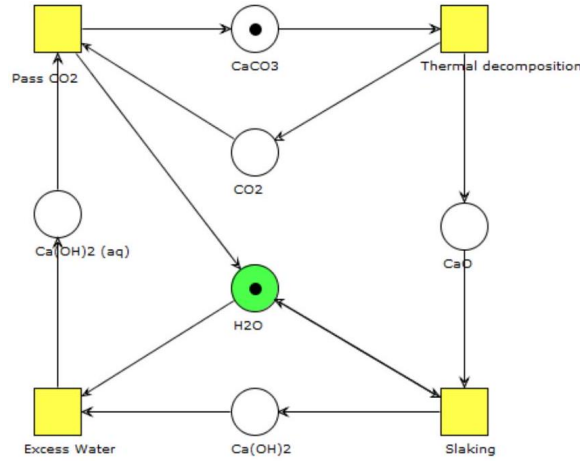


Figure 4.2: Petri net of lime cycle

Here,

$P = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ where the places denote CaCO_3 , Ca(OH)_2 (aq.), CaO , Ca(OH)_2 , CO_2 and H_2O respectively.

And $T = \{T_1, T_2, T_3, T_4\}$ where T_1 denotes the process of Thermal Decomposition, T_2 denotes Slaking, T_3 denotes Excess water and T_4 denotes the Carbon dioxide.

The reachability graph we obtained is-

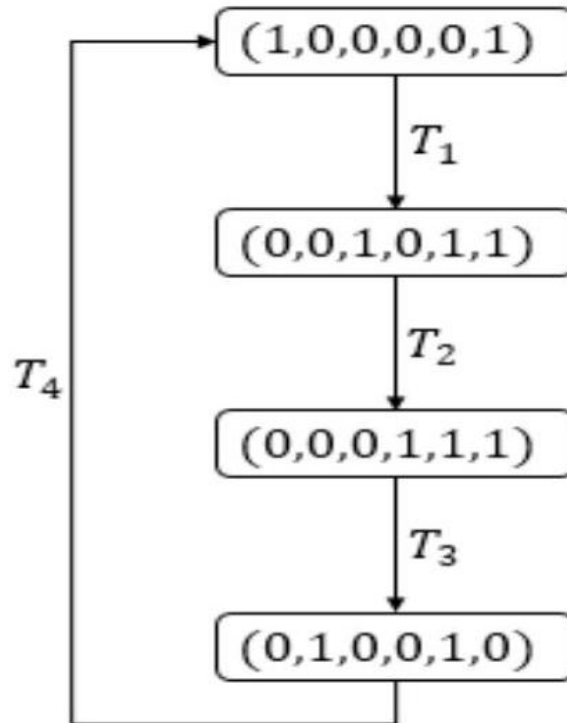


Figure 4.3: Reachability graph of Fig. 4.2.

4.2 Communication System

Consider the elements of the communication system. The elements include the collection of devices that work together in order to establish communication between sender and receiver. The block diagram for the example would be:

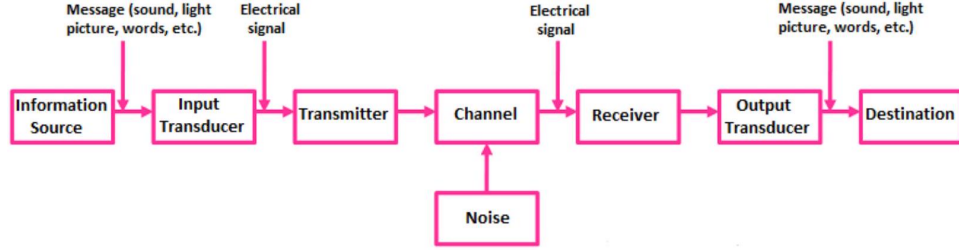


Figure 4.4: Elements of communication systems

Each of these elements has its own functioning that can be studied. Here, we have tried to model a Petri net (on WoPeD: A Petri net tool)[2,3] with the block diagram assuming that the information/message is reached correctly to the destination. We have shown the elements by the set of places and the condition of the message by the set of transitions.

This Petri net model has its own limitations but it can be updated according to the need of the user or situation.

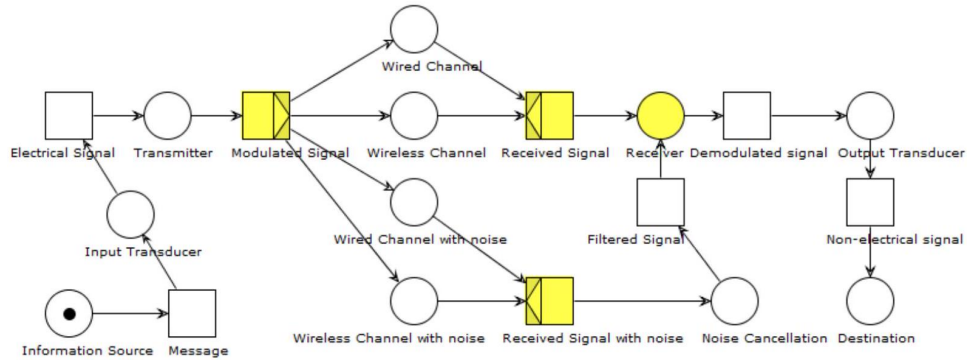


Figure 4.5: Petri net graph modeling of Fig. 4.4.

The model here presents to us the various stages and the possibilities of a communication system. When the message is received by the transmitter, the signal can travel to the receiver either through a wired or wireless channel. If the signal is not interrupted by the noise or the amount of noise is so little that it can entirely be ignored then the receiver can easily decode the message but if the message is accompanied by a constant noise that keeps interrupting the quality of the signal in such a manner that the message can entirely be lost in the process then there is a need to filter the signal to cure the noise and to receive the message correctly. These possibilities need their own individual path to reach the destination and these kinds of Petri net models can help us to reach out to every other possibility and can reduce the amount of error.

The model clearly states the initial marking, $M_0 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and the firing of transitions is according to the firing rule. The Petri net analysis gives us the reachability graph:

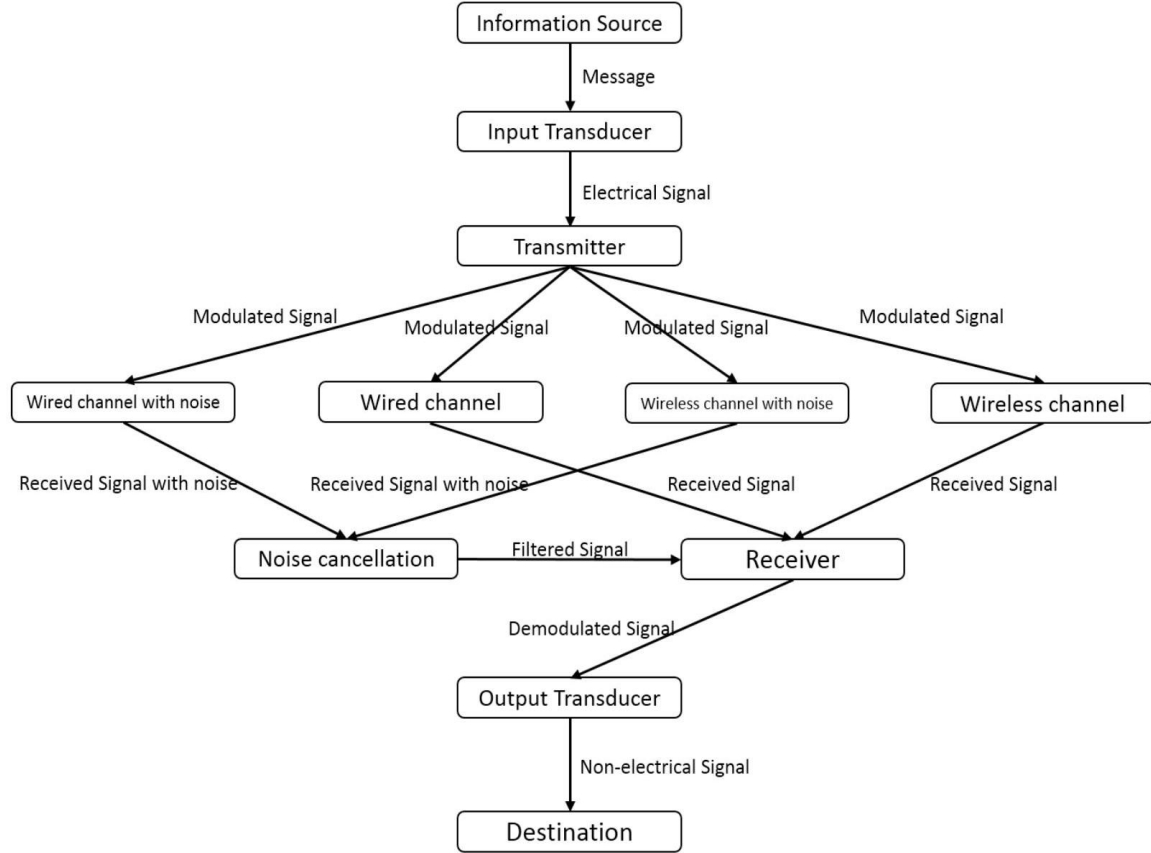


Figure 4.6: Reachability graph of Petri net in Fig. 4.5.

5 Conclusion, Discussion, and Outlook

There have been many mathematical models proposed to solve systems. Petri nets introduced in the 1960s changed the older methods and introduced some new modeling tools for the upcoming generations of mathematicians and scientists. The different types of Petri nets open the different ways in which this modeling tool can be used in our day-to-day life. It has definitely made the process easier to understand and easier to respond to. The types of Petri nets we compiled in this dissertation are a try to provide excess to, all the researchers and readers, the types in just one place which was not easily available earlier. We have discussed the application areas in this paper which is a better way to understand the topic and can definitely help through the application processes in the field. Even after my hard try, I might have missed a lot here in this work; as Petri nets, though interesting yet, is a wide area to work on. I hope this work extends some major help to the researchers.

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**FRACTIONAL DIFFERENTIAL INEQUALITIES WITH CAPUTO-FABRIZIO
DERIVATIVE IN CAPUTO-SENSE**

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Abstract

This research work aims to provide a comprehensive analysis of the comparison principle as applied to fractional differential inequalities, utilising the Caputo-Fabrizio fractional derivative in the Caputo sense, and shedding light on its theoretical foundations and practical implications.

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Keywords and Phrases: Fractional Calculus, Caputo-Fabrizio Derivative, Fractional Differential Inequalities, Comparison Principle, Non-singular Kernel.

1 Introduction

In the near past, fractional calculus has been able to fetch significant attention across varied scientific disciplines for its unmatched ability to model intricate phenomena. Fractional calculus, particularly fractional differential equations (*FDEs*), has gained substantial significance in physics, engineering and applied mathematics. Despite having a history spanning three hundred years, the applications of *FDEs* have seen a resurgence in the last twenty to thirty years [14, 26, 28]. While numerous analytical and numerical methods exist for solving integer-order differential equations, many times these methods often fall short when applied to their fractional counterparts. Because of this, there is a need to introduce new methods tailored to solve *FDEs*. Notably, obtaining analytical solutions for most *FDEs* is a challenging task, pushing researchers to work on various schemes.

The comparison principle is a common tool that is used to establish bounds and study other important properties of solutions to differential equations by comparing them to simpler, known solutions [21, 29]. Also, the comparison principle offers a powerful approach for understanding the behaviour of solutions to nonlinear inequalities [10, 12]. This can be extended to the domain of fractional calculus with the help of inequalities. Inequalities are also a powerful tool in mathematical analysis, providing insights into the properties of solutions to differential equations [11, 12, 16, 27]. In this research paper, we studied fractional differential inequalities. These types of inequalities are essential in the study of fractional calculus [1, 2, 3, 17, 18]. Fractional differential inequalities play a pivotal role in this area, offering a powerful framework to describe and analyze dynamic systems characterized by fractional-order derivatives. Particularly, the topic of uniqueness and existence of solution to the fractional differential system is essential in fractional calculus and the study of same can be done by using fractional differential inequalities [4, 13, 17, 18, 19].

Lakshmikantham and Vatsala [17, 18], investigated the comparison theorem for Riemann-Liouville-type fractional differential inequalities, the usefulness of which is also concretized in [10]. Regarding the Caputo fractional derivative, the comparison principle for fractional differential equations is presented by Lu and Zhu in their paper [19]. The generalized Riemann-Liouville and Caputo fractional derivatives are used to study the comparison principle for fractional differential inequalities by Al-Refai and Yuri Luchko in [3]. These results are useful in the theory of fractional calculus, as we can not directly apply them to the system having

the Caputo-fabrizio fractional derivative, we need new results which are suitable to our need. For that we reestablished the results in [3, 17, 18], using the Caputo-Fabrizio fractional derivative as it found various applications in applied mathematics, as mentioned in [9, 11, 15, 30]. The Caputo-Fabrizio fractional operator with a nonsingular kernel found many applications in the field of integral inequalities [7, 8, 23, 24, 25], due to its capability to model the mathematical problems accurately. Similar kind of attention need to be given to the fractional differential inequalities with Caputo-Fabrizio fractional derivative.

This research paper aims to provide a theoretical foundation parallel to the work of [3, 17, 18, 19], for the comparison principle in the context of fractional differential inequalities using the non-singular kernel of the Caputo-Fabrizio derivative in the Caputo sense, and investigating the conditions under which this principle can be effectively applied to draw meaningful conclusions about the behaviour of solutions of these inequalities. The outcomes of this study are expected to contribute to the development of theoretical knowledge in fractional calculus and provide valuable insights for researchers working with fractional-order systems having the Caputo-fabrizio derivative.

The paper is proceeded as follows: Section 2 provides a brief overview of some preliminaries that will be used in this paper. In Section 3, there are main results about the differential inequality involving Caputo-Fabrizio fractional derivatives. Finally, Section 4 summarizes the key findings, discusses potential areas for future research, and concludes the paper.

2 Preliminaries

Definition 2.1 ([6]). *Let $f \in H^1(a, b)$, $a < b$, $a \in (-\infty, x)$, $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative in the Caputo sense is defined by-*

$$({}^{CFC}\mathcal{D}_a^\alpha g)(x) = \frac{\mathcal{B}(\alpha)}{1-\alpha} \int_a^x g'(s) e^{\frac{-\alpha}{(1-\alpha)}(x-s)} ds, \quad (2.1)$$

where $\mathcal{B}(\alpha) > 0$ is a normalization function satisfying $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

By the above definition, when $g(x)$ is constant, ${}^{CFC}\mathcal{D}_a^\alpha$ is zero, but the kernel does not have a singularity for $x = s$.

Definition 2.2 ([20]). *Let $0 < \alpha < 1$, the corresponding Caputo-Fabrizio fractional integral of function g is given by*

$$({}^{CFC}\mathcal{J}_a^\alpha g)(x) = \frac{(1-\alpha)}{\mathcal{B}(\alpha)} g(x) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x g(s) ds. \quad (2.2)$$

Property 2.1 ([20]). *The relation between the Caputo-Fabrizio fractional derivative and the corresponding integral in the Caputo-sense is given by*

$$({}^{CFC}\mathcal{J}_a^\alpha)({}^{CFC}\mathcal{D}_a^\alpha)g(x) = g(x) - g(a). \quad (2.3)$$

Property 2.2 ([22]). *Let $\alpha \in (0, 1)$, and f, g, f', g' (f' and g' are derivatives of functions f and g , respectively) are non-negative functions on $(0, \infty]$. Moreover, f and g have the same sense of variation on $(0, \infty]$ such that*

- (i) $f(0) = g(0) = 0$,
- (ii) $fg \geq {}^{CFC}\mathcal{D}_0^\alpha(fg)$,

then

$${}^{CFC}\mathcal{D}_0^\alpha(fg)(t) \leq f(t) {}^{CFC}\mathcal{D}_0^\alpha(g(t)) + g(t) {}^{CFC}\mathcal{D}_0^\alpha(f(t)). \quad (2.4)$$

For more properties of the Caputo-Fabrizio fractional derivative, the reader may refer to [7, 8, 20, 22, 23, 24, 25, 31] and the references mentioned in them.

Definition 2.3 ((Lipschitz Condition) [21]). *Suppose $u(x)$ be continuous on the interval $[a, b]$ of the real line $\mathbb{R}$, and differentiable on $(a, b]$ and $f(x, u)$ is a continuous function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, then*

$$|f(x, u_1) - f(x, u_2)| \leq L|u_1 - u_2|, \quad (2.5)$$

where L is the Lipschitz constant.

Remark 2.1. Before proceeding to the next section, we need following lemmas, which are given in [3], [17], and [19] with respect to Riemann-Liouville, generalized, and Caputo fractional derivatives, respectively. However, we derive it for the Caputo-Fabrizio fractional derivative in the Caputo-sense.

Consider the fractional differential equation

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u), \quad u(a) = u_a, \quad (2.6)$$

where $f \in C([0, X] \times \mathbb{R}, \mathbb{R})$, and $0 \leq \alpha \leq 1$. The initial value problem (IVP) (2.6) is equivalent to the following Volterra fractional integral equation, which we get by using (2.1), (2.2) and (2.3) :

$$\begin{aligned} {}^{CFC}\mathcal{J}_a^\alpha {}^{CFC}\mathcal{D}_a^\alpha u &= {}^{CFC}\mathcal{J}_a^\alpha f(x, u), \\ u(x) - u(a) &= \frac{(1-\alpha)}{\mathcal{B}(\alpha)} f(x, u) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u) ds, \quad 0 < \alpha < 1, \\ u(x) &= u(a) + \frac{(1-\alpha)}{\mathcal{B}(\alpha)} f(x, u) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u) ds. \end{aligned} \quad (2.7)$$

Lemma 2.1. Let $u(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally Lipschitz continuous such that, for $x_1 \in (0, +\infty)$, $u(x_1) = 0$ and $u(x) \leq 0$ for $0 \leq x \leq x_1$. Then ${}^{CFC}\mathcal{D}_a^\alpha u(x_1) \geq 0$ for $\alpha \in (0, 1)$.

Proof. Let $\mu_\alpha = \frac{\alpha}{1-\alpha}$, then we know that

$$\begin{aligned} {}^{CFC}\mathcal{D}_a^\alpha u(x_1) &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} \int_a^{x_1} u'(s) e^{-\mu_\alpha(x_1-s)} ds, \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \int_a^{x_1} u'(s) e^{\mu_\alpha s} ds, \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[e^{\mu_\alpha s} \int_a^{x_1} u'(s) ds - \int_a^{x_1} \mu_\alpha e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[(e^{\mu_\alpha s} u(s))_{x=a}^{x_1} - \int_a^{x_1} \mu_\alpha e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[e^{\mu_\alpha x_1} u(x_1) - e^{\mu_\alpha a} u(a) - \mu_\alpha \int_a^{x_1} e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[0 - e^{\mu_\alpha a} u(a) - \mu_\alpha \int_a^{x_1} e^{\mu_\alpha s} u(s) ds \right], \\ &= -\frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha(x_1-a)} u(a) - \mu_\alpha \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \int_a^{x_1} e^{\mu_\alpha s} u(s) ds, \\ &\geq 0, \end{aligned}$$

as $u(x_1) \leq 0$ for $a \leq x \leq x_1$, make the terms on the right hand side positive. $\square$

Lemma 2.2. Let $u_\epsilon(x)$ be a family of continuous functions defined on $[a, X]$ each $\epsilon > 0$, where ${}^{CFC}\mathcal{D}_a^\alpha u_\epsilon(x) = f(x, u_\epsilon(x))$, $u_\epsilon(a) = u_a$, and $|f(x, u_\epsilon(x))| \leq M$ for $a \leq x \leq X$. Then the family $\{u_\epsilon(x)\}$ is equicontinuous on $a \leq x \leq X$.

Proof. For $a \leq x_1 \leq x_2 \leq X$, consider

$$\begin{aligned} |u_\epsilon(x_1) - u_\epsilon(x_2)| &= \left| u(a) + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x_1, u_\epsilon(x_1)) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_1} f(s, u_\epsilon(x_1)) ds - u(a) \right. \\ &\quad \left. - \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x_2, u_\epsilon(x_2)) - \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_2} f(s, u_\epsilon(x_2)) ds \right|, \\ &\leq \frac{1-\alpha}{\mathcal{B}(\alpha)} |f(x_1, u_\epsilon(x_1)) - f(x_2, u_\epsilon(x_2))| \\ &\quad + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_1} |f(s, u_\epsilon(x_1)) - f(s, u_\epsilon(x_2))| ds \\ &\quad - \frac{\alpha}{\mathcal{B}(\alpha)} \int_{x_1}^{x_2} |f(s, u_\epsilon(x_2))| ds, \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{\alpha}{\mathcal{B}(\alpha)} \int_{x_1}^{x_2} M ds, \\
&= -\frac{\alpha M}{\mathcal{B}(\alpha)} (x_2 - x_1), \\
&< \epsilon,
\end{aligned}$$

provided $|x_2 - x_1| < \delta = \frac{\mathcal{B}(\alpha)}{\alpha M}$. $\square$

3 Caputo-Fabrizio Fractional Differential Inequalities

The following results are inspired by the research work of Lakshmikantham [17, 18], but with the Caputo-Fabrizio fractional derivative in the Caputo-sense.

Theorem 3.1. *Let $v, w : [a, X] \rightarrow \mathbb{R}$ be locally Lipschitz continuous, $f \in C([a, X] \times \mathbb{R}, \mathbb{R})$ and*

$$i) {}^{CFC}\mathcal{D}_a^\alpha v(x) \leq f(x, v(x)), \quad ii) {}^{CFC}\mathcal{D}_a^\alpha w(x) \geq f(x, w(x)), \quad 0 \leq x \leq X,$$

one of the inequalities being strict. Then

$$v(a) < w(a) \tag{3.1}$$

implies

$$v(x) < w(x), \quad a \leq x \leq X. \tag{3.2}$$

Proof. Suppose that the conclusion (3.2) is not true. Let us suppose that the inequality (ii) is strict. Then setting $m(x) = v(x) - w(x)$, $a \leq x \leq x_1$ we find that $m(x) \leq 0$, $a \leq x \leq x_1$ and $m(x_1) = 0$. Then by Lemma 2.1, we get ${}^{CFC}\mathcal{D}_a^\alpha m(x_1)$, which yields, using (i) and (ii) and the definition of $m(x)$,

$$f(x_1, v(x_1)) \geq {}^{CFC}\mathcal{D}_a^\alpha v(x_1) \geq {}^{CFC}\mathcal{D}_a^\alpha w(x_1) > f(x_1, w(x_1)). \tag{3.3}$$

This is the contradiction since $v(x_1) = w(x_1)$. Hence, the conclusion (3.2) is valid and the proof is complete. $\square$

The next theorem is for non-strict fractional differential inequalities, which requires a one sided Lipschitz type condition.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 holds with non-strict inequalities (i) and (ii). Suppose further that*

$$f(x, v) - f(x, w) \leq \frac{L}{1 + e^{-\mu_\alpha x}} (v - w), \tag{3.4}$$

whenever $v \geq w$ and $x > a$. then $v(a) \leq w(a)$ implies,

$$v(x) < w(x), \quad a \leq x \leq X, \tag{3.5}$$

provided $Le^{\mu_\alpha x} \leq \frac{\mathcal{B}(\alpha)\mu_\alpha}{(1-\alpha)}$.

Proof. We set $w_\epsilon(x) = w(x) + \epsilon(1 + e^{-\mu_\alpha x})$, for small $\epsilon > 0$, so that we have

$$w_\epsilon(a) > w(a) \quad \text{and} \quad w_\epsilon(x) > w(x), \quad a \leq x \leq X. \tag{3.6}$$

Now,

$$\begin{aligned}
{}^{CFC}\mathcal{D}_a^\alpha w_\epsilon(x) &= {}^{CFC}\mathcal{D}_a^\alpha w(x) + \epsilon {}^{CFC}\mathcal{D}_a^\alpha (1 + e^{-\mu_\alpha x}), \\
&= {}^{CFC}\mathcal{D}_a^\alpha w(x) + \epsilon {}^{CFC}\mathcal{D}_a^\alpha (e^{-\mu_\alpha x}), \\
&\geq f(x, w(x)) + \epsilon \frac{\mathcal{B}(\alpha)}{1-\alpha} \int_a^x (e^{-\mu_\alpha s})' e^{-\mu_\alpha (x-s)} ds, \\
&> f(x, w_\epsilon(x)) - \epsilon \frac{\mathcal{B}(\alpha)\mu_\alpha}{1-\alpha} e^{-\mu_\alpha x} (x - a), \\
&> f(x, w_\epsilon(x)) - L\epsilon(x - a), \\
&> f(x, w_\epsilon(x)), \quad a \geq x \geq X.
\end{aligned}$$

Here, we have used relation (3.4), (3.6) and the assumptions $Le^{\mu_\alpha x} \leq \frac{\mathcal{B}(\alpha)\mu_\alpha}{(1-\alpha)}$. We can now apply Theorem 3.1 to v and $w_\epsilon(x)$ to get $v(x) < w_\epsilon(x)$, $a \leq x \leq X$. Since $\epsilon > 0$ is arbitrary, we conclude that (3.5) is true, and we are done. $\square$

To proceed further, we need Peano's type existence result for Caputo-Fabrizio fractional differential inequalities.

Theorem 3.3. Assume that $f \in C[R_a, \mathbb{R}]$ where $R_a = [(t, x) : a \leq x \leq b, \text{ and } |u - u_a| \leq c]$, and let $|f(x, u)| \leq M$ on R_a . Then there exists at least one solution for the IVP (2.6) on $a \leq x \leq \alpha$, where $\alpha = \min\left(b, \frac{c\mathcal{B}(\alpha)}{M(1-\alpha)}\right)$, $0 < \alpha < 1$.

Proof. Let $u_0(\eta)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$ such that $x_0(a) = x_a$, $|u_0(x) - u_a| \leq c$, and $|{}^{CFC}\mathcal{D}_a^\alpha u_a(x)| \leq M$, where ${}^{CFC}\mathcal{D}_a^\alpha u_a(x)$ is the continuous fractional derivative. For $0 < \eta \leq \delta$, we define the function $u_\eta(x) = u_a(x)$ on $[-\delta, 0]$ and

$$u_\eta(x) = u_a + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x, u_\eta) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u_\eta(s-\eta)) ds \quad (3.7)$$

on $[0, \alpha_1]$, where $\alpha_1 = \min(\alpha, \eta)$.

We observe that ${}^{CFC}\mathcal{D}_a^\alpha u_\eta(x)$ exists and

$$\begin{aligned} |u_\eta(x) - u_a| &\leq \frac{(1-\alpha)}{\mathcal{B}(\alpha)} |f(x, u_\eta)| + \int_a^x |f(s, u_\eta)| ds, \\ &\leq M \frac{(1-\alpha)}{\mathcal{B}(\alpha)} + M \int_a^x ds, \\ &= M \frac{(1-\alpha)}{\mathcal{B}(\alpha)} + M(x-a), \\ &= M \left[\frac{1-\alpha}{\mathcal{B}(\alpha)} + (x-a) \right] \leq c, \end{aligned} \quad (3.8)$$

because of the choice of α_1 . If $\alpha_1 < \alpha$, we can employ (14) to extend $u_\eta(x)$ as a continuously fractional differentiable function on $[-\delta, \alpha_2]$, $\alpha_2 = \min(\alpha, 2\eta)$ such that $|u_\eta(x) - x_a| \leq c$ holds. Continuing this process, we can define $u_\eta(x)$ over $[-\delta, \alpha]$ so that $|u_\eta(x) - x_a| \leq c$, it has a continuous fractional derivative and satisfies (3.7) on the same interval $[-\delta, \alpha]$. Moreover, $|{}^{CFC}\mathcal{D}_a^\alpha u_\eta(x)| \leq M$, since $|f(x, u_\eta(x-\eta))| \leq M$ on R_a , and therefore the family $\{u_\eta(x)\}$ forms an equicontinuous and uniformly bounded family of functions by Lemma 2.2. As application of Ascoli-Arzelà's theorem shows the existence of sequence $\{\eta_n\}$ such that $\eta_1 > \eta_2 > \eta_3 \dots > \eta_n \rightarrow 0$ as $n \rightarrow \infty$, and $u(x) = \lim_{n \rightarrow \infty} u_{\eta_n}(x)$ exists uniformly on $[-\delta, \alpha]$. Since f is uniformly continuous, we obtain $f(x, u_{\eta_n}(x-\eta_n))$ tends uniformly to $f(x, u(x))$ as $n \rightarrow \infty$, and hence term by term integration of (3.7) with $\eta = \eta_n$, $\alpha_1 = \alpha$ yields

$$u(x) = u_a + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x, u(x)) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u(s)) ds. \quad (3.9)$$

This proves that $u(x)$ is a solution of IVP (2.6), and the proof is complete. $\square$

Employing Theorems 3.1 and 3.3, we can now prove the existence of the extremal solutions for the IVP (2.6).

Theorem 3.4. Assume that $f \in C[R_a, \mathbb{R}]$ where $R_a = [(t, x) : a \leq x \leq b, \text{ and } |u - u_a| \leq c]$, and let $|f(x, u)| \leq M$ on R_a , then there exists extremal solutions for the IVP (2.6) on $a \leq x \leq \alpha_a$, $\alpha_a = \min\left(b, \left[\frac{c\mathcal{B}(\alpha)}{(2M+c)(1-\alpha)}\right]^{\frac{1}{\alpha}}\right)$.

Proof. We shall prove the existence of the maximal solution only, since the case of the minimal solution is very similar.

Let $0 < \epsilon \leq \frac{c}{2}$ and consider the IVP

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u) + \epsilon, \quad u(a) = u_a + \epsilon \quad (3.10)$$

Note that $f_\epsilon(x, u) = f(x, u) + \epsilon$ is defined and continuous on

$$R_\epsilon = \left[(t, x) : a \leq x \leq b \text{ and } |x - (x_a + \epsilon)| \leq \frac{c}{2}\right], \quad (3.11)$$

$R_\epsilon \subset R_a$ and $|f_\epsilon(x, u)| \leq M + \frac{c}{2}$ on R_ϵ .

We then deduce from Theorem 3.3 that IVP (3.10) has a solution $u(x, \epsilon)$ on $a \leq x \leq \alpha_a$. Now, for $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we have

$$\begin{aligned} u(a, \epsilon_2) &< u(a, \epsilon_1), \\ {}^{CFC}\mathcal{D}_a^\alpha u(a, \epsilon_2) &\leq f(x, u(x, \epsilon_2)) + \epsilon_2, \\ {}^{CFC}\mathcal{D}_a^\alpha u(a, \epsilon_1) &> f(x, u(x, \epsilon_1)) = \epsilon_2, \quad \text{on } a \leq x \leq \alpha_a. \end{aligned} \quad (3.12)$$

We apply Theorem 3.1 to get

$$u(x, \epsilon_2) < u(x, \epsilon_1), \quad a \leq x \leq \alpha_a. \quad (3.13)$$

Consider the family of continuous functions $\{u(x, \epsilon)\}$ on $a \leq x \leq \alpha_a$. Then it follows that

$$\begin{aligned} |u(x, \epsilon) - u(a, \epsilon)| &\leq \frac{1-\alpha}{\mathcal{B}(\alpha)} |f(x, u(s, \epsilon))| + \int_a^x |f(s, u(s, \epsilon))| ds, \\ &\leq \frac{(1-\alpha)}{\mathcal{B}(\alpha)} \left(M + \frac{c}{2}\right) + \left(M + \frac{c}{2}\right) (x - a), \\ &\leq \left(M + \frac{c}{2}\right) \left(\frac{1-\alpha}{\mathcal{B}(\alpha)} (1 + \epsilon)\right), \\ &\leq \frac{c}{2} \leq c. \end{aligned} \quad (3.14)$$

Showing that the family is uniformly bounded. Since $|{}^{CFC}\mathcal{D}_a^\alpha u(x, \epsilon)| = |f(x, u(x, \epsilon))| \leq M + \frac{\epsilon}{2}$, the family is also equicontinuous by Lemma 2.2. Hence, there exists a sequence ϵ_n with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and the uniform limit $\eta(x) = \lim_{n \rightarrow \infty} u(x, \epsilon_n)$ exists on $[a, \alpha_a]$. Clearly, $\eta(a) = u_a$. The uniform continuity of f gives, as before (as in Theorem 3.3), that $\eta(x)$ is a solution of IVP (2.6).

Next, we show that $\eta(x)$ is the required maximal solution of (2.6). Let $u(x)$ be any solution of (2.6) on $0 \leq x \leq \alpha_a$, then we have

$$\begin{aligned} u_a &< u_a + \epsilon = u(a, \epsilon), \\ {}^{CFC}\mathcal{D}_a^\alpha u(x) &< f(x, u(x)) + \epsilon, \\ {}^{CFC}\mathcal{D}_a^\alpha u(x, \epsilon) &\geq f(x, u(x, \epsilon)) + \epsilon. \end{aligned} \quad (3.15)$$

By Theorem 3.1, we obtain $u(x) < u(x, \epsilon)$, $a \leq x \leq \alpha_a$ for every $\epsilon > 0$. The uniqueness of maximal solution shows that $u(x, \epsilon)$ tends to $\eta(x)$ on $[a, \alpha_a]$ as $\epsilon \rightarrow 0$. The proof is therefore complete. $\square$

We need the following comparison theorem to prove the global existence of a solution.

Theorem 3.5. Assume that $m \in C([a, X], \mathbb{R}_+)$, locally Lipschitz continuous, $g \in C([a, X] \times \mathbb{R}_+, \mathbb{R}_+)$, and

$${}^{CFC}\mathcal{D}_a^\alpha m(x) \leq g(x, m(x)), \quad a \leq x \leq X. \quad (3.16)$$

Let $\eta(x)$ be the maximal solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) = g(x, v(x)), \quad u(a) = u_a \geq a, \quad (3.17)$$

existing on $[a, X)$ such that $m(a) \leq v_a$. Then we have

$$m(x) \leq \eta(x), \quad a \leq x \leq X. \quad (3.18)$$

Proof. In view of the definition of the maximal solution $\eta(x)$, it is enough to prove, to conclude (3.18), that

$$m(x) < v(x, \epsilon), \quad a \leq x < X, \quad (3.19)$$

where $v(x, \epsilon)$ is any solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) \leq g(x, v(x)) + \epsilon, \quad v(a) = v_a + \epsilon, \quad \epsilon > 0. \quad (3.20)$$

Now it follows from (3.20) that

$${}^{CFC}\mathcal{D}_a^\alpha v(x, \epsilon) > g(x, v(x, \epsilon)). \quad (3.21)$$

Then applying Theorem 3.1, we get immediately (3.19) and since $\lim_{\epsilon \rightarrow 0} v(x, \epsilon) = \eta(x)$ uniformly on each $a \leq x \leq X_a < X$, the proof is complete. $\square$

We are now ready to prove the global existence result.

Theorem 3.6. *Assume that $f \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, $g \in C([a, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$, $g(x, v)$ is nondecreasing in u for each x and*

$$|f(x, u)| \leq g(x, |u|). \quad (3.22)$$

Suppose that we have local existence of solutions $u(x, u_a)$ of

$${}^{CFC}\mathcal{D}_a^\alpha u(x) = f(x, u(x)), \quad u(a) = u_a. \quad (3.23)$$

and the maximal solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) = g(x, v(x)), \quad v(a) = v_a \geq a, \quad (3.24)$$

exists on $[a, \infty)$. Then the largest interval of existence of any solution $u(x, u_a)$ of (3.23) such that $|u_a| \leq v_a$ is $[a, \infty)$.

Proof. Let $u(x, x_a)$ be any solution of (3.23) with $|u_a| \leq v_a$, which exists on $[a, \beta)$, $\beta < \infty$ and the value of β cannot be increased further. Set $m(x) = |u(x, x_a)|$ for $a \leq x < \beta$. Then, using the assumption (3.22), we get

$$\begin{aligned} {}^{CFC}\mathcal{D}_a^\alpha m(x) &\leq |{}^{CFC}\mathcal{D}_a^\alpha u(x, u_a)|, \\ &= |f(x, u(x, u_a))|, \\ &\leq |g(x, u(x, u_a))|, \\ &\leq g(x, m(x)), \quad a \leq x < \beta, \end{aligned} \quad (3.25)$$

and $m(a) \leq u_a$. Applying Theorem 3.5, we obtain

$$m(x) \leq \eta(x), \quad a \leq x \leq \beta, \quad (3.26)$$

and therefore $|{}^{CFC}\mathcal{D}_a^\alpha u(x, u_a)| \leq g(x, m(x)) \leq g(x, \eta(x)) \leq M$ on $a \leq x \leq \beta$, since $\eta(x)$ is assumed to exist on $(a, \infty]$, it follows that $g(x, \eta(x)) \leq M$. Now for $a \leq x_1 \leq x_2 < \beta$ we find by Lemma 2.2,

$$|u(x_1, u_a) - u(x_2, u_a)| \leq \frac{2M\mathcal{B}(\alpha)}{1 - \alpha}(x_2 - x_1). \quad (3.27)$$

Letting $x_1, x_2 \rightarrow \beta^-$ and using Cauchy criterion, it follows that $\lim_{x \rightarrow \beta^-} u(x, u_a)$ exists.

We denote $u(\beta, u_a) = \lim_{x \rightarrow \beta^-} u(x, u_a)$ and consider the new IVP

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u), \quad u(\beta) = u(\beta, u_a). \quad (3.28)$$

By the assumed local existence, we find that $u(x, u_a)$ can be continued beyond β , contradicting our assumption. Hence every solution $u(x, u_a)$ of (3.23) exists on $[a, \infty)$ and completes the proof. $\square$

4 Conclusion

The new definition of fractional derivative, mentioned in [6, 20, 22], overcome the problem of singular kernel found in Riemann-Liouville and Caputo fractional derivative. By using this definition we reestablished the results in [3, 17, 18] for new conditions, that are seen in statements and proofs of Theorems 3.2, 3.3, and 3.4. This research investigates *FDEs* with the help of fractional differential inequalities and the comparison principle with the Caputo-Fabrizio fractional derivative in the Caputo-sense. The study's significance lies in analyzing solutions to fractional differential inequalities, showing its usefulness as a fundamental tool in fractional analysis. As fractional calculus advances, these findings provide valuable contributions to theory and its practical applications, guiding future research in the field. This work can be extended to the stochastic or the multi-term fractional differential system, which feature nonsingular kernels. Also one can extend the results for some other definitions of fractional derivative, for example Atangaba-Baleanu fractional derivative.

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CLASSIFICATION OF DIFFERENT EDUCATION BOARDS IN INDIA USING FCA

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Abstract

Formal Concept Analysis (*FCA*) is an effective mathematical method employed for knowledge extraction from data in a structured manner. Based on the lattice theory, *FCA* discovers significant groupings of objects and their common attributes as formal concepts. India has acknowledged education as a tool for national growth and establishing a new social structure. The idea of curriculum gained popularity when education was established as a formal system of human society for learning, and it has continued to enjoy the considerable significance ever since. The different educational boards in India are *CBSE* (Central Board of Secondary Education), *ICSE* (Indian Certificate of Secondary Education), *NIOS* (National Institute of Open Schooling), State Board (*SB*), *IGCSE* (International General Certificate of Secondary Education), and *IB* (International Baccalaureate), and many more. Each board has a unique structure. The researchers have examined all six boards by considering the examination pattern and the analysis of the 10th standard subject. *FCA* was used to classify the 10th standard subjects and assessments in different school boards in India. The readers will clearly understand the information about different school boards, subjects, and assessments.

2020 Mathematical Sciences Classification: 54A05

Keywords and Phrases: *CBSE*, *ICSE*, *NIOS*, *SB*, *IGCSE*, *IB*, School boards, *FCA*.

1 Introduction

Formal Concept Analysis (*FCA*) is an effective tool for data analysis. The main objective of this study is to demonstrate the effectiveness of *FCA* in determining the underlying knowledge from real-life datasets. The basic details of *FCA* is available in [1, 5, 16] and a review of *FCA* is covered in many articles [10, 11, 12, 14]. An experiment was conducted on the dataset from different school boards in India. Data was collected from the Kaggle platform. And finally, the researchers will discuss the importance of schools and major school boards in India.

Education is the only important tool for changing the society. Schools are the places which not only serves to enhance one's learning but also, they contain the future of individuals and societies. Schools are essential for personal growth (educated, responsible, and successful citizens) and national development. Different educational boards follow different curricula across India and also it remains the same globally. Here, six major boards like *CBSE*, *ICSE*, *NIOS*, *SB*, *IGCSE* and *IB*. The importance of schools are academic learning and knowledge, character building and discipline, social development and communication skills, career preparation and future opportunities, personal growth and confidence, reducing social inequality, citizenship and social responsibility, health and well-being, and physical fitness. In this study, the researchers discuss a brief explanation about school board-based articles.

Aida Kairien discussed the understanding of the micropolitics in schools by *FCA*. Context were framed like school heads and the entire school community, the importance of leadership for teachers. The relationships between the attributes aids in the better understanding of a school's micropolitics which in turn, improve the community relationships for the implementation of the schools strategies [9]. Anirban Roy Chowdhury

et al. [2] analysed three English medium Indian school boards, *ICSE*, *CBSE*, and *NIOS* and discussed on the mode of exams they conduct and focuses mainly on the areas and topics. Usha *et al.* [15] analysed Indian school boards State, *CBSE*, and *ICSE*, which clearly shows inequality in terms of curriculum, examination, and fee structure of these three boards "Education brings Equality" and "One Nation and One Education System". Gharami *et al.* [6] analysed educational contents and the sope of language development in secondary level English under several school boards.

Marlies Honingh *et al.* [7] reviewed on school boards and educational quality and also identified which school boards contribute to education quality. Honingh *et al.* [8] discussed Dutch secondary school-based measurements of board characteristics, pupil achievements, and board behaviour. The study leaves gap to write more articles based on school boards depending on variables such as board behaviour and corporate governance. Dervarics *et al.* [3] mentioned and explained eight characteristics of effective school boards. According to the work and analysis of Sell *et al.* [13] the best way to manage a successful school district is to make sure that school boards create a vision for student learning and successfully convey that goal to the superintendent /cite13. Ford *et al.* [4] analysed school boards and their use of different models of governance. School board articles were discussed so far, and next we will discuss the outline of this paper. The author strategized to classify the different school boards in India. Understand the different subjects among the different board systems. The first section gives the information about *FCA*, the importance of school, and the different types of boards in India and reviewed many articles. In Section 2, the top six board-based subjects in India were collected. Classified the different boards and subjects by the use of *FCA* were discussed in section 3. Section 4 concludes the article.

2 Data Description

The *CBSE*, *ICSE*, *NIOS*, *SB*, *IGCSE*, and *IB* are the most reputable boards where students can establish a basis for the future. Each of these boards, however, has different courses, evaluations, curricula, material, costs, goals, and methods for instructing students [2]. This paper particularly deals with only the subject and assessment in the 10th standard in different boards with the use of *FCA*. The following Table 2.1 contains the six school boards, subjects, sub-subjects and assessments.

Table 2.1: 10th standard subjects and assessments in *CBSE* board schools in India

Board	Subject	Subject (detailed)
1. <i>CBSE</i>	First Language	English
	Second Language	Hindi / Sanskrit / French
	Mathematics	Standard & Basic
	Science	IS Integrated Sciences (Physics, Chemistry, Biology)
	Social Science	History, Civics, Geography, Economics
	Optional Subject	Computer Science (CS), Artificial Intelligence (AI), Home Science (HS), Arts, Information Science (IS)
	Vocational Subjects	Information Technology (IT), AI, Coding
	Assessment Pattern	Year-end board exams
2. <i>ICSE</i>	First Language	English
	Second Language	Hindi / Sanskrit / French
	Mathematics	Maths
	Science	SS-Separable Subjects (Physics, Chemistry, Biology)
	Social Science	History, Civics, Geography
	Optional Subject	Computer Application (CA), Arts, Deama, Physical Education (PE), HS
	Vocational Subjects	Commercial Studies, Environmental Science (ES)
	Assessment Pattern	Internal and board exams
3. <i>NIOS</i>	First Language	English/Hindi
	Second Language	Hindi, Urdu, Sanskrit, etc.
	Mathematics	Maths
	Science	Science & Technology (ST)
	Social Science	Social Science
	Optional Subject	Data Entry, HS, Vocational Courses (VC)

therefore more specific. The concept 3 is more general concept than all other concepts, which means that this concept has less objects with more attributes and thus it is a more general concept. The concept 25 contains more objects and less attribute. Next, showed the extent and intent of the concepts in Figure 2.1.

Table 2.2: Concepts with their Intents and Extents for concept lattices from Figure 2.1

Concept	Extent	Intent
1	{ }	{Regional Language, English, Hindi, Sanskrit, French, Urdu, Spanish, German, Other, General Maths, Standard & Basic, Core/Extended, IS, SS, ST, IS or SS, History, Civics, Geography, Economics, General Social Science, GP, Individual Societies, CS, CA, AI, HS, Arts, IT, Drama, PE, Data Entry, VC, ES1, BS, Economics, PHE, Design, IS, AI, Coding, Commercial Studies, ES, HS, BS, SPVC, ICT, Business, TT, DT, Annual, Internal, Open School, Flexible, Cambridge, Project Based}
2	{SB}	{Regional Language, English, Hindi, General Maths, IS or SS, History, Civics, Geography, CS, ES1, SPVC, Annual}
3	{CBSE}	{English, Hindi, Sanskrit, French, Standard & Basic, IS, History, Civics, Geography, Economics, CS, AI, HS, Arts, IS, IT, AI, Coding, Annual}
4	{ ICSE }	{English, Hindi, Sanskrit, French, General, Maths, SS, History, Civics, Geography, CA, AI, HS, Drama, PE, Commercial Studies, ES, Annual, Internal}
5	{IGCSE}	{English, French, Spanish, German, Other, Core/Extended, SS, GP, Arts, Drama, BS, Economics, ICT, Business, TT}
6	{NIOS}	{English, Hindi, Sanskrit, Urdu, Other, General Maths, ST, General Social Science, HS, Data Entry, VC, HS, BS, Open School, Flexible}
7	{IB}	{English, French, Spanish, Other, General Maths, IS, individual Societies, Arts, PHE, Design, DT, Project Based}
8	{SB, CBSE}	{English, Hindi, History, Civics, Geography, CS, Annual}
9	{SB, ICSE }	{English, Hindi, General Maths, History, Civics, Geography, Annual}
10	{ ICSE , CBSE }	{English, Hindi, Sanskrit, French, History, Civics, Geography, AI, HS, Annual}
11	{ ICSE , IGCSE }	{English, French, SS, Drama}
12	{ ICSE , NIOS }	{English, Hindi, Sanskrit, General Maths, HS}
13	{ ICSE , IB }	{English, French, General Maths}
14	{CBSE, IB}	{English, French, IS, Arts}
15	{IGCSE, IB}	{English, French, Sanskrit, Other, Arts}
16	{NIOS, IB}	{English, Other, General Maths}
17	{SB, CBSE, ICSE }	{English, Hindi, History, Civics, Geography, Annual}
18	{CBSE, ICSE , NIOS}	{English, Hindi, Sanskrit, HS }
19	{SB, ICSE , NIOS}	{English, Hindi, General Maths}
20	{CBSE, IGCSE, IB}	{English, French, Arts}
21	{IGCSE, NIOS, IB}	{English, Other}
22	{SB, CBSE, ICSE, NIOS}	{English, Hindi}
23	{CBSE, ICSE , IGCSE, IB}	{English, French}
24	{SB, ICSE , NIOS, IB}	{English, General Maths}
25	{SB, CBSE, ICSE, IGCSE, NIOS, IB}	{English}

All six school boards are at the same bottom level and are therefore considered equally important concerning concept 1. It is evident from the concept lattice diagram in Figure 2.1 that the attributes and objects associated with each concept can be viewed simultaneously within the hierarchical structure.

Table 2.3: Language-based formal context

	Regional Lan- guage	English	Hindi	Sanskrit	French	Urdu	Spanish	German	Other
<i>CBSE</i>		X	X	X	X				
<i>ICSE</i>		X	X	X	X				
<i>NIOS</i>		X	X	X		X			X
SB	X	X	X						
<i>IGCSE</i>		X			X		X	X	X
<i>IB</i>		X			X		X		X

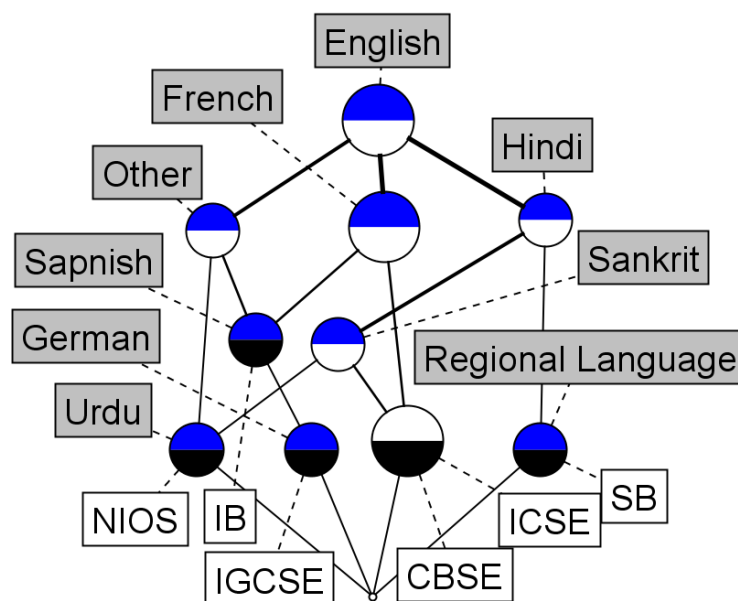


Figure 2.2: Language-based concept lattice from Table 2.3

From the concept lattice, Figure 2.2 contained 11 concepts whose intents and extents were also clearly mentioned. The numbering of concepts in the concept lattice starts from the bottom to the top and moves from right to left. Every 10th standard students study English as one of the subjects, and it is mandatory for all the boards. *CBSE* and *ICSE* boards both depend on the same language. The *IGCSE* board stands as a super-concept of the *IB*board; *IB*remains a sub-concept of *IGCSE*. The *NIOS* and *SB* boards contain Urdu and the regional language, respectively.

2.2 Mathematics

Table 2.4: Maths-based formal context

	General Maths	Standard & Basic	Core/Extended
<i>CBSE</i>		X	
<i>ICSE</i>	X		
<i>NIOS</i>	X		
<i>SB</i>	X		
<i>IGCSE</i>			X
<i>IB</i>	X		

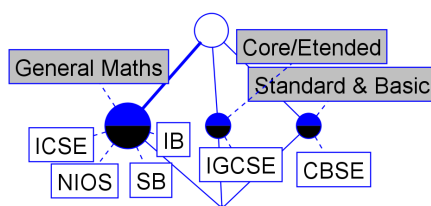


Figure 2.3: Maths-based concept lattice from Table 2.4

By considering the Figure 2.3, which contains 5 concepts: mathematics remains a compulsory subject for all the boards, with only small changes in the frame of syllabus setting. The *SB*, *NIOS*, *ICSE*, and *IB* boards follow the same math, but *IGCSE* and *CBSE* follow a different math. The subject-wise classification of 10th standard students makes it clear that maths is an essential subject and cannot be avoided.

2.3 Science

Table 2.5: Science-based formal context

	IS	SS	ST	IS or SS
<i>CBSE</i>	X			
<i>ICSE</i>		X		
<i>NIOS</i>			X	
<i>SB</i>				X
<i>IGCSE</i>		X		
<i>IB</i>	X			

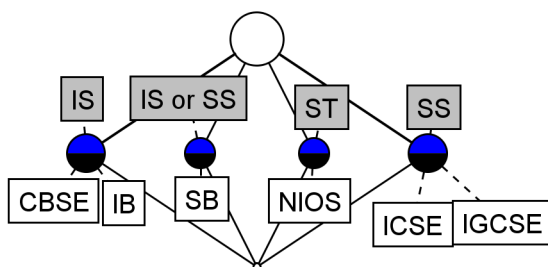


Figure 2.4: Science-based concept lattice from Table 2.5

Figure 2.4 contains 6 concepts; *IGCSE* stands as the super concept of *ICSE*. The *CBSE* and *IB* boards follow the same science syllabus. *NIOS* remains different from the other five boards of schools. Anirban Roy Chowdhury *et al.* analysed science and maths subjects in different school boards detailed in [8].

2.4 Social science

Table 2.6: Social science-based formal context

	History	Civics	Geography	Economics	General Social Science	GP	Individual Societies
<i>CBSE</i>	X	X	X	X			
<i>ICSE</i>	X	X	X				
<i>NIOS</i>					X		
<i>SB</i>	X	X	X				
<i>IGCSE</i>						X	
<i>IB</i>							X

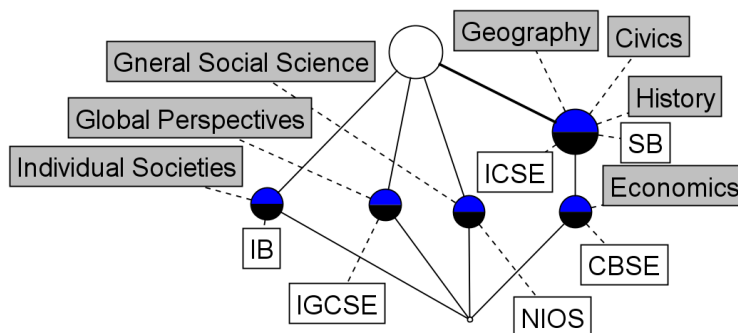


Figure 2.5: Social science-based concept lattice from Table 2.6

The researchers discussed the social science subject-based classification in Figure 2.5, which contains 7 concepts; it remained clear in the understanding of the subject. *CBSE* remained as a super concept of *SB*, and economics was the one extra part added to the *CBSE* syllabus. The other three boards have different styles of social studies that were followed.

2.5 Optional

Table 2.7: Optional subject-based formal context

	CS	CA	AI	HS	Arts	IS	Drama	PE	Data Entry	VC	ES1	BS	Econo	PHE	Design
<i>CBSE</i>	X		X	X	X	X									
<i>ICSE</i>		X	X	X			X	X							
<i>NIOS</i>									X	X					
<i>SB</i>	X											X			
<i>IGCSE</i>					X		X						X	X	
<i>IB</i>					X										X

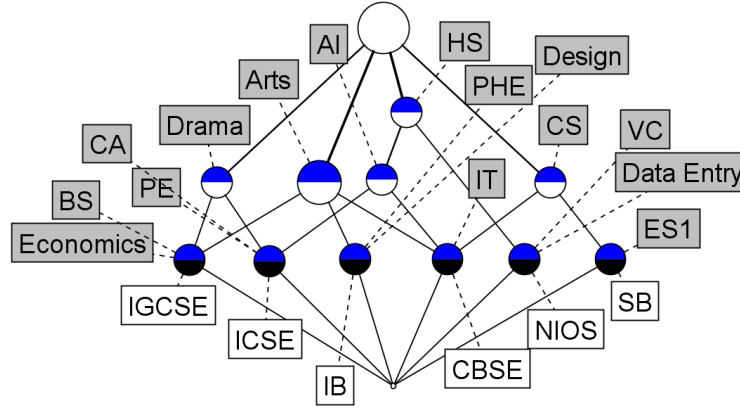


Figure 2.6: Optional subject-based concept lattice from Table 2.7

Figure 2.6 illustrated the optional subject-based classification, contains 13 concepts, and it remained feasible to understand the concepts. The optional subject is the only identified area of interest for a student in his(her) future to carry on and continue higher studies and it remained entirely different from the core papers. So, the students were interested, involved, learned quickly, and collaborated with the different class students, and they automatically improved their sociality and some other knowledge.

2.6 Vocational

Table 2.8: Vocational Subject-based formal context

	IT	AI	Coding	Commercial Studies	ES	HS	BS	SPVC	ICT	Business	TT	DT
<i>CBSE</i>	X	X	X									
<i>ICSE</i>				X	X							
<i>NIOS</i>						X	X					
<i>SB</i>								X				
<i>IGCSE</i>									X	X	X	
<i>IB</i>												X

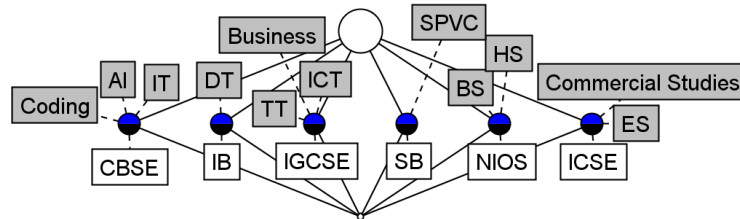


Figure 2.7: Vocational Subject-based concept lattice from Table 2.8

And finally, Figure 2.7 contained 8 concepts on the vocational-based subject. This also clearly showed the concepts and subjects of the different boards. This is also like an optional subject. All the school boards followed different vocational subjects, but the students had the freedom to select the subjects. Automatically, students got the knowledge from different areas and joined some other classes. This made them to enhance their learning and also they gained knowledge other than their academical.

2.7 Assessment

Table 2.9: Assessment-based formal context

	Annual	Internal	Open School	Flexible	Cambridge	Project based
<i>CBSE</i>	X					
<i>ICSE</i>	X	X				
<i>NIOS</i>			X	X		
<i>SB</i>	X					
<i>IGCSE</i>					X	
<i>IB</i>						X

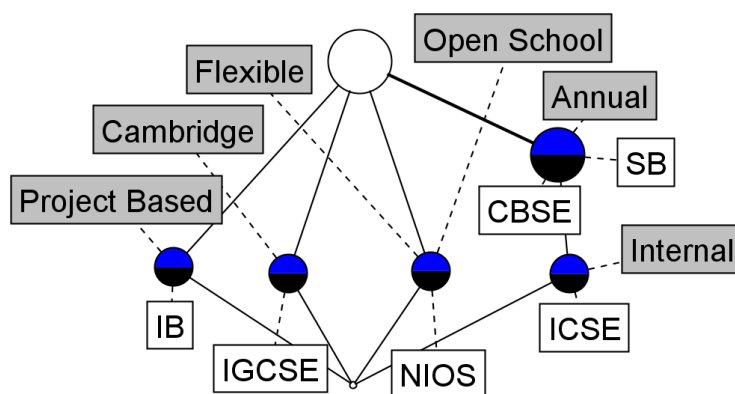


Figure 2.8: Assessment-based concept lattice from Table 2.9

The assessment has been made at this stage, which remained critical because the process ends with the success or output of the one year. Figure 2.8 contained 7 concepts; *ICSE* is the super concept of *CBSE* and *CB*. Cambridge-based assessment also had a year-end exam but written, oral, coursework, and practical. On conducting assesment twice a year, in June and November, results were issued in August and January (Grade A* to G). Mostly, the schools followed the year-end assessments, while a few boards followed in a different manner. Thus, the study suggest that people or student should be aware of the differences among the school board's subjects, assessments, fee structure, etc.

3 Discussion

The overall classification of this particular work revealed the outline of different school boards, 10th standard subjects, and assessments covered. The selection of language mostly remained as the first option, which was either English or the mother tongue and in *SB* it remained as regional language. In India State Board has 10 divisions for the 10th standard students. The list of *SB* in India are Tamil Nadu State Board of School Examination(*TNBSE*), Maharashtra State Board of Secondary and Higher Secondary Education (*MSBSHSE*), Bihar School Examination Board (*BSEB*), Uttar Pradesh Madhyamik Shiksha Parishad (*UPMSP*), Board of Secondary Education, Rajasthan (*RBSE/BSER*), Gujarat Secondary and Higher Secondary Education Board (*GSEB*), Board of Secondary Education, Odisha (*BSE Odisha*), Karnataka Secondary Education Examination Board (*KSEEB*), West Bengal Board of Secondary Education (*WBBSE*), and Board of Secondary Education, Andhra Pradesh (*BSEAP*). Based on the above boards, the following language serves as their mother tongue, which is the first language; the second language remains also optional, but the first option is English or Hindi. As our national language is Hindi, they both stands above all the other regional languages in India for the communication purposes which made English or Hindi for being the compulsory language to learn.

Mathematics, science, and social science were the compulsory subjects for all the school boards in 10th standard, but the higher secondary schools were different in choosing the subjects. Optional and vocational subjects were rooted in extracurricular activities, making it essential for students to choose their higher secondary and college courses wisely, as these serve as the foundation for their future. Continuous engagement in academics occasionally made the students feel tedious, involving in these activities offered them a refreshing break, allowing students to return to their core subjects with ease and clarity, leading to a more stress-free learning experience. Optional and vocational activities were often hands-on, engaging both the mind and body, which benefits the overall health.

The key aspect of all school board *PT* periods was to enhance the students' holistic development. Particularly, *CBSE* school boards have *PT* periods on regular basis. These days, students participate in *PT* and engage in additional activities such as yoga, basic floor exercises, and track and field events. Nowadays, *PT* periods are made compulsory for all the schools and standards, but they lack necessary equipments and sufficient grounds for all the students. Scheduling physical training (*PT*) between 11 AM and 3 PM is not ideal for students' health, however, having *PT* for two to three hours per week is feasible. Ideally, sessions should be scheduled after 3 PM and when the climate is more suitable for outdoor activities and there is no time restriction or schedule for indoor games.

The researcher recommended that the government should look after the schools with no grounds and take necessary action to give the strict order to develop the ground. In the current scenario of stressful environment, students have heavy competition with other students, schools, parents, and teachers. Only these extra curricular activities can make them fit physically, mentally, and emotionally.

Parents and students, before entering the schools or selecting the schools, kindly refer to the full details of the schools, subjects, fees, etc., and then decide. Homes, schools, and the society are the three main factors where a student can learn, grow, and enhance themselves which helps them for their better future and also the countrys development.

FCA makes it feasible to analyse and understand the data clearly. Through the analysis of the review articles based on *FCA* has discussed clustering, attribute reduction, knowledge reduction, concept reduction, etc. Thus, it is evident that only this current study has analysed based on the data model.

4 Conclusion

In the current study, the different school boards in 10th standard and subjects in India were classified by the use of *FCA*. Thus, *FCA* serve as one of the best data analysis tools in scientific research. Which aids people to know more about the school boards and subjects without any further search outdoors. This research leaves a gap to collect and analyse 11th and 12th boards and subjects using *FCA* and different data analysis methods, such as *CART* and neural networks for its future endeavours (scope).

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GROWTH ANALYSIS OF COMPOSITE ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES IN THE UNIT POLYDISC BASED ON THE RELATIVE (p, q) -th ORDER, RELATIVE (p, q) -th TYPE AND RELATIVE (p, q) -th WEAK TYPE

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Abstract

In this paper, we study some growth properties of composite analytic functions of several complex variables in the unit polydisc. This paper establishes new results concerning the relative (p, q) -th order, relative (p, q) -th type and relative (p, q) -th weak type of an analytic function with respect to an entire function, where p and q are any two positive integers.

2020 Mathematical Sciences Classification: 30D35, 46S10, 30D20, 32A15

Keywords and Phrases: Growth, Composition, Analytic Function, Polydisc, Relative (p, q) -th Order, Relative (p, q) -th Type and Relative (p, q) -th Weak Type.

1 Introduction

Let $\mathcal{A}(\mathbb{K})$ represents the set of analytic functions. So it is formed $\mathbb{K}$ -algebra of analytic functions in $\mathbb{K}$, i.e. the set of power series with an infinite radius of convergence. For more details on the study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we can read the following articles [1, 7, 8]. Suppose $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$ then $|f|(r) = \sup\{|f| : |x| = r\}$ represents the multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, for a non-constant analytic function f , $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ whenever r tends to $+\infty$. Therefore, its inverse function exists, and we can define it in the following way $\widehat{|f|} : (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim_{s \rightarrow \infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log(\log^{[k-1]} x)$ and $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$, where $\mathbb{N}$ denotes the set of all positive integers.

To start our paper we just recall the following definitions:

2 Definitions

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the unit disc $D = \{z : |z| < 1\}$ and $M(r, f)$ be the maximum of $|f(z)|$ on $|z| = r$. Sons [9] defined the order $\rho(f)$ and the lower order $\lambda(f)$ as

$$\rho(f) = \limsup_{r \rightarrow 1} \frac{\log \log M(r, f)}{-\log(1-r)} \quad \text{and} \quad \lambda(f) = \liminf_{r \rightarrow 1} \frac{\log \log M(r, f)}{-\log(1-r)}.$$

Considering the unit polydisc $D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\}$, Banerjee and Dutta [2] introduced the definition of order and lower order of functions of two complex variables analytic in the unit polydisc, which are as follows:

Definition 2.1. Let

$$f(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} a_{n_1 n_2} z_1^{n_1} z_2^{n_2}$$

be a non-constant analytic function of two complex variables z_1 and z_2 in the closed unit polydisc

$$D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\}$$

and let

$$M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, i = 1, 2\}.$$

Then order and lower order of $f(z_1, z_2)$ are denoted by $\rho(f)$ and $\lambda(f)$ respectively and defined as

$$\rho(f) = \limsup_{r_1, r_2 \rightarrow 1} \frac{\log \log M_f(r_1, r_2)}{\log \left(\frac{1}{(1-r_1)(1-r_2)} \right)} \quad \text{and} \quad \lambda(f) = \liminf_{r_1, r_2 \rightarrow 1} \frac{\log \log M_f(r_1, r_2)}{\log \left(\frac{1}{(1-r_1)(1-r_2)} \right)}.$$

Generalising this notion, Dutta [6] introduced the definitions of (p, q) -th order and (p, q) -th lower order of functions of two complex variables analytic in the unit polydisc in this way:

Definition 2.2 ([6]). Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 in the closed unit polydisc

$$D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\},$$

then (p, q) -th order $\rho^{(p, q)}(f)$ and (p, q) -th lower order $\lambda^{(p, q)}(f)$ of $f(z_1, z_2)$ are defined as:

$$\rho^{(p, q)}(f) = \limsup_{r_1, r_2 \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)} \right)} \quad \text{and} \quad \lambda^{(p, q)}(f) = \liminf_{r_1, r_2 \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)} \right)},$$

where p and q are positive integers with $p \geq q \geq 1$.

Extending this notion, one may introduce (p, q) -th order and (p, q) -th lower order for functions of n complex variables analytic in a unit polydisc as follows:

$$\rho^{(p, q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2) \dots (1-r_n)} \right)},$$

and

$$\lambda^{(p, q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2) \dots (1-r_n)} \right)},$$

where $p, q \in \mathbb{N}$ and $f(z_1, z_2, \dots, z_n)$ is a non-constant analytic function of n -complex variables $z_1, z_2, \dots, z_{n-1}, z_n$ in the unit polydisc

$$D = \{(z_1, z_2, \dots, z_n) : |z_i| \leq 1, i = 1, 2, \dots, n; r_1 > 0, r_2 > 0, \dots, r_n > 0\}.$$

Define

$$M_f(r_1, r_2, \dots, r_n) = \max\{|f(z_1, z_2, \dots, z_n)| : |z_i| \leq 1, i = 1, 2, \dots, n; r_1 > 0, r_2 > 0, \dots, r_n > 0\}.$$

The above definition avoids the restriction $p \geq q \geq 1$ from the original definition of the (p, q) -th order (respectively (p, q) -th lower order) of functions of two complex variables holomorphic in the unit polydisc, as introduced by Dutta [6].

In this connection, we recall the following definition:

Definition 2.3. A non-constant analytic function of n -complex variables $f(z_1, z_2, \dots, z_n)$ is said to have index-pair (p, q) if

$$b < \rho^{(p, q)}(f) < \infty \quad \text{and} \quad \rho^{(p-1, q-1)}(f)$$

is not a nonzero finite number, where $b = 1$ if $p = q$, and $b = 0$ otherwise.

Moreover, if $0 < \rho^{(p, q)}(f) < \infty$, then:

$$\begin{cases} \rho^{(p-n, q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p, q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n, q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

Similarly, for $0 < \lambda^{(p, q)}(f) < \infty$, one can easily verify that:

$$\begin{cases} \lambda^{(p-n, q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p, q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n, q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

If $\rho^{(p, q)}(f) = \lambda^{(p, q)}(f)$ then $f(z_1, z_2, \dots, z_n)$ is of regular growth; otherwise, this function is of irregular growth.

Somasundaram and Thamizharasi [10] introduced the notions of L -order and L -lower order for entire functions of a single variable, where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant a . In the line of Somasundaram and Thamizharasi, one may introduce the definition of the $(p, q, t)L$ -th order and $(p, q, t)L$ -th lower order for functions of n complex variables analytic in a unit polydisc:

$$\rho^{(p,q,t)L}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right) + \exp^{[t]} L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \dots, \frac{1}{1-r_n} \right)}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right) + \exp^{[t]} L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \dots, \frac{1}{1-r_n} \right)}$$

where $p, q \in \mathbb{N}$, $t \in \mathbb{N} \cup \{-1, 0\}$.

Definition 2.4. If $f(z_1, z_2, \dots, z_n)$ is an analytic function in D and $g(z_1, z_2, \dots, z_n)$ is an entire function of n complex variables, then the relative (p, q) -th order of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\rho_g^{(p,q)}(f)$, is defined by:

$$\rho_g^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)},$$

where p and q are any two positive integers.

Similarly, for any two positive integers p and q , the relative (p, q) -th lower order of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\lambda_g^{(p,q)}(f)$, is given by:

$$\lambda_g^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)}.$$

We now introduce relative (p, q) -th type and relative (p, q) -th lower type in the unit polydisc as follows:

Definition 2.5 ([3]). Let $f(z_1, z_2, \dots, z_n)$ is an analytic function in D then (p, q) -th type and the (p, q) -th lower type of f having finite positive order are defined by

$$\sigma^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\rho^{(p,q)}(f)}},$$

and

$$\bar{\sigma}^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\rho^{(p,q)}(f)}},$$

respectively.

Definition 2.6 ([3]). Let $f(z_1, z_2, \dots, z_n)$ is an analytic function in D then (p, q) -th weak type of f having finite positive lower-order is defined by

$$\tau^{(p,q)}(f) = \liminf_{(r_1, r_2, \dots, r_n) \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda^{(p,q)}(f)}},$$

and (p, q) -th lower weak type of f , having finite positive lower order is defined by

$$\bar{\tau}^{(p,q)}(f) = \limsup_{(r_1, r_2, \dots, r_n) \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda^{(p,q)}(f)}}.$$

Definition 2.7. If $f(z_1, z_2, \dots, z_n)$ be an analytic function in D and $g(z_1, z_2, \dots, z_n)$ is an entire function of n complex variables, then the relative (p, q) -th weak type of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\tau_g^{(p,q)}(f)$, is defined by:

$$\tau_g^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda_g^{(p,q)}(f)}},$$

where p and q are any two positive integers.

Similarly, for any two positive integers p and q , the relative (p, q) -th lower weak type of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\bar{\tau}_g^{(p,q)}(f)$, is given by:

$$\bar{\tau}_g^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda_g^{(p,q)}(f)}}.$$

3 Known Results

In this section, firstly, we present some known results which will be needed in the sequel.

Theorem 3.1 ([2]). *Let f, g and h be the analytic functions in the unit polydisc D such that the relative index pair of f with respect to h and the index pair of g are (p, q) and (m, n) respectively. Then*

- (i) *The relative index-pair of $f \circ g$ is (p, n) when $q = m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\lambda_h^{(p,q)}(f)\rho^{(m,n)}(g) \leq \rho_h^{(p,n)}(f \circ g) \leq \rho_h^{(p,q)}(f)\rho^{(m,n)}(g)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(p,q)}(f)\lambda^{(m,n)}(g) \leq \rho_h^{(p,n)}(f \circ g) \leq \rho_h^{(p,q)}(f)\rho^{(m,n)}(g)$ if $\lambda^{(p,q)}(f) > 0$.
- (ii) *The relative index-pair of $f \circ g$ is $(p, q + n - m)$ when $q > m$ and either $\lambda_h(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\lambda_h^{(p,q)}(f) \leq \rho^{(p,q+n-m)}(f \circ g) \leq \rho_h^{(p,q)}(f)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(p,q+n-m)}(f \circ g) = \rho_h^{(p,q)}(f)$ if $\lambda^{(m,n)}(g) > 0$.
- (iii) *The relative index-pair of $f \circ g$ is $(p + m - q, n)$ when $q < m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\rho_h^{(p+m-q,n)}(f \circ g) = \rho^{(m,n)}(g)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(m,n)}(g) \leq \rho_h^{(p+m-q,n)}(f \circ g) \leq \rho^{(m,n)}(g)$ if $\lambda^{(m,n)}(g) > 0$.

Theorem 3.2 ([2]). *Let f, g and h be the analytic functions in the unit polydisc D such that the relative index pair of f with respect to h and the index pair of g are (p, q) and (m, n) respectively. Then*

- (i) $\lambda_h^{(p,q)}(f)\lambda^{(m,n)}(g) \leq \lambda_h^{(p,n)}(f \circ g) \leq \min\{\lambda_h^{(p,q)}(f)\lambda^{(m,n)}(g), \lambda_h^{(p,q)}(f)\rho^{(m,n)}(g)\}$ if $q = m$, $\lambda_h^{(p,q)}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$;
- (ii) $\lambda_h^{(p,q+n-m)}(f \circ g) = \lambda_h^{p,q}(f)$ if $q > m$, $\lambda_h^{(p,q)}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$, and
- (iii) $\lambda_h^{(p+m-q,n)}(f \circ g) = \lambda^{(m,n)}(g)$, if $q < m$, $\lambda_h^{p,q}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$.

4 Main Results

In this section, we present the main results of the paper.

Theorem 4.1. *Let f, g, h and k be the analytic functions of several complex variables such that the index pairs of f, g, h and k are (p, q) , (m, n) , (a, b) and (c, d) respectively and*

$$\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[n]}(r_1, r_2, \dots, r_n)}{\log^{[n]}(r_1 r_2 \dots r_n)} = 1.$$

- (i) *If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda^{(p,q)}(f) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$ then*

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \min \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \max \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)},
\end{aligned}$$

and

(ii) If $(q > m, a = c = p)$, $\lambda^{(p,q)}(f) > 0$ and $0 < \lambda_h^{(b,q+n-m)}(f \circ g) < \infty$, then

$$\begin{aligned}
\frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\end{aligned}$$

Proof. Suppose either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ hold and $\lambda^{(p,q)}(f) > 0$. Then in view of Theorem 3.1, the index-pair of $f \circ g$ is (p, n) or $(p + m - q, n)$ respectively and therefore by above definitions, $\rho_h^{(b,n)}(f \circ g)$ and $\lambda_h^{(b,n)}(f \circ g)$ respectively exist.

Since the index-pair of f and k are (p, q) and (c, d) respectively, and $\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[q]}(r_1, r_2, \dots, r_n)}{\log^{[q]}(r_1 r_2 \dots r_n)} = 1$. Therefore, the relative index-pair of f with respect to k is (d, q) . Now from the definition of $\lambda_h^{(b,n)}(f \circ g)$, $\rho_h^{(b,n)}(f \circ g)$, $\lambda_k^{(d,q)}(f)$ and $\rho_k^{(d,q)}(f)$; for arbitrary positive ϵ and for all values of $(r_1, r_2, \dots, r_n)$ in the unit disc; we obtain the following

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \geq (\lambda_h^{(b,n)}(f \circ g) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.1)$$

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \leq (\rho_h^{(b,n)}(f \circ g) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.2)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \geq (\lambda_k^{(d,q)}(f) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.3)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \leq (\rho_k^{(d,q)}(f) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n). \quad (4.4)$$

When $(r_1, r_2, \dots, r_n) \rightarrow 1$ then

$$\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n)) \leq (\lambda_h^{(b,n)}(f \circ g) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.5)$$

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \geq (\rho_h^{(b,n)}(f \circ g) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.6)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \leq (\lambda_k^{(d,q)}(f) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.7)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \geq (\rho_k^{(d,q)}(f) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.8)$$

Now, from (4.1) and (4.4), and for all values of $(r_1, r_2, \dots, r_n)$ in the unit disc, we obtain the following inequality-

$$\frac{\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{(\lambda_h^{(b,n)}(f \circ g) - \epsilon)}{(\rho_k^{(d,q)}(f) + \epsilon)}$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (4.9)$$

Similarly, from (4.3) and (4.5), we get

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.10)$$

From (4.1) and (4.7), we derive

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.11)$$

From (4.2) and (4.3), we obtain

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.12)$$

From (4.2) and (4.8), we establish

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}, \quad (4.13)$$

While from (4.4) and (4.6), we derive the following

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (4.14)$$

Thus, the first part of the theorem follows from (4.9), (4.10), (4.11), (4.12), (4.13) and (4.14).

Similarly, we can prove the second part of the theorem. $\square$

Remark 4.1. If we replace the conditions $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$, $q > m, a = c = p$ and $\lambda^{(p,q)}(f) > 0$ by $(q = m = c, a = p)$ or $(q < m = c, a = p + m - q)$, $q > m = c, a = p$ and $\lambda^{(m,n)}(g, \phi) > 0$ respectively and other conditions remain same, then first part of theorem still valid for $\rho_k^{(d,n)}(g)$, $\lambda_k^{(d,n)}(g)$ and $\log^{[d]} M_k^{-1}(M_g(r_1, r_2, \dots, r_n))$ instead of $\rho_k^{(d,q)}(f)$, $\lambda_k^{(d,q)}(f)$ and $\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))$ respectively. Similarly second part of theorem still valid for $\rho_k^{(d,n)}(f)$, $\lambda_k^{(d,n)}(g)$ and $\log^{[d]} M_k^{-1}(M_g(r_1, r_2, \dots, r_n))$ instead of $\rho_k^{(d,q)}(f)$, $\lambda_k^{(d,q)}(f)$ and $\log^{[d]} M_k^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))$ respectively.

Theorem 4.2. Let f, g, h and k be the analytic functions of several complex variables in the unit disc D such that the index pairs of f, g, h and k are (p, q) , (m, n) , (a, b) and (c, d) respectively and $\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[n-1]}(r_1, r_2, \dots, r_n)}{\log^{[n-1]}(r_1 r_2 \dots r_n)} = 1$.

(i) If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda^{(p,q)}(f) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \min \left\{ \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b, q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \max \left\{ \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b, q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \end{aligned}$$

and

$$\begin{aligned}
(ii) \text{ If } q > m, a = c = p, 0 < \lambda_h^{(p,q+n-m)}(f \circ g) \leq \rho_h^{(b,q+n-m)}(f \circ g) < \infty, \\
0 < \lambda_k^{(d,q)}(f) \leq \rho_k^{(d,q)}(f) < \infty \text{ and } \rho_h^{(b,q+n-m)}(f \circ g) = \rho_k^{(d,q)}(f), \text{ then} \\
\frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} \log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\end{aligned}$$

Proof. The proof can be followed as similar to Theorem 4.1. $\square$

Remark 4.2. The Theorem 4.2 will also hold if we replace $\lambda_h^{(b,n)}(f \circ g)$, $\rho_h^{(b,n)}(f \circ g)$, $\lambda_k^{(d,q)}(f)$, $\rho_k^{(d,q)}(f)$, $\lambda_h^{(b,q+n-m)}(f \circ g)$ and $\rho_h^{(b,q+n-m)}(f \circ g)$ by $\tau_h^{(b,n)}(f \circ g)$, $\bar{\tau}_h^{(b,n)}(f \circ g)$, $\tau_k^{(d,q)}(f)$, $\bar{\tau}_k^{(d,q)}(f)$, $\tau_h^{(b,q+n-m)}(f \circ g)$ and $\bar{\tau}_h^{(b,q+n-m)}(f \circ g)$ respectively.

5 Conclusion

In this paper, we established inequalities between the relative order and the relative lower order of composite analytic functions of several complex variables in the unit polydisc.

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**A GRAPH-THEORETIC APPROACH TO CHATURANGA TURANGA BANDHĀM IN
ANCIENT INDIAN KĀVYĀ**

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Abstract

Ancient Indian scriptures are a vast ocean of knowledge, embedding profound insights within verses, shlokas, and sutras. These texts often connect deeply with disciplines such as mathematics, medicine, astronomy, astrology, cosmology, and philosophy. This paper aims to explore a specific verse that discloses the existence of graph-theoretic concepts in ancient Indian text, dating back to the 9th century A.D. Through an analytical study of a shloka arranged in the poetic structure of Kāvya, which stands out for its intellectual and visual appeal, we uncover a representation of the Knights Tour on a chessboard traversing each cell exactly once without repetition. This poetic construct mirrors the properties of a Hamiltonian path and introduces the foundational idea of what is now formalized as a Hamiltonian graph, thus showcasing a remarkable early instance of graph-theoretic thought in ancient Indian tradition.

2020 Mathematical Sciences Classification: 05C45 (Primary), 05C38, 68R10, 01A32 (Secondary)

Keywords and Phrases: : Kāvya, Chitrakāvya, Chaturanga, Turanga, Shlokas, Hamiltonian Path, Bipartite graph and Regular graph.

1 Introduction:

Chaturanga Turanga (Knight) Bandhām in Ancient India is a smart solution to a mathematical puzzle, which solves the knight tour problem. The knight tour problem states that the chess knight travels each and every cell exactly once, which was once deeply rooted in our ancient Indian work. Knight tour problem is beautifully explained in our ancient Indian text with the help of poem or Kāvya around 9th century, and as a mathematical problem it was first noticed by Leonhard Euler in 18th Century [7]. This paper provides the exploration of the knight tour problem in our Indian texts along with the graphical theoretical approach.

2 Meaning of Chaturanga Turanga Bandhām:

Chaturanga means chess board & Turanga means horse or knight which is a coin of a chess game. Chaturanga Turanga Bandhām, a technique which cleverly describe the movement of the Turanga in chaturanga to solve knight tour problem.

3 Origin of the Knight Tour Problem:

Early known mathematician to work on Knight problem was Leonhard Euler [7, 1759], he attempted to find a knight tour that will allow knight to visit each grid once and come back to the origin. Euler was not successful in getting these results but it became a base for the mathematicians who came after him like De Moivre [6], Vandermonde [13, 1774], H. C. Von Warnsdorff [1, 1823], Pratt [11, 1825], Legendre [8, 1830] and De Lavernade [5, 1839]. Warnsdorff's method [1] was to place the knight on the first cell of an empty 8x8 board and move as per knights L steps rule, ensuring that each cell is visited exactly once.

0	59	38	33	30	17	8	63
37	34	31	60	9	62	29	16
58	1	36	39	32	27	18	7
35	48	41	26	61	10	15	28
42	57	2	49	40	23	6	19
47	50	45	54	25	20	11	14
56	43	52	3	22	13	24	5
51	46	55	44	53	4	21	12

Figure 3.1: Warnsdorffs Knight tour Problem.

According to Warnsdorffs Rule, Knight follows the following steps:

- The Knight can start from any initial cell; in the above figure, the starting cell is marked as 0.
- The first L step of knight leads to the cell marked 1, the second step to 2, and so on.
- For each step, the Knight must move L steps to an unvisited cell/grid, selecting the cell with the minimal degree.
- The process will be followed until the Knight has visited all the cell exactly once.

In 20th Century the research work connected to Knight tour took a new direction, problem was extended to square and rectangle of different sizes and dimension to get different possible solution. The knight tour problem was treated as an instance for Hamiltonian path problem in graph theory.

4 Poetry meets Mathematics- Knight journey in ancient Indian texts:

Sanskrit is an old classical language of Hindus, and lot of our ancient work is written in Sanskrit script. Sanskrit poetry or Kāvyaṃ [9] is rich for its uniqueness and diversity which flows from Mahakāvyaṃ (epic Kavyas), devotional kāvyāṃ and chitra kāvyāṃ (figure or image poetry).

Before stating the knight tour in ancient India, it is worth going through the background of kāvyāṃ. Let us provide with the brief introduction about sabda alankara and chitrakāvyaṃ. Sabda alankara namely word-based ornamentation consists of well recognized alliteration and rhyme, which in Sanskrit resembles to anuprasa and yamaka. Anuprasa (alliteration) corresponds consonant which is repeatedly used in line of verse, and where as yamaka (rhyme) is associated with end of the line in a verse.

Chitrakāvyaṃ, is categorized into śabda-chitra (sound-based patterns), artha-citra (meaning-based patterns), and ubhaya-citra (mixture of both). It includes subtypes like bandha -geometric or object-based patterns, gati-movement-based patterns such as gomūtrika (cows zigzag path), rathapada (chariot movement), and turagapada (knights movement). This work sometimes represented as Adhama kāvyāṃ inferiority poetry, which targets the emotion and used to demonstrate the poetic and mathematical powers together.

5 Rudrata 4 x 8 knight tour:

In India the solution for the knight tour problem was known as early as 9th century AD which were a pair of verses in the form of kāvyām. The earliest example of this can be seen in Sanskrit treatise Kāvyaṭaṅkāra by Rudrata (9th Century). This type of kāvyā in sanskrit comes under chitrakāvyaṃ which means figure or image poetry. Rudrata Kāvyaṭaṅkāra, a sanskrit kāvyām works gives the pattern of knight steps on the chess board. It is a four-line verse read from left to right or by the path followed by the knight, where each syllable represents the single square of grid/cell [4]. Rudrata Kāvyaṃ verses are as follows:

अथ तुरगपदपाठः—

सेना लीलीलीना नाली लीनाना नानालीलीली ।

नालीनालीले नालीना लीलीली नानानानाली ॥१५॥

→ Sutra 1

The following translation is based on Choudharys Kāvyaṭaṅkāra [4], verse 15: I, who is a truthful well-read man, a group leader and helpful to my servants, praise the army whose leader a man who praises playful persons.

Rudratas verse is a kāvyām in which the syllables of the verses are first arranged in an 8x4 grid from left to right. Remarkably, the same syllables of the verse can be read again from the beginning by following the knights L moves, forming a Hamiltonian path. According to the syllabic arrangement of the verse, the knights follows the following path stated in the Figure 5.1:

१ से	३० ना	६ ली	२० ली	३ ली	२४ ना	११ ना	२६ ली
१६ ली	१९ ना	२ ना	२९ ना	१० ना	२७ ली	४ ली	२३ ली
३१ न	८ ली	१७ ना	१४ ली	२१ ले	६ ना	२५ ली	१२ ना
१८ ली	१५ ली	३२ ली	७ ना	२८ ना	१३ ना	२२ ना	५ ली

Figure 5.1: Representation of Turagapadabandha on 8x4 grids.

According to Figure 5.1, Firstly the syllable of the verse is arranged in 8x4 grid form (i.e. 8 columns and 4 rows), Now by taking the syllable of verses from left to right exactly the way it appears in an 8x4 grid, Turaga starts from से which is marked as 1, from here the first L step of knight moves to ना which is marked as 2. Following the syllable of the verses, Knight move from ना to ली which is marked as 3, then from ली to ली, marked as 4 and so on, the Turga follows the syllable of sutra 1 verses and moves on without repeating any of the cell or revisiting any of the cell, this uninterrupted movement of knight forms the bases for Hamiltonian path, which is a fundamental concept of graph theory. Thus Rudrata Kāvyaṭaṅkāra verses not only gives the literary classiness but also states the movement of Knight, landing to an example of Hamiltonian path.

6 Desikas 8x4 knight tour problem:

In 13th century Swami Vedanta Desika [2], who was an Indian scholar, expertise in poetic work of philosophical & religious framed a pair of verses for 8x4 knight tour problem -Chaturanga Turanga Bandhm, which is a poetic technic used in the Paduka Sahasram, meaning thousand verses on the sandals God Ranganatha, it is a Sanskrit 1008 verse kāvyām. Swami Vedanta Desika composed works in Sanskrit, Tamil and Manipravalam (a Sanskritised form of Tamil). He is regarded as a great Āchārya of his era and one of a true polymath, excelling in numerous skills including mathematics, civil, law, Medicine, kāvyām, philosophy, teaching, sports, acting, magic, linguistics, cryptography and many more.

Unlike Rudrata single verse solution, Desikas pair has two specific 32 syllable verses from Paduka sahasram i.e. 929 and 930, which were emphasized to the solution of knight problem about 500 years before Euler. The verses were meaningful, aligned with the approach of traditional chitrakāvya style and presented in 8x4 grid form. The movement or path of the knight is reflected in the way the verses are structured and read. The verse 929 talks about the filling up of half chess board i.e. 8x4 grids where the other one specifies the movement of the knight on the chess board. Following verses highlights, the beauty of literature, Mathematics, and the game of chess. The verses listed below carries a poetic or kvym feel, intertwined with the Mathematical classiness of Knights path.

स्थिरागसां सदाराध्या विहताकततामता ।
सत्पादुके सरासा मा रङ्गराजपदं नय ॥²¹

→ Sutra 2 (929 verse)

स्थिता समयराजत्पागतरा मादके गवि ।
दुरंहसां सन्नतादा साध्यातापकरासरा ॥

→ Sutra 3 (930 verse)

The following is a translation from Swami Vedanta Desika [2]: Sutra 2 (929 verse)- Oh sacred sandals of the Brahman, you are worshipped by those who have committed unforgiveable sins; you eliminate all that is sorrowful and unpleasant; you lead me to jingling/joyfully to the feet of Lord Rangaraja.

Sutra 3 (930 verse)- O sandals, you protect those who shine by their right attitude, whose place is the center of the sun blissful rays, you abolish the intoxication of the distressed; your radiance brings peace to those who take refuge in you, which move everywhere, may! those sandals which are golden and radiating lead me to the feet of Lord Rangaraja.

Figure stated below gives the 8x4 grid formation along with the knight path, formed with the help of two verses 929 and 930, which have to be read together as one set,

स्थि	रा	ग	सां	स	दा	रा	ध्या
1	30	9	20	3	24	11	26
वि	ह	ता	क	त	ता	म	ता
16	19	2	29	10	27	4	23
स	त्पा	दु	के	स	रा	सा	मा
31	8	17	14	21	6	25	12
रं	ग	रा	ज	प	द	न्न	य
18	15	32	7	28	13	22	5

Figure 6.1: Desikas Knight Tour Problem.

Working of Desikas verses: Desikas pair consists of two specific 32 syllable verses-929 and 930, Using verse 929, he arranges the syllables in an 8 x 8 grid from left to right. Then, with the assistance of verse 930, the path of Turanga (knight) is stated. The knight follows the sequence of syllables from verse 930, moving through the grid without repeating any of the cell and covering each and every cell once. This sequence forms the basis for a Hamiltonian path and a bipartite graph. Bhoja is another important poet of that century, who has given knights tour problem solution in his book on poetics, Sarasvatīkahābharaam. He described the chess puzzle as a mathematical problem and a literary challenge [9].

7 Ratnakara 4 x 8 Knight tour:

Rajanaka Ratnakara [9], a 9th-century Kashmiri poet, is best known for his epic Sanskrit kāvyām Hārāvijāyām meaning victory of Hara (God shiva) over the demon Andhaka. This kāvyām consists of 50 sargas (cantos or chapters) that gives in the details of the heroic deeds of Lord Shiva. His work is famous for its complexity, rich metaphors and the vivid description of Shivas celestial city and his powerful Tāṇḍava dance.

Rudrata, Bhoja and Vedanta Desikas approach was different from Ratnakaras approach, He employed a different knight tour path that was closed, meaning the knight returns to its initial point, forming a Hamiltonian cycle. He used 64 syllables for this structure. The verses of Ratnakara from his Hārāvijāyām kāvyām are as follows:

[तामभ्ययुङ्क्त यासङ्क्ष्यसङ्ख्यावद्धिर्वृता भटैः ।

सूचिरोम्णः चमूनदिः सूचितप्रलयागमा ॥ १४४ ॥

भासेद्धाहाततनना सादिमायादसेविता ।

→ Sutra 4

सासहस्ततयाशान्तसाराहितसमिद्युता ॥ १४५ ॥

सहसादितसेनाशा तयासेसाहितद्युता ।

याता वित्रस्तरासाद्धा सभामातदनन्तसित् ॥ १४६ ॥]

→ Sutra 5

The following translation is based on Rajanaka Ratnakaras Sanskrit kāvyām Hārāvijāyām (verse 144,145 & 146), translated by G. S. S Murthy [9]:

Fearless roar of Suchiromanhs Army, which was served by horsemen and magicians, having swords in hand, and comprised of soldiers protected by the large fearless army, whose sounds indicated the arrival of total destruction and which attacked Kali, and the great force was defeated by her.

The three verses together form a solo sentence. The second verse (145) is the main one, which arranges the syllables in a 4 x 8 grid, whereas the third verse (146) provides the knight tour path.

	A	B	C	D	E	F	G	H
1	भा26	से11	द्वा24	हा5	त20	त9	न30	ना7
2	सा23	दि4	मा27	या10	द29	से6	वि19	ता16
3	सा12	स25	ह2	स्त21	त14	या17	शा8	न्त31
4	सा3	रा22	हि13	त28	स1	मि32	द्यु15	ता18

Figure 7.1: Ratnakara 4 x 8 Knight Tour Problem.

8 Graphical Framework for the Knights Tour Problem:

Graph Construction for the Knights Tour Problem is as follows:

- The knight placed on any cell of an empty board, is moved on the basis of knight rule.
- For building knight tour graph take each cell of chess board as node or vertices.
- Each L (legal) steps by the knight can be marked as edge in the graph.
- L steps moves knight one square along one axis and two steps along the other axis.
- For $n \times m$ knight tour graph the total number of vertices is $n \times m$ only.
- For $n \times n$ knight tour graph the total number of vertices is $n \times n = n^2$ and the total number of edges is $4(n-2)(n-1)$.
- Initial condition for knight tour path is $n \geq 5$ Knight follows L move and steps on each cell only once and cover each and every nodes presenting Hamiltonian path.
- Knight moves alternate nodes between white and black, example if the first knight step is in black cell the second step will be in white, third jump will be in black and so on, hence knight tour graph is bipartite graph along with Hamiltonian path. It is k -partite graph with value of $k = 2$.
- Knight graph follows two colorable graph and its chromatic number is 2.
- Degree of each vertex in knight graph is 2, therefore it is also known as regular bipartite graph, as degree is same for each vertex and in general every bipartite graph is perfect.
- It is a perfect graph as for every induced subgraph, chromatic number (Minimum number of colors needed) is equal to clique number (Maximum number of mutually connected vertices) i.e. $\chi(G) = \omega(G) = 2$
- Knight path can be represented as tree (Tree is a special type of graph which is undirected, connected and acyclic, and there is only one path between two nodes), with each node as position on chess board and the movement is represented by edges. The root of the tree is the starting position which follows the path and reaches the leaf node.

9 Conclusion

The Knights Tour problem, though widely associated with mathematics and chess, finds its origins in ancient intellectual traditions, including early Indian and Arabic mathematical texts. These historical verses reveal not just a fascination with the knight's steps movement, but also a deeper understanding of structure, logic, and strategy long before modern graph theory framed such problems. By expressing the Knights Tour as a graph, where each square of the chessboard is a vertex or node and each valid knight move is an edge. We bridge ancient curio with modern mathematical frameworks. This transformation helps us to apply powerful graph theory concepts such as Hamiltonian paths, Bipartite graph, tree, Perfect graph and algorithmic traversal strategies, presenting both analytical depth and computational possibilities.

Ultimately, the Knights Tour stands as an endless example of how ancient mathematical thinking in the form of shlokas and verses, which when interpreted through modern graphical lenses, continues to inspire exploration, innovation and bridging ancient Indian scholarship with modern analysis. This exploration not

only celebrates India's intellectual heritage but also underlines the cross-cultural appreciation of mathematical heritage.

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APPLICATION OF ARTIFICIAL INTELLIGENCE ALGORITHMS FOR ELECTRIC VEHICLE-CONNECTED POWER SYSTEMS TO CONTROL ELECTRICAL LOAD FREQUENCY

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Abstract

There are a few challenges confronting electric vehicle producers. This study looks at the influence of electrically powered vehicles on load frequency deviation. However, the load fluctuates throughout time and cannot be constant. Multiple remedies have been proposed in academic literature to tackle the deficiencies of traditional controllers. This study presents an innovative method for creating a multi-load frequency controller for a single-zone power system by utilizing various artificial intelligence (AI) optimization methods such as fuzzy logic and ANN tuners. Research on controllers has demonstrated enhancements in frequency offset signal, peak crossover, and frequency output signal resolution. The functionality of the suggested system is confirmed through the utilization of *MATLAB/SIMULINK* tools.

2020 Mathematical Sciences Classification: 93C05, 34A30

Keywords and Phrases: Artificial Intelligence (AI) , Electric Vehicles (EV) , Fuzzy Logic Control (FLC) , Load Frequency Control (LFC).

1 Introduction

Power system stability requires a balance between total output, total load demand and system losses. Due to the change in load demand, the balance of real and reactive power is disturbed. This results in system frequency and tie-line power deviations from the designed value. Excessive fluctuations in system frequency can cause system failure. For the efficient operation of the electrical system, the frequency must remain constant [2, 9]. Several problems related to frequency fluctuations can directly affect the reliability, efficiency and effectiveness of an electrical system. Critical frequency deviations can impair load performance, harm equipment, overburden transmission lines, and disrupt system protection mechanisms [3,10]. These frequent deviations at a high rate can in the long run result in the collapse of the system [5, 15]. Changes in frequency can adversely affect the operation and speed control of both asynchronous and synchronous motors [1, 2].

2 Need for Load Frequency Control

Load frequency control (LFC) is critical in a large-scale power system with interconnected regions to keep the system frequency and interconnection power close to expected levels Load Frequency Control [2,9]. A well-designed power system should be able to handle changes in demand and disruptions in the system [9, 11]. It also needs to produce the right levels of high-quality energy while maintaining safe voltage and frequency [2, 5]. Load frequency control is an important control element in the operation of electrical systems [15]. Load frequency control (LFC) has long been used in electrical systems as part of the AGC (Automatic

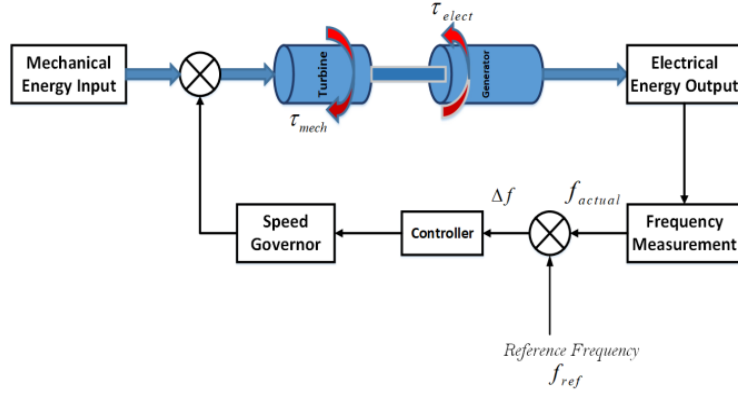


Figure 2.1: The AGC system

Generation Control) system [2, 10]. Maintaining the system frequency at its nominal value of 50 Hz is one of AGC's goals [2, 9].

The electrical system consists of a main controller, turbine and generator with generator input [1, 3]. The electrical system also includes the generator phase load. The purpose of the load frequency controller is to simultaneously perform frequency control and actual current change through the transmission line [6, 10]. The power frequency is detected by the frequency sensor [12]. The alteration in rotor angle δ can be used to measure the variations in line frequency and real power [5]. The load frequency controller enhances and transforms the error signal, specifically [4]. The change in the current power control signal ΔP_{ci} is transmitted to the power source via the controller (valve mechanism control).

3 Methodology

A complete block diagram representation involving the turbine, generator, governor, and load is typically modeled by individual components [1, 3, 5]. Because of the system's actual performance as a result of a step change in load, the controller does not take any action in this instance [10]. In this case, no controller action is applied because of actual system performance due to a step change in load [2, 10]. The dynamic behavior for the nominal system, as reflected in the change in frequency, consists of the hydraulic amplifier, turbine, and generator model as shown in Fig. 2.1.

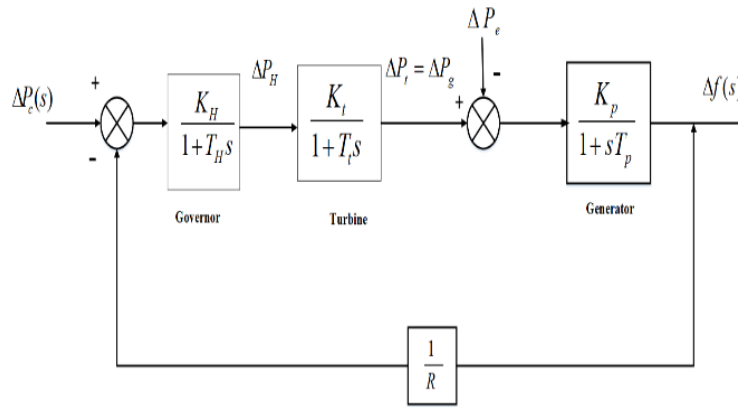


Figure 3.1: LFC Block Diagram

An alternative way to represent Fig. 3.1 is to use the matrix equation in state space form.

$$\dot{x} = Ax + Bu + F\rho. \quad (3.1)$$

Here,

$$A = \begin{bmatrix} -\frac{1}{T_p} & \frac{K_p}{T_p} & 0 \\ 0 & -\frac{1}{T_t} & \frac{1}{T_t} \\ \frac{1}{RT_H} & 0 & -\frac{1}{T_H} \end{bmatrix}, \quad (3.2)$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_H} \end{bmatrix}, \quad (3.3)$$

$$F = \begin{bmatrix} -\frac{K_p}{T_p} \\ 0 \\ 0 \end{bmatrix}, \quad (3.4)$$

$$\rho = [\Delta P_E]. \quad (3.5)$$

Here, x^T = State vectors, $[\Delta f \quad \Delta P_T \quad \Delta P_H]$,
 u = When not under control, the control vector equals zero.

4 Introduction to Electric Vehicles

Mechanized vehicles utilize a significant portion of the oil supply, and their operation leads to considerable air pollution, which is harmful to the environment [9, 11]. Technologies such as battery-powered Plug-in Electric Vehicles (*PEV*) and Plug-in Hybrid Electric Vehicles (*PHEV*) are gaining momentum to combat greenhouse gas emissions and air pollution, especially in regions where the vehicle industry is heavily dependent on petroleum-based fuels [9, 14]. Device regulators powered by Electric Vehicle Supply Equipment (*EVSE*) are essential for linking *EVs* to the electrical grid [7]. The *AC* to *DC* converter, with single or three-stage connectors, is a key component of these regulators. Recently, there has been a significant number of new advancements in *EV* charging technologies being developed and supported worldwide [8]. Furthermore, if numerous electric vehicles charge simultaneously from the grid, it can lead to increased peak load demands and system harmonics, potentially causing operational issues for the power system [7, 14]. It is important to analyze how multiple *EV* charging stations impact the electric power distribution system *EPDS* [9]. Because of the high demand for *EV* charging, this study focuses on the main challenges that *EPDS* operators are likely to face [9, 11].

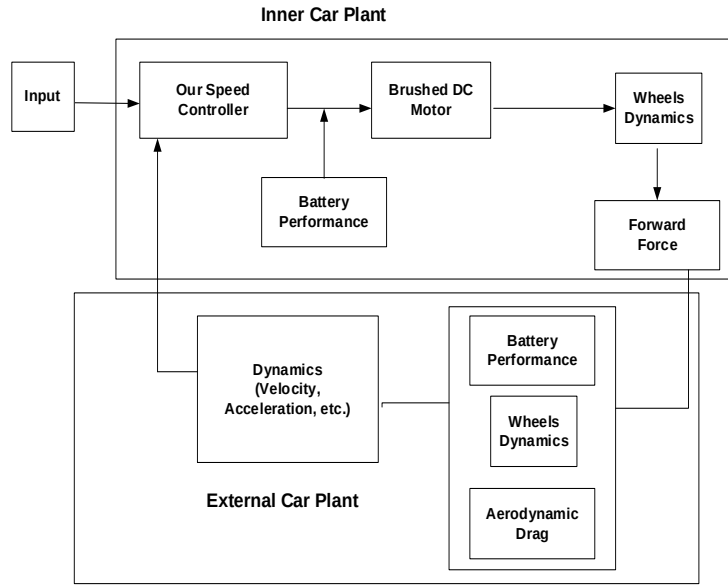


Figure 4.1: Model Diagram for EV

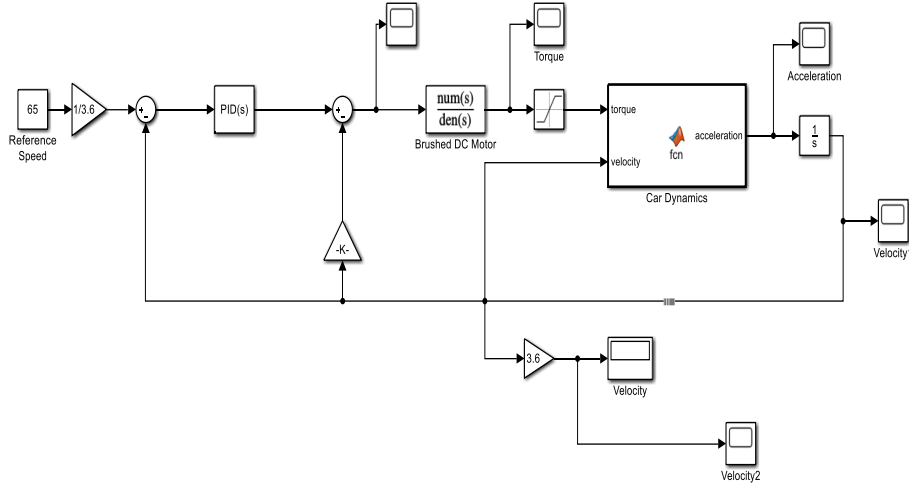


Figure 4.2: Simulink Model for EV

5 Mathematical Modeling of EV

With K_E the back *EMF* constant and (t) the rotational speed of the motor in rad/s. Hence, we can write:

$$V(t) = I(t) * R + L * \frac{dI}{dt} + K_E * \omega(t) . \quad (5.1)$$

Transforming this equation to the Laplace domain, we get:

$$V(s) = R * I(s) + sL * I(s) + K_E * \omega(s) . \quad (5.2)$$

We can also apply this to the torque equation to obtain:

$$T(s) = K_T * I(s) . \quad (5.3)$$

Hence, we can write:

$$I(s) = \frac{V(s) - K_E * \omega(s)}{sL + R} . \quad (5.4)$$

Replacing to get Torque as a function of Voltage Input and rotational velocity:

$$T(s) = K_T * \frac{V(s) - K_E * \omega(s)}{sL + R} . \quad (5.5)$$

The control methodology proposed in this work is the *ANFIS*, which stands for Adaptive Neuro-Fuzzy Inference System, combines neural networks and fuzzy logic to create artificial intelligence (AI) [4, 11]. This type of soft computing method is capable of predicting or determining outcomes for complex nonlinear systems using input data [4, 6]. *ANFIS* proves particularly useful in situations involving complex systems, uncertainty, dynamic behavior, and nonlinear relationships, where traditional rule-based or statistical models may be inadequate [11, 14].

The parameters that can be adjusted to modify the shape of the membership function generally determine the structure and behavior of the membership function in *ANFIS* models [3, 4].

6 Mathematical Modeling of ANFIS

While there are several algorithms to train the *ANN/ANFIS* model [13], one of the fastest and most efficient algorithms is the Levenberg Marquardt (*LM*) algorithm which is a modified version of the steepest gradient descent. Taking a look at the nature of the steepest descent algorithm, it is concluded that it is 1st order derivative of the minimization function which makes it an asymptotic convergence[11]. Considering the gradient of the error concerning weights for loss minimization, we obtain the derivative gradient as:

$$g = \frac{\partial E(x, w)}{\partial x} = \left[\frac{\partial E}{\partial w_1} \frac{\partial E}{\partial w_2} \quad \cdots \quad \frac{\partial E}{\partial w_N} \right]^T . \quad (6.1)$$

Considering the loss/cost function as the mean squared error, the weight update rule for the (*LM*) algorithm is:

$$w_{k+1} = w_k - [J_k J_k^T + \mu I]^{-1} J_k^T e_k . \quad (6.2)$$

Here, k is the iteration number,

w_{k+1} is the weight of the next iteration,

w_k is the weight of the present iteration,

J_k is the Jacobian Matrix and is given by the terms $J_k = \frac{\partial^2 e}{\partial w^2}$ i.e. the second-order rate of change of errors concerning weights,

J_k^T is Transpose of Jacobian Matrix,

e_k is an error of the Present Iteration,

μ is step size i.e. amount by which weight changes in each iteration,

I is an identity matrix, with all diagonal elements equal to 1 and other elements 0.

The training is truncated as the cost or loss function is stable or the maximum iterations are over. The cost or loss function is defined as:

$$MSE = \frac{1}{N} \sum_{t=1}^N e_t^2 \quad H < return > . \quad (6.3)$$

The mean absolute percentage error can also be computed as:

$$MAPE = \frac{100}{N} \sum_{t=1}^N \frac{|V_t - \hat{V}_t|}{V_t} . \quad (6.4)$$

Here,

N is the number of predicted samples,

V is the predicted value,

$\hat{V}_t$ is the actual value,

e is the error value.

The complete implementation and validation of the results can be performed by sequentially implementing the steps in the flowchart of the proposed system:

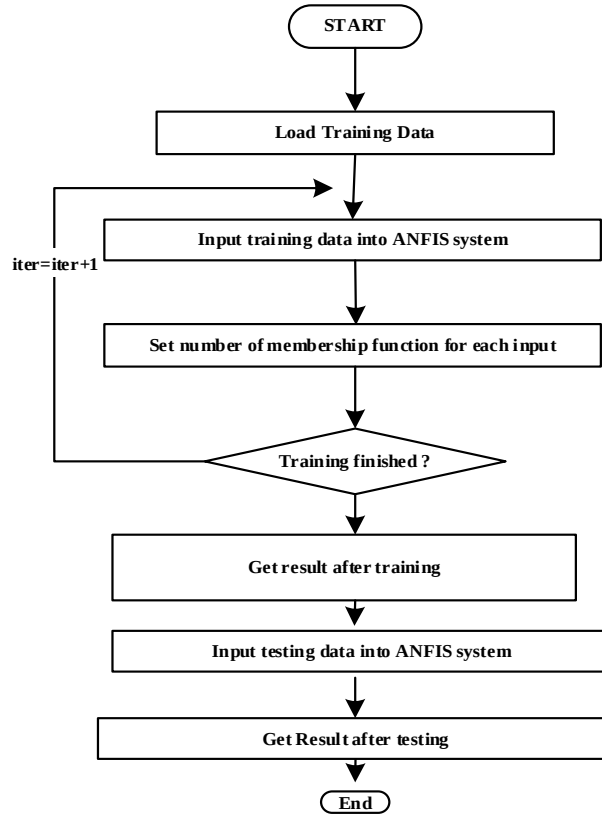


Figure 6.1: ANFIS Flowchart

7 Results and Discussions

The system has been designed on *MATLAB/SIMULINK*. Figure 7.3 displays the output signal of Change in frequency response using the *ANFIS* controller.

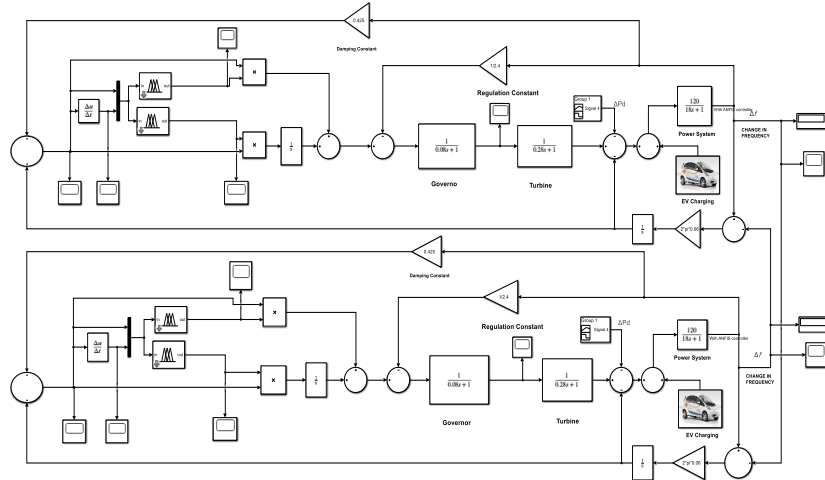


Figure 7.1: *ANFIS Controlled LFC Model*

Table 7.1: *Summary of Results*

Parameter	ANN	ANFIS
Rise Time	1.5593×10^{-5} sec	1.3832×10^{-5} sec
Settling Time	11.8526 sec	11.0145 sec
Settling Min	-0.5747 Hz	-0.3966 Hz
Settling Max	1.8975 Hz	0.6361 Hz
Overshoot	7.2653×10^6	6.6417×10^5
Undershoot	2.2007×10^6	3.2790×10^6
Peak	1.8975 Hz	1.9579 Hz
Peak Time	5.5281 sec	5.4959 sec

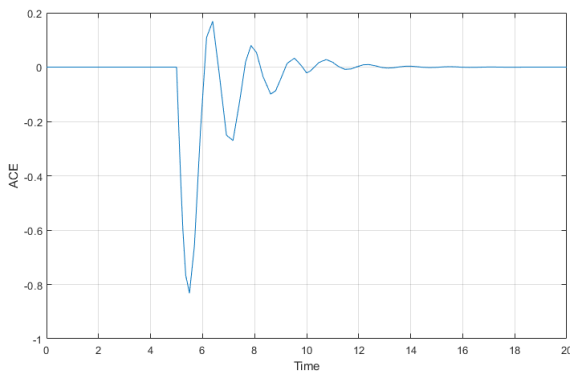


Figure 7.2: *ACE for ANFIS Controller*

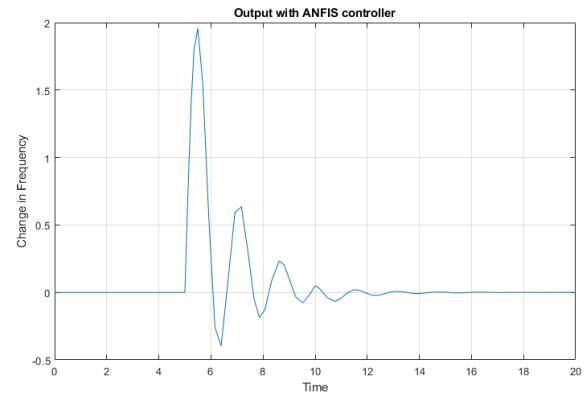


Figure 7.3: *Change in Frequency for ANFIS Controller*

Table 7.1 shows the value of *LFC* response parameters for the *ANFIS* controller. Based on these parameters comparison between the two controllers will be done to check their performance.

8 Conclusion

Load frequency control, or *LFC*, is utilized to maintain and adjust the frequency signal of the electric power output within a specific region for electric vehicles when there are fluctuations in system loads. This article discusses the impact of electric vehicles on load frequency deviation. This project demonstrates how to design a controller that can sustain optimal performance in a two-area power system even when the system's input parameters are changed. Two approaches to load frequency control were examined while taking an isolated power system into account.

The controller's performance is confirmed using *MATLAB/SIMULINK* tools. In comparison, *ANFIS* exhibited the smallest percentage overshoot and shortest settling time.

In the future scope, a signal processing tool can be utilized to check their effect on results.

Some of the Hybrid techniques like Wavelet Neural Network etc. can be checked to see improvement in results.

The system will appear more realistic and nonlinear if generation control and governor deadband are added.

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ANALYZING SOCIAL NETWORKS STRUCTURALLY USING PETRI NETS

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Abstract

Social networks are commonly expressed, modelled and analyzed using discrete mathematics as graph theory. This paper proposes a directed bipartite based model in the framework of Petri-net for behavioral and structural analysis of social works using places, transitions, directed arcs, and marking vectors. The approach establishes a formal relationship between graph-based social network structures and Petri-nets, enabling the analysis of connectivity, surfaces, P and T- invariants, marked graph, liveness, reachability, and structural properties. Illustrative examples demonstrate the applicability of Petri nets for representing and analyzing social network structures.

2020 Mathematical Sciences Classification: 05C82, 68R10, 68M11, 68Q85

Keywords and Phrases: Social networks, Graph theory, Petri nets, Reachability, Invariants, Firing Rule, Marking Vector.

1 Introduction

Social networks have been studied for a long-time using graph theory and it has been proven a powerful way to analyze such networks in different ways. Even we use the term 'graph' and 'network' interchangeably very oftenly. A graph is nothing but a set of vertices (nodes) linked with a set of arcs (edges). Nodes in a graph depicts the entities that we want to model and edges in the graph represent the connection between the nodes in our network. For instance, suppose we want to show the relation of 'friendship' among a group of people. The graph constructed is such that each node in the graph will depict the persons among which we want to show the relationship. And links are shown between only the persons who are friends with each other by the undirected arcs. Petri nets, A formalized modeling language which goes back around 1960s, has been used to describe and model discrete event dynamic systems (*DEDS*). These are topologically bipartite multi-digraphs, where we study and use some well-defined couples of rules with the digraph and in turn helps in analyzing *DEDS*. Bipartite directed graph means, we have two nonempty disjoint sets of nodes denoting using oval/circles (called places) and rectangles/bar (called transitions) and flow relation is denoted by directed arc in the graph. The arcs must be used to connect oval/circles and bar/rectangles but in such a way that no direct connection in between place to place or transition to transition. An edge which is directed from circle/place to rectangle/transition is the pre place of that bar/rectangle and vice-versa.

Marking/Token vector in a Petri Net: We assign dots/ token or write number to the places, called as marking vector. These tokens/markings are further used for change of state/execution of Petri nets. Theoretically, marking or n -vector = $(M_1, M_2, M_3, \dots, M_n)$, where n is the cardinality of number of circles/places in the Petri net structure and each $M_i \in \{0, 1, 2, 3, \dots, n\}$. In many cases, we assign some non-negative integer to the arcs in a Petri net, which we call as weight of the arc. and if weight is not assigned, we take it 1 by default.

Formally we define Petri net as a 5-tuple $PN = (\text{Place Set } P, \text{Transition Set } T, \text{Flow Relation } F, \text{Weight}$

Function W , Initial Marking M_0) (See the Fig 1.1), where:

$P = \{p_1, p_2, p_3, \dots, p_m\}$: A nonempty set of places,

$T = \{t_1, t_2, t_3, \dots, t_n\}$: A nonempty of transitions,

$F \subseteq (P \times T) \cup (T \times P)$: Flow-relation/arc set,

$W = F \rightarrow \{1, 2, 3, \dots\}$: Weight function,

$M_0 = P \rightarrow \{0, 1, 2, 3, \dots\}$: Marking Function called the initial marking vector.

Here $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

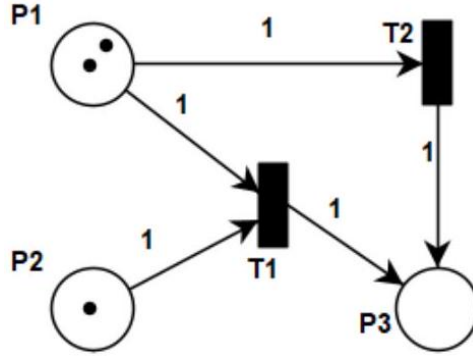


Figure 1.1: An illustration for a marked Petri net with initial marking vector $(2,1,0)$.

2 Petri net analysis using reachability

Firing-rule in Petri Net: Consuming of $w(p, t)$ tokens weight from pre place of a transition and producing $w(t, p)$ tokens weight into post place of the transition, responsible for the Petri Nets's change of state, called the transition firing rule. Here, $w(p, t)$ place to transition arc weight and similarly $w(t, p)$ means transition to place arc weight. In the following illustration shown in Fig 2.1 and Fig 2.2 (Reachability tree), we consider equal arc weight $w(p, t) = w(t, p) = 1$ for all the transitions.

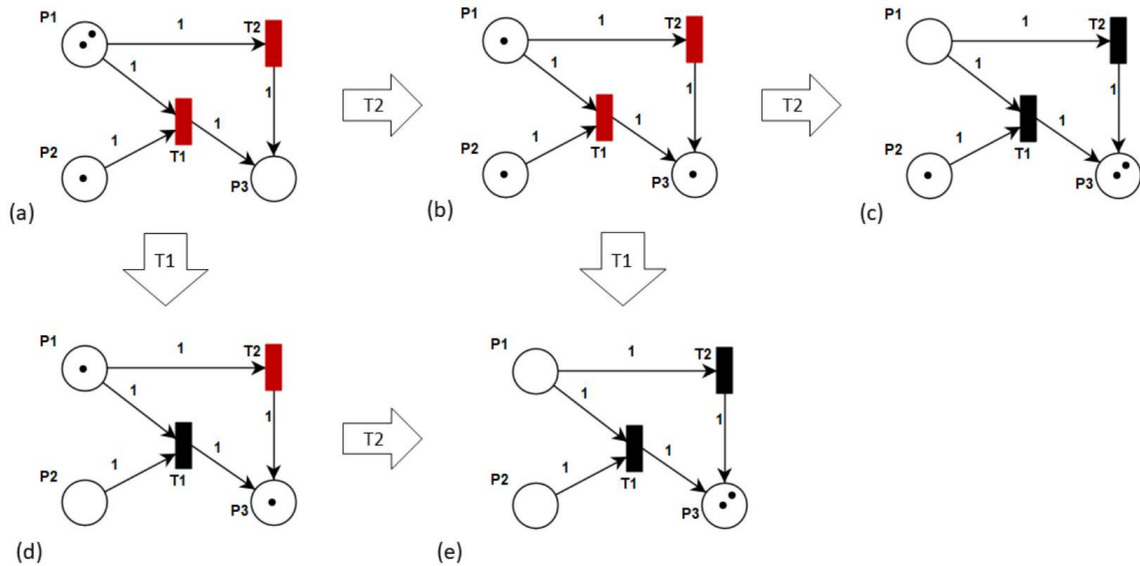


Figure 2.1: Illustration of firing rule in Petri net depicted in Fig.1.1.

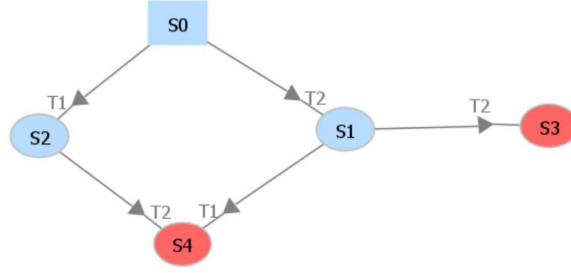


Figure 2.2: Reachability/Coverability graph of Petri net in Fig.1.1.

In the concept of social network analysis, we wish to study on how many different ways there are for a person to create connections with any other person in the community or an organization if they are unknown to each other directly using Petri nets.

3 Methodology and Construction

We have chosen a very small network, a group of seven people in an organization. We have shown the number of people in the network using a set of places in the Petri net. Now there can be multiple conditions one can think of to study the network-

- i. Initially when there is no interconnection among any person in the organization, there is one manager who is leading the organization and who recruits all the other employees into the organization. Such networks can be easily understood as similar to star networks. (Fig. 3.1.)
- ii. Other possibilities that can be studied are there can be multiple connections among employees before joining the organization. Creating and removing the links between the star networks can easily be understood for visualizing.

Studying the first case, we have a recruiter of the employees in an organization and we have depicted here the graphical model in the figure below. P0 place in the figure shows there is a recruiter of the organization.

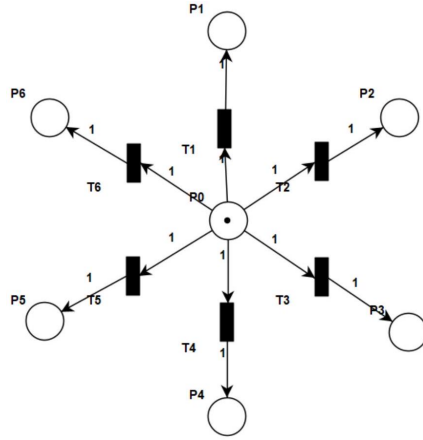


Figure 3.1: Petri net illustrating a kind of social network

This Petri net can be studied as a dual of the Safe Star Petri net as defined by Kansal *et al.* [3] in Boolean Petri nets. The social network as discussed can be studied using reachability-based analysis of this Petri net. For validation, we have shown here the generated reachability/coverability graph of the Petri net. The reachability graph shows us the connections that are there among the employees of the organization (see Fig. 3.2).

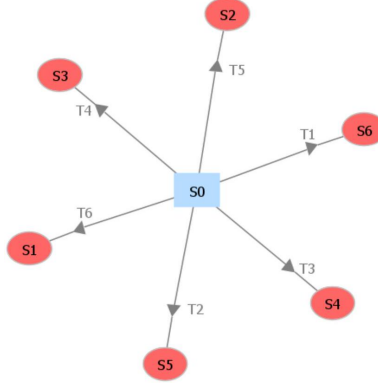


Figure 3.2: Reachability/Coverability graph of Petri net in Fig.3.1.

4 Reachability/Coverability Graph Results

Generating Reachability graph took 0.082s

Constructing it took 0.064s

Total time was 0.146s

We can clearly see from the results that the computation time is very fast. Besides computation time, there are many more results obtained with the Petri net analysis results that are shown below. Creating Incidence matrix for the state space analysis and looking for all possible firings and markings of the Petri net, we have used here the free software available for Petri nets, Platform Independent Petri Net Editor (*PIPE* v4.3.0). Besides this, we can also study the structural and behavioral properties of the Petri net. For e.g. Boundedness, Liveness, Safeness, Home State and Reversibility, etc. using the software.

Our aim was to look for possible shortest connection among the community, this is depicted using the shortest path to deadlock. The figure only depicts one path to deadlock3

5 Petri net state space analysis results

Combined incidence matrix /

	T1	T2	T3	T4	T5	T6
P0	-1	-1	-1	-1	-1	-1
P1	1	0	0	0	0	0
P2	0	1	0	0	0	0
P3	0	0	1	0	0	0
P4	0	0	0	1	0	0
P5	0	0	0	0	1	0
P6	0	0	0	0	0	1

(a)

Marking

	P0	P1	P2	P3	P4	P5	P6
Initial	0	1	0	0	0	0	0
Current	1	0	0	0	0	0	0

Enabled transitions

T1	T2	T3	T4	T5	T6
yes	yes	yes	yes	yes	yes

(b)

Petri net state space analysis results

Bounded	true
Safe	true
Deadlock	true

Shortest path to deadlock: T1

(c)

Figure 5.1: PIPE v4.3.0 analysis results of Petri net in Fig. 3.1.

There are another analysis techniques also available in the Petri net literature besides Reachability, that are Matrix and State Equations. One can study Invariant analysis results and Petri net classification results shown in the figure below (see Fig 5.1, Fig 5.2 and Fig 6.1).

Petri net classification results

State Machine	true
Marked Graph	false
Free Choice Net	true
Extended Free Choice Net	true
Simple Net	true
Extended Simple Net	true

Figure 5.2: Classification analysis results of Petri net in Fig. 3.1

6 Petri net invariant analysis results

Petri net invariant analysis results

T-Invariants

T1 T2 T3 T4 T5 T6

The net is not covered by positive T-Invariants, therefore we do not know if it is bounded and live.

P-Invariants

P0 P1 P2 P3 P4 P5 P6
1 1 1 1 1 1 1

The net is covered by positive P-Invariants, therefore it is bounded.

P-Invariant equations

$$M(P0) + M(P1) + M(P2) + M(P3) + M(P4) + M(P5) + M(P6) = 1$$

Analysis time: 0.0s

Figure 6.1: Invariant analysis results of Petri net in Fig. 3.1.

7 Conclusions and Scopes

In this article, we have shown graphical model for social relationships with the help of graph theory, Petri nets and readability. This idea may be applied to study in various biological pathways and process where there is a possibility of interconnection of nodes.

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