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**FRACTIONAL DIFFERENTIAL INEQUALITIES WITH CAPUTO-FABRIZIO
DERIVATIVE IN CAPUTO-SENSE**

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Abstract

This research work aims to provide a comprehensive analysis of the comparison principle as applied to fractional differential inequalities, utilising the Caputo-Fabrizio fractional derivative in the Caputo sense, and shedding light on its theoretical foundations and practical implications.

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1 Introduction

In the near past, fractional calculus has been able to fetch significant attention across varied scientific disciplines for its unmatched ability to model intricate phenomena. Fractional calculus, particularly fractional differential equations (*FDEs*), has gained substantial significance in physics, engineering and applied mathematics. Despite having a history spanning three hundred years, the applications of *FDEs* have seen a resurgence in the last twenty to thirty years [14, 26, 28]. While numerous analytical and numerical methods exist for solving integer-order differential equations, many times these methods often fall short when applied to their fractional counterparts. Because of this, there is a need to introduce new methods tailored to solve *FDEs*. Notably, obtaining analytical solutions for most *FDEs* is a challenging task, pushing researchers to work on various schemes.

The comparison principle is a common tool that is used to establish bounds and study other important properties of solutions to differential equations by comparing them to simpler, known solutions [21, 29]. Also, the comparison principle offers a powerful approach for understanding the behaviour of solutions to nonlinear inequalities [10, 12]. This can be extended to the domain of fractional calculus with the help of inequalities. Inequalities are also a powerful tool in mathematical analysis, providing insights into the properties of solutions to differential equations [11, 12, 16, 27]. In this research paper, we studied fractional differential inequalities. These types of inequalities are essential in the study of fractional calculus [1, 2, 3, 17, 18]. Fractional differential inequalities play a pivotal role in this area, offering a powerful framework to describe and analyze dynamic systems characterized by fractional-order derivatives. Particularly, the topic of uniqueness and existence of solution to the fractional differential system is essential in fractional calculus and the study of same can be done by using fractional differential inequalities [4, 13, 17, 18, 19].

Lakshmikantham and Vatsala [17, 18], investigated the comparison theorem for Riemann-Liouville-type fractional differential inequalities, the usefulness of which is also concretized in [10]. Regarding the Caputo fractional derivative, the comparison principle for fractional differential equations is presented by Lu and Zhu in their paper [19]. The generalized Riemann-Liouville and Caputo fractional derivatives are used to study the comparison principle for fractional differential inequalities by Al-Refai and Yuri Luchko in [3]. These results are useful in the theory of fractional calculus, as we can not directly apply them to the system having

the Caputo-fabrizio fractional derivative, we need new results which are suitable to our need. For that we reestablished the results in [3, 17, 18], using the Caputo-Fabrizio fractional derivative as it found various applications in applied mathematics, as mentioned in [9, 11, 15, 30]. The Caputo-Fabrizio fractional operator with a nonsingular kernel found many applications in the field of integral inequalities [7, 8, 23, 24, 25], due to its capability to model the mathematical problems accurately. Similar kind of attention need to be given to the fractional differential inequalities with Caputo-Fabrizio fractional derivative.

This research paper aims to provide a theoretical foundation parallel to the work of [3, 17, 18, 19], for the comparison principle in the context of fractional differential inequalities using the non-singular kernel of the Caputo-Fabrizio derivative in the Caputo sense, and investigating the conditions under which this principle can be effectively applied to draw meaningful conclusions about the behaviour of solutions of these inequalities. The outcomes of this study are expected to contribute to the development of theoretical knowledge in fractional calculus and provide valuable insights for researchers working with fractional-order systems having the Caputo-fabrizio derivative.

The paper is proceeded as follows: Section 2 provides a brief overview of some preliminaries that will be used in this paper. In Section 3, there are main results about the differential inequality involving Caputo-Fabrizio fractional derivatives. Finally, Section 4 summarizes the key findings, discusses potential areas for future research, and concludes the paper.

2 Preliminaries

Definition 2.1 ([6]). *Let $f \in H^1(a, b)$, $a < b$, $a \in (-\infty, x)$, $0 < \alpha < 1$, the Caputo-Fabrizio fractional derivative in the Caputo sense is defined by-*

$$({}^{CFC}\mathcal{D}_a^\alpha g)(x) = \frac{\mathcal{B}(\alpha)}{1-\alpha} \int_a^x g'(s) e^{\frac{-\alpha}{(1-\alpha)}(x-s)} ds, \quad (2.1)$$

where $\mathcal{B}(\alpha) > 0$ is a normalization function satisfying $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

By the above definition, when $g(x)$ is constant, ${}^{CFC}\mathcal{D}_a^\alpha$ is zero, but the kernel does not have a singularity for $x = s$.

Definition 2.2 ([20]). *Let $0 < \alpha < 1$, the corresponding Caputo-Fabrizio fractional integral of function g is given by*

$$({}^{CFC}\mathcal{J}_a^\alpha g)(x) = \frac{(1-\alpha)}{\mathcal{B}(\alpha)} g(x) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x g(s) ds. \quad (2.2)$$

Property 2.1 ([20]). *The relation between the Caputo-Fabrizio fractional derivative and the corresponding integral in the Caputo-sense is given by*

$$({}^{CFC}\mathcal{J}_a^\alpha)({}^{CFC}\mathcal{D}_a^\alpha)g(x) = g(x) - g(a). \quad (2.3)$$

Property 2.2 ([22]). *Let $\alpha \in (0, 1)$, and f, g, f', g' (f' and g' are derivatives of functions f and g , respectively) are non-negative functions on $(0, \infty]$. Moreover, f and g have the same sense of variation on $(0, \infty]$ such that*

- (i) $f(0) = g(0) = 0$,
- (ii) $fg \geq {}^{CFC}\mathcal{D}_0^\alpha(fg)$,

then

$${}^{CFC}\mathcal{D}_0^\alpha(fg)(t) \leq f(t) {}^{CFC}\mathcal{D}_0^\alpha(g(t)) + g(t) {}^{CFC}\mathcal{D}_0^\alpha(f(t)). \quad (2.4)$$

For more properties of the Caputo-Fabrizio fractional derivative, the reader may refer to [7, 8, 20, 22, 23, 24, 25, 31] and the references mentioned in them.

Definition 2.3 ((Lipschitz Condition) [21]). *Suppose $u(x)$ be continuous on the interval $[a, b]$ of the real line $\mathbb{R}$, and differentiable on $(a, b]$ and $f(x, u)$ is a continuous function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, then*

$$|f(x, u_1) - f(x, u_2)| \leq L|u_1 - u_2|, \quad (2.5)$$

where L is the Lipschitz constant.

Remark 2.1. Before proceeding to the next section, we need following lemmas, which are given in [3], [17], and [19] with respect to Riemann-Liouville, generalized, and Caputo fractional derivatives, respectively. However, we derive it for the Caputo-Fabrizio fractional derivative in the Caputo-sense.

Consider the fractional differential equation

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u), \quad u(a) = u_a, \quad (2.6)$$

where $f \in C([0, X] \times \mathbb{R}, \mathbb{R})$, and $0 \leq \alpha \leq 1$. The initial value problem (IVP) (2.6) is equivalent to the following Volterra fractional integral equation, which we get by using (2.1), (2.2) and (2.3) :

$$\begin{aligned} {}^{CFC}\mathcal{J}_a^\alpha {}^{CFC}\mathcal{D}_a^\alpha u &= {}^{CFC}\mathcal{J}_a^\alpha f(x, u), \\ u(x) - u(a) &= \frac{(1-\alpha)}{\mathcal{B}(\alpha)} f(x, u) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u) ds, \quad 0 < \alpha < 1, \\ u(x) &= u(a) + \frac{(1-\alpha)}{\mathcal{B}(\alpha)} f(x, u) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u) ds. \end{aligned} \quad (2.7)$$

Lemma 2.1. Let $u(x) : \mathbb{R}_+ \rightarrow \mathbb{R}$ be locally Lipschitz continuous such that, for $x_1 \in (0, +\infty)$, $u(x_1) = 0$ and $u(x) \leq 0$ for $0 \leq x \leq x_1$. Then ${}^{CFC}\mathcal{D}_a^\alpha u(x_1) \geq 0$ for $\alpha \in (0, 1)$.

Proof. Let $\mu_\alpha = \frac{\alpha}{1-\alpha}$, then we know that

$$\begin{aligned} {}^{CFC}\mathcal{D}_a^\alpha u(x_1) &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} \int_a^{x_1} u'(s) e^{-\mu_\alpha(x_1-s)} ds, \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \int_a^{x_1} u'(s) e^{\mu_\alpha s} ds, \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[e^{\mu_\alpha s} \int_a^{x_1} u'(s) ds - \int_a^{x_1} \mu_\alpha e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[(e^{\mu_\alpha s} u(s))_{x=a}^{x_1} - \int_a^{x_1} \mu_\alpha e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[e^{\mu_\alpha x_1} u(x_1) - e^{\mu_\alpha a} u(a) - \mu_\alpha \int_a^{x_1} e^{\mu_\alpha s} u(s) ds \right], \\ &= \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \left[0 - e^{\mu_\alpha a} u(a) - \mu_\alpha \int_a^{x_1} e^{\mu_\alpha s} u(s) ds \right], \\ &= -\frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha(x_1-a)} u(a) - \mu_\alpha \frac{\mathcal{B}(\alpha)}{(1-\alpha)} e^{-\mu_\alpha x_1} \int_a^{x_1} e^{\mu_\alpha s} u(s) ds, \\ &\geq 0, \end{aligned}$$

as $u(x_1) \leq 0$ for $a \leq x \leq x_1$, make the terms on the right hand side positive. $\square$

Lemma 2.2. Let $u_\epsilon(x)$ be a family of continuous functions defined on $[a, X]$ each $\epsilon > 0$, where ${}^{CFC}\mathcal{D}_a^\alpha u_\epsilon(x) = f(x, u_\epsilon(x))$, $u_\epsilon(a) = u_a$, and $|f(x, u_\epsilon(x))| \leq M$ for $a \leq x \leq X$. Then the family $\{u_\epsilon(x)\}$ is equicontinuous on $a \leq x \leq X$.

Proof. For $a \leq x_1 \leq x_2 \leq X$, consider

$$\begin{aligned} |u_\epsilon(x_1) - u_\epsilon(x_2)| &= \left| u(a) + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x_1, u_\epsilon(x_1)) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_1} f(s, u_\epsilon(x_1)) ds - u(a) \right. \\ &\quad \left. - \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x_2, u_\epsilon(x_2)) - \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_2} f(s, u_\epsilon(x_2)) ds \right|, \\ &\leq \frac{1-\alpha}{\mathcal{B}(\alpha)} |f(x_1, u_\epsilon(x_1)) - f(x_2, u_\epsilon(x_2))| \\ &\quad + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^{x_1} |f(s, u_\epsilon(x_1)) - f(s, u_\epsilon(x_2))| ds \\ &\quad - \frac{\alpha}{\mathcal{B}(\alpha)} \int_{x_1}^{x_2} |f(s, u_\epsilon(x_2))| ds, \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{\alpha}{\mathcal{B}(\alpha)} \int_{x_1}^{x_2} M ds, \\
&= -\frac{\alpha M}{\mathcal{B}(\alpha)} (x_2 - x_1), \\
&< \epsilon,
\end{aligned}$$

provided $|x_2 - x_1| < \delta = \frac{\mathcal{B}(\alpha)}{\alpha M}$. $\square$

3 Caputo-Fabrizio Fractional Differential Inequalities

The following results are inspired by the research work of Lakshmikantham [17, 18], but with the Caputo-Fabrizio fractional derivative in the Caputo-sense.

Theorem 3.1. *Let $v, w : [a, X] \rightarrow \mathbb{R}$ be locally Lipschitz continuous, $f \in C([a, X] \times \mathbb{R}, \mathbb{R})$ and*

$$i) {}^{CFC}\mathcal{D}_a^\alpha v(x) \leq f(x, v(x)), \quad ii) {}^{CFC}\mathcal{D}_a^\alpha w(x) \geq f(x, w(x)), \quad 0 \leq x \leq X,$$

one of the inequalities being strict. Then

$$v(a) < w(a) \tag{3.1}$$

implies

$$v(x) < w(x), \quad a \leq x \leq X. \tag{3.2}$$

Proof. Suppose that the conclusion (3.2) is not true. Let us suppose that the inequality (ii) is strict. Then setting $m(x) = v(x) - w(x)$, $a \leq x \leq x_1$ we find that $m(x) \leq 0$, $a \leq x \leq x_1$ and $m(x_1) = 0$. Then by Lemma 2.1, we get ${}^{CFC}\mathcal{D}_a^\alpha m(x_1)$, which yields, using (i) and (ii) and the definition of $m(x)$,

$$f(x_1, v(x_1)) \geq {}^{CFC}\mathcal{D}_a^\alpha v(x_1) \geq {}^{CFC}\mathcal{D}_a^\alpha w(x_1) > f(x_1, w(x_1)). \tag{3.3}$$

This is the contradiction since $v(x_1) = w(x_1)$. Hence, the conclusion (3.2) is valid and the proof is complete. $\square$

The next theorem is for non-strict fractional differential inequalities, which requires a one sided Lipschitz type condition.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 holds with non-strict inequalities (i) and (ii). Suppose further that*

$$f(x, v) - f(x, w) \leq \frac{L}{1 + e^{-\mu_\alpha x}} (v - w), \tag{3.4}$$

whenever $v \geq w$ and $x > a$. then $v(a) \leq w(a)$ implies,

$$v(x) < w(x), \quad a \leq x \leq X, \tag{3.5}$$

provided $Le^{\mu_\alpha x} \leq \frac{\mathcal{B}(\alpha)\mu_\alpha}{(1-\alpha)}$.

Proof. We set $w_\epsilon(x) = w(x) + \epsilon(1 + e^{-\mu_\alpha x})$, for small $\epsilon > 0$, so that we have

$$w_\epsilon(a) > w(a) \quad \text{and} \quad w_\epsilon(x) > w(x), \quad a \leq x \leq X. \tag{3.6}$$

Now,

$$\begin{aligned}
{}^{CFC}\mathcal{D}_a^\alpha w_\epsilon(x) &= {}^{CFC}\mathcal{D}_a^\alpha w(x) + \epsilon {}^{CFC}\mathcal{D}_a^\alpha (1 + e^{-\mu_\alpha x}), \\
&= {}^{CFC}\mathcal{D}_a^\alpha w(x) + \epsilon {}^{CFC}\mathcal{D}_a^\alpha (e^{-\mu_\alpha x}), \\
&\geq f(x, w(x)) + \epsilon \frac{\mathcal{B}(\alpha)}{1-\alpha} \int_a^x (e^{-\mu_\alpha s})' e^{-\mu_\alpha (x-s)} ds, \\
&> f(x, w_\epsilon(x)) - \epsilon \frac{\mathcal{B}(\alpha)\mu_\alpha}{1-\alpha} e^{-\mu_\alpha x} (x - a), \\
&> f(x, w_\epsilon(x)) - L\epsilon(x - a), \\
&> f(x, w_\epsilon(x)), \quad a \geq x \geq X.
\end{aligned}$$

Here, we have used relation (3.4), (3.6) and the assumptions $Le^{\mu_\alpha x} \leq \frac{\mathcal{B}(\alpha)\mu_\alpha}{(1-\alpha)}$. We can now apply Theorem 3.1 to v and $w_\epsilon(x)$ to get $v(x) < w_\epsilon(x)$, $a \leq x \leq X$. Since $\epsilon > 0$ is arbitrary, we conclude that (3.5) is true, and we are done. $\square$

To proceed further, we need Peano's type existence result for Caputo-Fabrizio fractional differential inequalities.

Theorem 3.3. Assume that $f \in C[R_a, \mathbb{R}]$ where $R_a = [(t, x) : a \leq x \leq b, \text{ and } |u - u_a| \leq c]$, and let $|f(x, u)| \leq M$ on R_a . Then there exists at least one solution for the IVP (2.6) on $a \leq x \leq \alpha$, where $\alpha = \min\left(b, \frac{c\mathcal{B}(\alpha)}{M(1-\alpha)}\right)$, $0 < \alpha < 1$.

Proof. Let $u_0(\eta)$ be a continuous function on $[-\delta, 0]$, $\delta > 0$ such that $x_0(a) = x_a$, $|u_0(x) - u_a| \leq c$, and $|{}^{CFC}\mathcal{D}_a^\alpha u_a(x)| \leq M$, where ${}^{CFC}\mathcal{D}_a^\alpha u_a(x)$ is the continuous fractional derivative. For $0 < \eta \leq \delta$, we define the function $u_\eta(x) = u_a(x)$ on $[-\delta, 0]$ and

$$u_\eta(x) = u_a + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x, u_\eta) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u_\eta(s-\eta)) ds \quad (3.7)$$

on $[0, \alpha_1]$, where $\alpha_1 = \min(\alpha, \eta)$.

We observe that ${}^{CFC}\mathcal{D}_a^\alpha u_\eta(x)$ exists and

$$\begin{aligned} |u_\eta(x) - u_a| &\leq \frac{(1-\alpha)}{\mathcal{B}(\alpha)} |f(x, u_\eta)| + \int_a^x |f(s, u_\eta)| ds, \\ &\leq M \frac{(1-\alpha)}{\mathcal{B}(\alpha)} + M \int_a^x ds, \\ &= M \frac{(1-\alpha)}{\mathcal{B}(\alpha)} + M(x-a), \\ &= M \left[\frac{1-\alpha}{\mathcal{B}(\alpha)} + (x-a) \right] \leq c, \end{aligned} \quad (3.8)$$

because of the choice of α_1 . If $\alpha_1 < \alpha$, we can employ (14) to extend $u_\eta(x)$ as a continuously fractional differentiable function on $[-\delta, \alpha_2]$, $\alpha_2 = \min(\alpha, 2\eta)$ such that $|u_\eta(x) - x_a| \leq c$ holds. Continuing this process, we can define $u_\eta(x)$ over $[-\delta, \alpha]$ so that $|u_\eta(x) - x_a| \leq c$, it has a continuous fractional derivative and satisfies (3.7) on the same interval $[-\delta, \alpha]$. Moreover, $|{}^{CFC}\mathcal{D}_a^\alpha u_\eta(x)| \leq M$, since $|f(x, u_\eta(x-\eta))| \leq M$ on R_a , and therefore the family $\{u_\eta(x)\}$ forms an equicontinuous and uniformly bounded family of functions by Lemma 2.2. As application of Ascoli-Arzelà's theorem shows the existence of sequence $\{\eta_n\}$ such that $\eta_1 > \eta_2 > \eta_3 \dots > \eta_n \rightarrow 0$ as $n \rightarrow \infty$, and $u(x) = \lim_{n \rightarrow \infty} u_{\eta_n}(x)$ exists uniformly on $[-\delta, \alpha]$. Since f is uniformly continuous, we obtain $f(x, u_{\eta_n}(x-\eta_n))$ tends uniformly to $f(x, u(x))$ as $n \rightarrow \infty$, and hence term by term integration of (3.7) with $\eta = \eta_n$, $\alpha_1 = \alpha$ yields

$$u(x) = u_a + \frac{1-\alpha}{\mathcal{B}(\alpha)} f(x, u(x)) + \frac{\alpha}{\mathcal{B}(\alpha)} \int_a^x f(s, u(s)) ds. \quad (3.9)$$

This proves that $u(x)$ is a solution of IVP (2.6), and the proof is complete. $\square$

Employing Theorems 3.1 and 3.3, we can now prove the existence of the extremal solutions for the IVP (2.6).

Theorem 3.4. Assume that $f \in C[R_a, \mathbb{R}]$ where $R_a = [(t, x) : a \leq x \leq b, \text{ and } |u - u_a| \leq c]$, and let $|f(x, u)| \leq M$ on R_a , then there exists extremal solutions for the IVP (2.6) on $a \leq x \leq \alpha_a$, $\alpha_a = \min\left(b, \left[\frac{c\mathcal{B}(\alpha)}{(2M+c)(1-\alpha)}\right]^{\frac{1}{\alpha}}\right)$.

Proof. We shall prove the existence of the maximal solution only, since the case of the minimal solution is very similar.

Let $0 < \epsilon \leq \frac{c}{2}$ and consider the IVP

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u) + \epsilon, \quad u(a) = u_a + \epsilon \quad (3.10)$$

Note that $f_\epsilon(x, u) = f(x, u) + \epsilon$ is defined and continuous on

$$R_\epsilon = \left[(t, x) : a \leq x \leq b \text{ and } |x - (x_a + \epsilon)| \leq \frac{c}{2}\right], \quad (3.11)$$

$R_\epsilon \subset R_a$ and $|f_\epsilon(x, u)| \leq M + \frac{c}{2}$ on R_ϵ .

We then deduce from Theorem 3.3 that IVP (3.10) has a solution $u(x, \epsilon)$ on $a \leq x \leq \alpha_a$. Now, for $0 < \epsilon_2 < \epsilon_1 \leq \epsilon$, we have

$$\begin{aligned} u(a, \epsilon_2) &< u(a, \epsilon_1), \\ {}^{CFC}\mathcal{D}_a^\alpha u(a, \epsilon_2) &\leq f(x, u(x, \epsilon_2)) + \epsilon_2, \\ {}^{CFC}\mathcal{D}_a^\alpha u(a, \epsilon_1) &> f(x, u(x, \epsilon_1)) = \epsilon_2, \quad \text{on } a \leq x \leq \alpha_a. \end{aligned} \quad (3.12)$$

We apply Theorem 3.1 to get

$$u(x, \epsilon_2) < u(x, \epsilon_1), \quad a \leq x \leq \alpha_a. \quad (3.13)$$

Consider the family of continuous functions $\{u(x, \epsilon)\}$ on $a \leq x \leq \alpha_a$. Then it follows that

$$\begin{aligned} |u(x, \epsilon) - u(a, \epsilon)| &\leq \frac{1-\alpha}{\mathcal{B}(\alpha)} |f(x, u(s, \epsilon))| + \int_a^x |f(s, u(s, \epsilon))| ds, \\ &\leq \frac{(1-\alpha)}{\mathcal{B}(\alpha)} \left(M + \frac{c}{2}\right) + \left(M + \frac{c}{2}\right) (x - a), \\ &\leq \left(M + \frac{c}{2}\right) \left(\frac{1-\alpha}{\mathcal{B}(\alpha)} (1 + \epsilon)\right), \\ &\leq \frac{c}{2} \leq c. \end{aligned} \quad (3.14)$$

Showing that the family is uniformly bounded. Since $|{}^{CFC}\mathcal{D}_a^\alpha u(x, \epsilon)| = |f(x, u(x, \epsilon))| \leq M + \frac{c}{2}$, the family is also equicontinuous by Lemma 2.2. Hence, there exists a sequence ϵ_n with $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and the uniform limit $\eta(x) = \lim_{n \rightarrow \infty} u(x, \epsilon_n)$ exists on $[a, \alpha_a]$. Clearly, $\eta(a) = u_a$. The uniform continuity of f gives, as before (as in Theorem 3.3), that $\eta(x)$ is a solution of IVP (2.6).

Next, we show that $\eta(x)$ is the required maximal solution of (2.6). Let $u(x)$ be any solution of (2.6) on $0 \leq x \leq \alpha_a$, then we have

$$\begin{aligned} u_a &< u_a + \epsilon = u(a, \epsilon), \\ {}^{CFC}\mathcal{D}_a^\alpha u(x) &< f(x, u(x)) + \epsilon, \\ {}^{CFC}\mathcal{D}_a^\alpha u(x, \epsilon) &\geq f(x, u(x, \epsilon)) + \epsilon. \end{aligned} \quad (3.15)$$

By Theorem 3.1, we obtain $u(x) < u(x, \epsilon)$, $a \leq x \leq \alpha_a$ for every $\epsilon > 0$. The uniqueness of maximal solution shows that $u(x, \epsilon)$ tends to $\eta(x)$ on $[a, \alpha_a]$ as $\epsilon \rightarrow 0$. The proof is therefore complete. $\square$

We need the following comparison theorem to prove the global existence of a solution.

Theorem 3.5. Assume that $m \in C([a, X], \mathbb{R}_+)$, locally Lipschitz continuous, $g \in C([a, X] \times \mathbb{R}_+, \mathbb{R}_+)$, and

$${}^{CFC}\mathcal{D}_a^\alpha m(x) \leq g(x, m(x)), \quad a \leq x \leq X. \quad (3.16)$$

Let $\eta(x)$ be the maximal solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) = g(x, v(x)), \quad u(a) = u_a \geq a, \quad (3.17)$$

existing on $[a, X)$ such that $m(a) \leq v_a$. Then we have

$$m(x) \leq \eta(x), \quad a \leq x \leq X. \quad (3.18)$$

Proof. In view of the definition of the maximal solution $\eta(x)$, it is enough to prove, to conclude (3.18), that

$$m(x) < v(x, \epsilon), \quad a \leq x < X, \quad (3.19)$$

where $v(x, \epsilon)$ is any solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) \leq g(x, v(x)) + \epsilon, \quad v(a) = v_a + \epsilon, \quad \epsilon > 0. \quad (3.20)$$

Now it follows from (3.20) that

$${}^{CFC}\mathcal{D}_a^\alpha v(x, \epsilon) > g(x, v(x, \epsilon)). \quad (3.21)$$

Then applying Theorem 3.1, we get immediately (3.19) and since $\lim_{\epsilon \rightarrow 0} v(x, \epsilon) = \eta(x)$ uniformly on each $a \leq x \leq X_a < X$, the proof is complete. $\square$

We are now ready to prove the global existence result.

Theorem 3.6. *Assume that $f \in C([a, \infty) \times \mathbb{R}, \mathbb{R})$, $g \in C([a, \infty) \times \mathbb{R}_+, \mathbb{R}_+)$, $g(x, v)$ is nondecreasing in u for each x and*

$$|f(x, u)| \leq g(x, |u|). \quad (3.22)$$

Suppose that we have local existence of solutions $u(x, u_a)$ of

$${}^{CFC}\mathcal{D}_a^\alpha u(x) = f(x, u(x)), \quad u(a) = u_a. \quad (3.23)$$

and the maximal solution of

$${}^{CFC}\mathcal{D}_a^\alpha v(x) = g(x, v(x)), \quad v(a) = v_a \geq a, \quad (3.24)$$

exists on $[a, \infty)$. Then the largest interval of existence of any solution $u(x, u_a)$ of (3.23) such that $|u_a| \leq v_a$ is $[a, \infty)$.

Proof. Let $u(x, x_a)$ be any solution of (3.23) with $|u_a| \leq v_a$, which exists on $[a, \beta)$, $\beta < \infty$ and the value of β cannot be increased further. Set $m(x) = |u(x, x_a)|$ for $a \leq x < \beta$. Then, using the assumption (3.22), we get

$$\begin{aligned} {}^{CFC}\mathcal{D}_a^\alpha m(x) &\leq |{}^{CFC}\mathcal{D}_a^\alpha u(x, u_a)|, \\ &= |f(x, u(x, u_a))|, \\ &\leq |g(x, u(x, u_a))|, \\ &\leq g(x, m(x)), \quad a \leq x < \beta, \end{aligned} \quad (3.25)$$

and $m(a) \leq u_a$. Applying Theorem 3.5, we obtain

$$m(x) \leq \eta(x), \quad a \leq x \leq \beta, \quad (3.26)$$

and therefore $|{}^{CFC}\mathcal{D}_a^\alpha u(x, u_a)| \leq g(x, m(x)) \leq g(x, \eta(x)) \leq M$ on $a \leq x \leq \beta$, since $\eta(x)$ is assumed to exist on $(a, \infty]$, it follows that $g(x, \eta(x)) \leq M$. Now for $a \leq x_1 \leq x_2 < \beta$ we find by Lemma 2.2,

$$|u(x_1, u_a) - u(x_2, u_a)| \leq \frac{2M\mathcal{B}(\alpha)}{1 - \alpha}(x_2 - x_1). \quad (3.27)$$

Letting $x_1, x_2 \rightarrow \beta^-$ and using Cauchy criterion, it follows that $\lim_{x \rightarrow \beta^-} u(x, u_a)$ exists.

We denote $u(\beta, u_a) = \lim_{x \rightarrow \beta^-} u(x, u_a)$ and consider the new IVP

$${}^{CFC}\mathcal{D}_a^\alpha u = f(x, u), \quad u(\beta) = u(\beta, u_a). \quad (3.28)$$

By the assumed local existence, we find that $u(x, u_a)$ can be continued beyond β , contradicting our assumption. Hence every solution $u(x, u_a)$ of (3.23) exists on $[a, \infty)$ and completes the proof. $\square$

4 Conclusion

The new definition of fractional derivative, mentioned in [6, 20, 22], overcome the problem of singular kernel found in Riemann-Liouville and Caputo fractional derivative. By using this definition we reestablished the results in [3, 17, 18] for new conditions, that are seen in statements and proofs of Theorems 3.2, 3.3, and 3.4. This research investigates *FDEs* with the help of fractional differential inequalities and the comparison principle with the Caputo-Fabrizio fractional derivative in the Caputo-sense. The study's significance lies in analyzing solutions to fractional differential inequalities, showing its usefulness as a fundamental tool in fractional analysis. As fractional calculus advances, these findings provide valuable contributions to theory and its practical applications, guiding future research in the field. This work can be extended to the stochastic or the multi-term fractional differential system, which feature nonsingular kernels. Also one can extend the results for some other definitions of fractional derivative, for example Atangaba-Baleanu fractional derivative.

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