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GROWTH ANALYSIS OF COMPOSITE ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES IN THE UNIT POLYDISC BASED ON THE RELATIVE (p, q) -th ORDER, RELATIVE (p, q) -th TYPE AND RELATIVE (p, q) -th WEAK TYPE

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Abstract

In this paper, we study some growth properties of composite analytic functions of several complex variables in the unit polydisc. This paper establishes new results concerning the relative (p, q) -th order, relative (p, q) -th type and relative (p, q) -th weak type of an analytic function with respect to an entire function, where p and q are any two positive integers.

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1 Introduction

Let $\mathcal{A}(\mathbb{K})$ represents the set of analytic functions. So it is formed $\mathbb{K}$ -algebra of analytic functions in $\mathbb{K}$, i.e. the set of power series with an infinite radius of convergence. For more details on the study of analytic functions inside a disk or in the whole field $\mathbb{K}$, we can read the following articles [1, 7, 8]. Suppose $f \in \mathcal{A}(\mathbb{K})$ and $r > 0$ then $|f|(r) = \sup\{|f| : |x| = r\}$ represents the multiplicative norm on $\mathcal{A}(\mathbb{K})$. Moreover, for a non-constant analytic function f , $|f|(r)$ is strictly increasing function of r and tends to $+\infty$ whenever r tends to $+\infty$. Therefore, its inverse function exists, and we can define it in the following way $\widehat{|f|} : (|f(0)|, \infty) \rightarrow (0, \infty)$ with $\lim_{s \rightarrow \infty} \widehat{|f|}(s) = \infty$.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log(\log^{[k-1]} x)$ and $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$, where $\mathbb{N}$ denotes the set of all positive integers.

To start our paper we just recall the following definitions:

2 Definitions

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in the unit disc $D = \{z : |z| < 1\}$ and $M(r, f)$ be the maximum of $|f(z)|$ on $|z| = r$. Sons [9] defined the order $\rho(f)$ and the lower order $\lambda(f)$ as

$$\rho(f) = \limsup_{r \rightarrow 1} \frac{\log \log M(r, f)}{-\log(1-r)} \quad \text{and} \quad \lambda(f) = \liminf_{r \rightarrow 1} \frac{\log \log M(r, f)}{-\log(1-r)}.$$

Considering the unit polydisc $D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\}$, Banerjee and Dutta [2] introduced the definition of order and lower order of functions of two complex variables analytic in the unit polydisc, which are as follows:

Definition 2.1. Let

$$f(z_1, z_2) = \sum_{n_1, n_2=0}^{\infty} a_{n_1 n_2} z_1^{n_1} z_2^{n_2}$$

be a non-constant analytic function of two complex variables z_1 and z_2 in the closed unit polydisc

$$D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\}$$

and let

$$M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, i = 1, 2\}.$$

Then order and lower order of $f(z_1, z_2)$ are denoted by $\rho(f)$ and $\lambda(f)$ respectively and defined as

$$\rho(f) = \limsup_{r_1, r_2 \rightarrow 1} \frac{\log \log M_f(r_1, r_2)}{\log \left(\frac{1}{(1-r_1)(1-r_2)} \right)} \quad \text{and} \quad \lambda(f) = \liminf_{r_1, r_2 \rightarrow 1} \frac{\log \log M_f(r_1, r_2)}{\log \left(\frac{1}{(1-r_1)(1-r_2)} \right)}.$$

Generalising this notion, Dutta [6] introduced the definitions of (p, q) -th order and (p, q) -th lower order of functions of two complex variables analytic in the unit polydisc in this way:

Definition 2.2 ([6]). Let $f(z_1, z_2)$ be a non-constant analytic function of two complex variables z_1 and z_2 in the closed unit polydisc

$$D = \{(z_1, z_2) : |z_i| \leq 1, i = 1, 2\},$$

then (p, q) -th order $\rho^{(p, q)}(f)$ and (p, q) -th lower order $\lambda^{(p, q)}(f)$ of $f(z_1, z_2)$ are defined as:

$$\rho^{(p, q)}(f) = \limsup_{r_1, r_2 \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)} \right)} \quad \text{and} \quad \lambda^{(p, q)}(f) = \liminf_{r_1, r_2 \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)} \right)},$$

where p and q are positive integers with $p \geq q \geq 1$.

Extending this notion, one may introduce (p, q) -th order and (p, q) -th lower order for functions of n complex variables analytic in a unit polydisc as follows:

$$\rho^{(p, q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2) \dots (1-r_n)} \right)},$$

and

$$\lambda^{(p, q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2) \dots (1-r_n)} \right)},$$

where $p, q \in \mathbb{N}$ and $f(z_1, z_2, \dots, z_n)$ is a non-constant analytic function of n -complex variables $z_1, z_2, \dots, z_{n-1}, z_n$ in the unit polydisc

$$D = \{(z_1, z_2, \dots, z_n) : |z_i| \leq 1, i = 1, 2, \dots, n; r_1 > 0, r_2 > 0, \dots, r_n > 0\}.$$

Define

$$M_f(r_1, r_2, \dots, r_n) = \max\{|f(z_1, z_2, \dots, z_n)| : |z_i| \leq 1, i = 1, 2, \dots, n; r_1 > 0, r_2 > 0, \dots, r_n > 0\}.$$

The above definition avoids the restriction $p \geq q \geq 1$ from the original definition of the (p, q) -th order (respectively (p, q) -th lower order) of functions of two complex variables holomorphic in the unit polydisc, as introduced by Dutta [6].

In this connection, we recall the following definition:

Definition 2.3. A non-constant analytic function of n -complex variables $f(z_1, z_2, \dots, z_n)$ is said to have index-pair (p, q) if

$$b < \rho^{(p, q)}(f) < \infty \quad \text{and} \quad \rho^{(p-1, q-1)}(f)$$

is not a nonzero finite number, where $b = 1$ if $p = q$, and $b = 0$ otherwise.

Moreover, if $0 < \rho^{(p, q)}(f) < \infty$, then:

$$\begin{cases} \rho^{(p-n, q)}(f) = \infty & \text{for } n < p, \\ \rho^{(p, q-n)}(f) = 0 & \text{for } n < q, \\ \rho^{(p+n, q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

Similarly, for $0 < \lambda^{(p, q)}(f) < \infty$, one can easily verify that:

$$\begin{cases} \lambda^{(p-n, q)}(f) = \infty & \text{for } n < p, \\ \lambda^{(p, q-n)}(f) = 0 & \text{for } n < q, \\ \lambda^{(p+n, q+n)}(f) = 1 & \text{for } n = 1, 2, \dots \end{cases}$$

If $\rho^{(p, q)}(f) = \lambda^{(p, q)}(f)$ then $f(z_1, z_2, \dots, z_n)$ is of regular growth; otherwise, this function is of irregular growth.

Somasundaram and Thamizharasi [10] introduced the notions of L -order and L -lower order for entire functions of a single variable, where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant a . In the line of Somasundaram and Thamizharasi, one may introduce the definition of the (p, q, t) -th order and (p, q, t) -th lower order for functions of n complex variables analytic in a unit polydisc:

$$\rho^{(p,q,t)L}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right) + \exp^{[t]} L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \dots, \frac{1}{1-r_n} \right)}$$

and

$$\lambda^{(p,q,t)L}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right) + \exp^{[t]} L \left(\frac{1}{1-r_1}, \frac{1}{1-r_2}, \dots, \frac{1}{1-r_n} \right)}$$

where $p, q \in \mathbb{N}$, $t \in \mathbb{N} \cup \{-1, 0\}$.

Definition 2.4. If $f(z_1, z_2, \dots, z_n)$ is an analytic function in D and $g(z_1, z_2, \dots, z_n)$ is an entire function of n complex variables, then the relative (p, q) -th order of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\rho_g^{(p,q)}(f)$, is defined by:

$$\rho_g^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)},$$

where p and q are any two positive integers.

Similarly, for any two positive integers p and q , the relative (p, q) -th lower order of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\lambda_g^{(p,q)}(f)$, is given by:

$$\lambda_g^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)}.$$

We now introduce relative (p, q) -th type and relative (p, q) -th lower type in the unit polydisc as follows:

Definition 2.5 ([3]). Let $f(z_1, z_2, \dots, z_n)$ is an analytic function in D then (p, q) -th type and the (p, q) -th lower type of f having finite positive order are defined by

$$\sigma^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\rho^{(p,q)}(f)}},$$

and

$$\bar{\sigma}^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\rho^{(p,q)}(f)}},$$

respectively.

Definition 2.6 ([3]). Let $f(z_1, z_2, \dots, z_n)$ is an analytic function in D then (p, q) -th weak type of f having finite positive lower-order is defined by

$$\tau^{(p,q)}(f) = \liminf_{(r_1, r_2, \dots, r_n) \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda^{(p,q)}(f)}},$$

and (p, q) -th lower weak type of f , having finite positive lower order is defined by

$$\bar{\tau}^{(p,q)}(f) = \limsup_{(r_1, r_2, \dots, r_n) \rightarrow 1} \frac{\log^{[p-1]} M_f(r_1, r_2, \dots, r_n)}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda^{(p,q)}(f)}}.$$

Definition 2.7. If $f(z_1, z_2, \dots, z_n)$ be an analytic function in D and $g(z_1, z_2, \dots, z_n)$ is an entire function of n complex variables, then the relative (p, q) -th weak type of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\tau_g^{(p,q)}(f)$, is defined by:

$$\tau_g^{(p,q)}(f) = \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda_g^{(p,q)}(f)}},$$

where p and q are any two positive integers.

Similarly, for any two positive integers p and q , the relative (p, q) -th lower weak type of $f(z_1, z_2, \dots, z_n)$ with respect to $g(z_1, z_2, \dots, z_n)$, denoted by $\bar{\tau}_g^{(p,q)}(f)$, is given by:

$$\bar{\tau}_g^{(p,q)}(f) = \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[p-1]} M_g^{-1}(M_f(r_1, r_2, \dots, r_n))}{\log^{[q-1]} \left(\frac{1}{(1-r_1)(1-r_2)\dots(1-r_n)} \right)^{\lambda_g^{(p,q)}(f)}}.$$

3 Known Results

In this section, firstly, we present some known results which will be needed in the sequel.

Theorem 3.1 ([2]). *Let f, g and h be the analytic functions in the unit polydisc D such that the relative index pair of f with respect to h and the index pair of g are (p, q) and (m, n) respectively. Then*

- (i) *The relative index-pair of $f \circ g$ is (p, n) when $q = m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\lambda_h^{(p,q)}(f)\rho^{(m,n)}(g) \leq \rho_h^{(p,n)}(f \circ g) \leq \rho_h^{(p,q)}(f)\rho^{(m,n)}(g)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(p,q)}(f)\lambda^{(m,n)}(g) \leq \rho_h^{(p,n)}(f \circ g) \leq \rho_h^{(p,q)}(f)\rho^{(m,n)}(g)$ if $\lambda^{(p,q)}(f) > 0$.
- (ii) *The relative index-pair of $f \circ g$ is $(p, q + n - m)$ when $q > m$ and either $\lambda_h(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\lambda_h^{(p,q)}(f) \leq \rho^{(p,q+n-m)}(f \circ g) \leq \rho_h^{(p,q)}(f)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(p,q+n-m)}(f \circ g) = \rho_h^{(p,q)}(f)$ if $\lambda^{(m,n)}(g) > 0$.
- (iii) *The relative index-pair of $f \circ g$ is $(p + m - q, n)$ when $q < m$ and either $\lambda_h^{(p,q)}(f) > 0$ or $\lambda^{(m,n)}(g) > 0$. Also*
 - (a) $\rho_h^{(p+m-q,n)}(f \circ g) = \rho^{(m,n)}(g)$ if $\lambda_h^{(p,q)}(f) \geq 0$ and
 - (b) $\rho_h^{(m,n)}(g) \leq \rho_h^{(p+m-q,n)}(f \circ g) \leq \rho^{(m,n)}(g)$ if $\lambda^{(m,n)}(g) > 0$.

Theorem 3.2 ([2]). *Let f, g and h be the analytic functions in the unit polydisc D such that the relative index pair of f with respect to h and the index pair of g are (p, q) and (m, n) respectively. Then*

- (i) $\lambda_h^{(p,q)}(f)\lambda^{(m,n)}(g) \leq \lambda_h^{(p,n)}(f \circ g) \leq \min\{\lambda_h^{(p,q)}(f)\lambda^{(m,n)}(g), \lambda_h^{(p,q)}(f)\rho^{(m,n)}(g)\}$ if $q = m$, $\lambda_h^{(p,q)}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$;
- (ii) $\lambda_h^{(p,q+n-m)}(f \circ g) = \lambda_h^{p,q}(f)$ if $q > m$, $\lambda_h^{(p,q)}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$, and
- (iii) $\lambda_h^{(p+m-q,n)}(f \circ g) = \lambda^{(m,n)}(g)$, if $q < m$, $\lambda_h^{p,q}(f) > 0$ and $\lambda^{(m,n)}(g) > 0$.

4 Main Results

In this section, we present the main results of the paper.

Theorem 4.1. *Let f, g, h and k be the analytic functions of several complex variables such that the index pairs of f, g, h and k are (p, q) , (m, n) , (a, b) and (c, d) respectively and*

$$\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[n]}(r_1, r_2, \dots, r_n)}{\log^{[n]}(r_1 r_2 \dots r_n)} = 1.$$

- (i) *If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda^{(p,q)}(f) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$ then*

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \min \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \max \left\{ \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)},
\end{aligned}$$

and

(ii) If $(q > m, a = c = p)$, $\lambda^{(p,q)}(f) > 0$ and $0 < \lambda_h^{(b,q+n-m)}(f \circ g) < \infty$, then

$$\begin{aligned}
\frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
&\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n)}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
&\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\end{aligned}$$

Proof. Suppose either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ hold and $\lambda^{(p,q)}(f) > 0$. Then in view of Theorem 3.1, the index-pair of $f \circ g$ is (p, n) or $(p + m - q, n)$ respectively and therefore by above definitions, $\rho_h^{(b,n)}(f \circ g)$ and $\lambda_h^{(b,n)}(f \circ g)$ respectively exist.

Since the index-pair of f and k are (p, q) and (c, d) respectively, and $\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[q]}(r_1, r_2, \dots, r_n)}{\log^{[q]}(r_1 r_2 \dots r_n)} = 1$. Therefore, the relative index-pair of f with respect to k is (d, q) . Now from the definition of $\lambda_h^{(b,n)}(f \circ g)$, $\rho_h^{(b,n)}(f \circ g)$, $\lambda_k^{(d,q)}(f)$ and $\rho_k^{(d,q)}(f)$; for arbitrary positive ϵ and for all values of $(r_1, r_2, \dots, r_n)$ in the unit disc; we obtain the following

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \geq (\lambda_h^{(b,n)}(f \circ g) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.1)$$

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \leq (\rho_h^{(b,n)}(f \circ g) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.2)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \geq (\lambda_k^{(d,q)}(f) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.3)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \leq (\rho_k^{(d,q)}(f) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n). \quad (4.4)$$

When $(r_1, r_2, \dots, r_n) \rightarrow 1$ then

$$\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n)) \leq (\lambda_h^{(b,n)}(f \circ g) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.5)$$

$$\log^{[b]} M_h^{-1} M_{f \circ g}(r_1, r_2, \dots, r_n) \geq (\rho_h^{(b,n)}(f \circ g) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.6)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \leq (\lambda_k^{(d,q)}(f) + \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.7)$$

$$\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n)) \geq (\rho_k^{(d,q)}(f) - \epsilon) \log^{[n]}(r_1 r_2 \dots r_n), \quad (4.8)$$

Now, from (4.1) and (4.4), and for all values of $(r_1, r_2, \dots, r_n)$ in the unit disc, we obtain the following inequality-

$$\frac{\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{(\lambda_h^{(b,n)}(f \circ g) - \epsilon)}{(\rho_k^{(d,q)}(f) + \epsilon)}$$

As $\epsilon > 0$ is arbitrary, we obtain that

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1} (M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1} (M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (4.9)$$

Similarly, from (4.3) and (4.5), we get

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.10)$$

From (4.1) and (4.7), we derive

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\lambda_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.11)$$

From (4.2) and (4.3), we obtain

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}. \quad (4.12)$$

From (4.2) and (4.8), we establish

$$\liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \leq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}, \quad (4.13)$$

While from (4.4) and (4.6), we derive the following

$$\limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \geq \frac{\rho_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)}. \quad (4.14)$$

Thus, the first part of the theorem follows from (4.9), (4.10), (4.11), (4.12), (4.13) and (4.14).

Similarly, we can prove the second part of the theorem. $\square$

Remark 4.1. If we replace the conditions $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$, $q > m, a = c = p$ and $\lambda^{(p,q)}(f) > 0$ by $(q = m = c, a = p)$ or $(q < m = c, a = p + m - q)$, $q > m = c, a = p$ and $\lambda^{(m,n)}(g, \phi) > 0$ respectively and other conditions remain same, then first part of theorem still valid for $\rho_k^{(d,n)}(g)$, $\lambda_k^{(d,n)}(g)$ and $\log^{[d]} M_k^{-1}(M_g(r_1, r_2, \dots, r_n))$ instead of $\rho_k^{(d,q)}(f)$, $\lambda_k^{(d,q)}(f)$ and $\log^{[d]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))$ respectively. Similarly second part of theorem still valid for $\rho_k^{(d,n)}(f)$, $\lambda_k^{(d,n)}(g)$ and $\log^{[d]} M_k^{-1}(M_g(r_1, r_2, \dots, r_n))$ instead of $\rho_k^{(d,q)}(f)$, $\lambda_k^{(d,q)}(f)$ and $\log^{[d]} M_k^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))$ respectively.

Theorem 4.2. Let f, g, h and k be the analytic functions of several complex variables in the unit disc D such that the index pairs of f, g, h and k are (p, q) , (m, n) , (a, b) and (c, d) respectively and $\lim_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[n-1]}(r_1, r_2, \dots, r_n)}{\log^{[n-1]}(r_1 r_2 \dots r_n)} = 1$.

(i) If either $(q = m, a = c = p, q \geq n)$ or $(q < m, c = p, a = p + m - q, q \geq n)$ holds and $\lambda^{(p,q)}(f) > 0$, $0 < \lambda_h^{(b,n)}(f \circ g) \leq \rho_h^{(b,n)}(f \circ g) < \infty$, then

$$\begin{aligned} \frac{\lambda_h^{(b,n)}(f \circ g)}{\rho_k^{(d,q)}(f)} &\leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \min \left\{ \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b, q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \max \left\{ \frac{\lambda_h^{(b, q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b, q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\ &\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\ &\leq \frac{\rho_h^{(b,n)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \end{aligned}$$

and

$$\begin{aligned}
(ii) \text{ If } q > m, a = c = p, 0 < \lambda_h^{(p,q+n-m)}(f \circ g) \leq \rho_h^{(b,q+n-m)}(f \circ g) < \infty, \\
0 < \lambda_k^{(d,q)}(f) \leq \rho_k^{(d,q)}(f) < \infty \text{ and } \rho_h^{(b,q+n-m)}(f \circ g) = \rho_k^{(d,q)}(f), \text{ then} \\
\frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \leq \liminf_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
\leq \min \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
\leq \max \left\{ \frac{\lambda_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}, \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\rho_k^{(d,q)}(f)} \right\} \\
\leq \limsup_{r_1, r_2, \dots, r_n \rightarrow 1} \frac{\log^{[b-1]} \log^{[b-1]} M_h^{-1}(M_{f \circ g}(r_1, r_2, \dots, r_n))}{\log^{[d-1]} M_k^{-1}(M_f \exp^{[q-n]}(r_1, r_2, \dots, r_n))} \\
\leq \frac{\rho_h^{(b,q+n-m)}(f \circ g)}{\lambda_k^{(d,q)}(f)}.
\end{aligned}$$

Proof. The proof can be followed as similar to Theorem 4.1. □

Remark 4.2. The Theorem 4.2 will also hold if we replace $\lambda_h^{(b,n)}(f \circ g)$, $\rho_h^{(b,n)}(f \circ g)$, $\lambda_k^{(d,q)}(f)$, $\rho_k^{(d,q)}(f)$, $\lambda_h^{(b,q+n-m)}(f \circ g)$ and $\rho_h^{(b,q+n-m)}(f \circ g)$ by $\tau_h^{(b,n)}(f \circ g)$, $\bar{\tau}_h^{(b,n)}(f \circ g)$, $\tau_k^{(d,q)}(f)$, $\bar{\tau}_k^{(d,q)}(f)$, $\tau_h^{(b,q+n-m)}(f \circ g)$ and $\bar{\tau}_h^{(b,q+n-m)}(f \circ g)$ respectively.

5 Conclusion

In this paper, we established inequalities between the relative order and the relative lower order of composite analytic functions of several complex variables in the unit polydisc.

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