ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, **Vol. 55(2) (2025), 90-95**

FIXED POINTS THEOREMS USING SPECIAL TYPES OF CONTRACTIONS IN MULTIPLICATIVE METRIC SPACES

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(Received: April 20, 2024; In format: May 14, 2024; Revised: October 09, 2025; Accepted: October 10, 2025)

DOI: https://doi.org/10.58250/jnanabha.2025.55211

Abstract

In this work, we construct fixed point theorems utilizing the extendibility of the Banach iteration approach to specific special contraction mappings with changes in domains has been studied. More precisely, we study an expanding sequence of subsets of a complete multiplicative metric space such that one item of the sequence is mapped into the next member by a map such that a contraction condition is fulfilled. Additionally, multiple fixed point findings are derived for different multiplicative contraction maps with multiplicative closed graphs.

2020 Mathematical Sciences Classification: 47H10, 54H25
Keywords and Phrases: Multiplicative metric spaces, Multiplicative closed graphs, Iteration technique.

1 Introduction and Preliminaries

Kirk *et al.* [8] proposed using a cycle of domains to develop distinct fixed point theorems for metric spaces. A metric space (Ω, Θ) including progressively more subsets $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ and a map $\Delta : \Omega \to \Omega$ that meets a contraction requirement, such that $\Delta(\Omega_i) \subseteq \Omega_{i+1}, \forall i$, and $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ were investigated by Moorthy and Raj in [9]. Furthermore, some works [15, 17] broaden the fixed point conclusions of [9]. The set of positive real numbers $\mathbb{R}^+$ is incomplete in standard metrics. In order to overcome this challenge, multiplicative metric spaces were first proposed in the following way by Bashirov *et al.* [2] in 2008.

Definition 1.1 ([2, 13]). *Given a function Θ and a non-empty set Ω, let $\Theta : \Omega \times \Omega \to \mathbb{R}^+$ completing the requirements listed below:*
(i) $\Theta(\zeta, \eta) \geq 1$ for all $\zeta, \eta \in \Omega$ and $\Theta(\zeta, \eta) = 1$ if and only if $\zeta = \eta$;
(ii) $\Theta(\zeta, \eta) = \Theta(\eta, \zeta)$ for all $\zeta, \eta \in \Omega$;
(iii) $\Theta(\zeta, \eta) \leq \Theta(\zeta, z).\Theta(z, \eta)$ for all $\zeta, \eta, z \in \Omega$ (multiplicative triangle inequality).
Θ is called a multiplicative metric on Ω and (Ω, Θ) is a multiplicative metric space.

Example 1.1 ([10]). Suppose $\mathbb{R}^n+$ be the collection of all n-tuples of positive real numbers. Suppose $\Theta^* : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined as follows:

$$\Theta^*(\zeta, \eta) = \left(\left| \frac{\zeta_1}{\eta_1} \right|^* \cdot \left| \frac{\zeta_2}{\eta_2} \right|^* \cdots \left| \frac{\zeta_n}{\eta_n} \right|^* \right),$$

where $\zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^{n+}$and $|| : .\mathbb{R}_+ \to \mathbb{R}_+$is defined by

$$|a|^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1 \end{cases}.$$

This makes it clear that the multiplicative metric's constraints are all fulfilled.

Example 1.2 ([14])**.** Suppose $\Theta : \mathbb{R} \times \mathbb{R} \to [1, \infty)$ be defined as $\Theta(\zeta, \eta) = a^{|\zeta-\eta|}$, where $\zeta, \eta \in \mathbb{R}$ and $a > 1$. Then $\Theta(\zeta, \eta)$ is referred to as a multiplicative metric space, and (Ω, Θ) is a multiplicative metric. These may be referred to as the usual multiplicative metric spaces.

Example 1.3 ([14])**.** Suppose (Ω, Θ) be a multiplicative metric space . Define a mapping Θ_a on Ω by $\Theta_a(\zeta, \eta) = a^{\Theta(\zeta,\eta)}$,where $a > 1$ is a real number and
$\Theta_a(\zeta, y) = a^{\Theta(\zeta,\eta)} = \begin{cases} 1 & \text{if } \zeta = \eta \\ a & \text{if } \zeta \neq \eta. \end{cases}$ When Ω and Θ_a are combined then $\Theta_a(\zeta, \eta)$ becomes a discrete multiplicative metric space .

Example 1.4 ([1])**.** Assume that all real-valued multiplicative continuous functions over $[a, b] \subseteq \mathbb{R}^+$ are collected in $\Omega = C^*[a, b]$.Then, for each $f, g \in \Omega$, $\Theta(f, g) = \sup_{x\in[a,b]} \left|\frac{f(x)}{g(x)}\right|$ is a multiplicative metric space with Θ specified. For a thorough explanation of multiplicative metric topology, see ([10]).

Definition 1.2 ([2])**.** *Let (Ω, Θ) be a metric space that is multiplicative.A sequence $\{\zeta_n\}$ in Ω is said to be a*
(i) multiplicative convergent sequence to ζ, if, for any multiplicative open ball
$B_\epsilon(\zeta) = \{\eta \mid \Theta(\zeta, \eta) < \epsilon\}, \epsilon > 1$, there exists a natural number N such that $\zeta_n \in B_\epsilon(\zeta)$ for all $n \geq N$, i. e, $\Theta(\zeta_n, \zeta) \to 1$ as $n \to \infty$.
(ii) multiplicative Cauchy sequence if for all $\epsilon > 1$, there exists $N \in \mathbb{N}$ such that
$\Theta(\zeta_n, \zeta_m) < \epsilon$ for all $m, n > N$ i. e, $\Theta(\zeta_n, \zeta_m) \to 1$ as $n \to \infty$.
(iii) If all multiplicative Cauchy sequences in Ω are multiplicatively convergent to $\zeta \in \Omega$, then Ω is the complete multiplicative metric spaces.

The notion of multiplicative contraction mapping was introduced and various fixed point theorems for these mappings in complete multiplicative metric spaces were shown by Ozavsar [10] in 2012.

Definition 1.3 ([8])**.** *Suppose (Ω, Θ) be a multiplicative metric space. The map $f : \Omega \to \Omega$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0, 1)$ such that*

$$\Theta(f(\zeta_1), f(\zeta_2)) \leq \Theta(\zeta_1, \zeta_2)^\lambda \text{ for all } \zeta, \eta \in \Omega.$$

Definition 1.4 ([10])**.** *Suppose that $\Delta : (\Omega, \Theta, s) \to (\Omega, \Theta, s)$ is a self-mapping on a $b-$ multiplicative metric space (Ω, Θ, s). If whenever $\kappa_n \to \kappa_0$ and $\Delta\kappa_n \to \iota_0$ for some sequence $\{\kappa_n\}$ in Ω and some κ_0, ι_0 in Ω, we have $\iota_0 = \Delta\kappa_0$, then Δ is said to have a multiplicative closed graph.*

2 Main Results

Theorem 2.1. *Let (Ω, Θ) be a complete multiplicative metric space and suppose that the multiplicative closed graph of $\Delta : \Omega \to \Omega$ exists. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = U_{j=1}^{\infty}\Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and*

$$\Theta(\Delta\kappa, \Delta z) \leq \Theta(\kappa, z)^{a_i} \cdot (\Theta(\Delta z, t))^{b_i} \cdot (\Theta(\Delta\kappa, z))^{c_i} \cdot (\Theta(\Delta z, z))^{d_i} \cdot$$
$$\left[\frac{\Theta(z, \Delta z)\Theta(\kappa, \Delta\kappa)}{\Theta(\kappa, z)}\right]^{e_i} \cdot \left[\frac{\Theta(z, \Delta\kappa)\Theta(\kappa, \Delta\kappa)}{\Theta(\kappa, z)\Theta(z, \Delta z)}\right]^{f_i}$$

for all $\kappa, z \in \Omega_i$, for all i, where $a_i, b_i, c_i, d_i, e_i, f_i \in \left(0, \frac{1}{2}\right)$ are real positive constants such that $a_i + 2b_i + 2c_i + d_i + e_i + f_i < 1$, for all i, and $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{a_{n+1}+c_{n+1}+d_{n+1}-f_{n+1}}{1-c_{n+1}-e_{n+1}-2f_{n+1}}$, for all i.
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n\kappa_1\}$ multiplicative converges to the unique fixed point.

Proof. Fix $\kappa_1 \in \Omega_1$, and set $\kappa_{n+1} = \Delta\kappa_n = \Delta^n\kappa_1, \forall n = 1, 2, 3, \ldots$ Then we have

$$\Theta\left(\Delta^{n+1}\kappa_1, \Delta^n\kappa_1\right) \leq \Theta\left(\Delta^n\kappa_1, \Delta^{n-1}\kappa_1\right)^{a_{n+1}} \cdot \left(\Theta\left(\Delta^n\kappa_1, \Delta^n\kappa_1\right)\right)^{b_{n+1}}$$
$$\cdot \left(\Theta\left(\Delta^{n+1}\kappa_1, \Delta^{n-1}\kappa_1\right)\right)^{c_{n+1}} \cdot \left(\Theta\left(\Delta^n\kappa_1, \Delta^{n-1}\kappa_1\right)\right)^{d_{n+1}} \cdot$$
$$\left[\frac{\Theta\left(\Delta^{n-1}\kappa_1, \Delta^n\kappa_1\right)\Theta\left(\Delta^n\kappa_1, \Delta^{n+1}\kappa_1\right)}{\Theta\left(\Delta^n\kappa_1, \Delta^{n-1}\kappa_1\right)}\right]^{e_{n+1}} \cdot \left[\frac{\Theta\left(\Delta^{n-1}\kappa_1, \Delta^{n+1}\kappa_1\right)\Theta\left(\Delta^n\kappa_1, \Delta^{n+1}\kappa_1\right)}{\Theta\left(\Delta^n\kappa_1, \Delta^{n-1}\kappa_1\right)\Theta\left(\Delta^{n-1}\kappa_1, \Delta^n\kappa_1\right)}\right]^{f_{n+1}}.$$

Since $\Theta\left(\Delta^n\kappa_1,\Delta^n\kappa_1\right)=1$, we get

$$\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\leq\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{a_{n+1}}\cdot\left(\Theta\left(\Delta^{n+1}\kappa_1,\Delta^{n-1}\kappa_1\right)\right)^{c_{n+1}}\cdot\left(\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right)^{d_{n+1}}\cdot$$

$$\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\right]^{e_{n+1}}\cdot\left[\frac{\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)}{\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)}\right]^{f_{n+1}}$$

$$\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\leq\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{a_{n+1}}\cdot\left(\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right)^{c_{n+1}}\cdot\left(\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right)^{d_{n+1}}\cdot$$

$$\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\right]^{e_{n+1}}\cdot\left[\frac{\left(\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\right)^2}{\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)}\right]^{f_{n+1}}$$

Now, we get

$$\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\leq\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{\left(\frac{a_{n+1}+c_{n+1}+d_{n+1}-f_{n+1}}{1-c_{n+1}-e_{n+1}-2f_{n+1}}\right)}$$

$$i.e.\quad\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{p_{n+1}}\leq\Theta\left(\Delta\kappa_1,\kappa_1\right)^{p_{n+1}p_np_{n-1}\dots p_2}.$$

Further, for $1\leq n<m$, we have

$$\Theta\left(\Delta^m\kappa_1,\Delta^n\kappa_1\right)\leq\Theta\left(\Delta^m\kappa_1,\Delta^{m-1}\kappa_1\right)^{m-1}\Theta\left(\Delta^{m-1}\kappa_1,\Delta^{m-2}\kappa_1\right)^{m-2}\dots\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)^n$$
$$\leq\Theta\left(\Delta\kappa_1,\kappa_1\right)\left(\sum_{i=n}^{m-1}p_2p_3\dots p_{i+1}\right).$$

Therefore, $\Theta\left(\Delta^m\kappa_1,\Delta^n\kappa_1\right)\to 1(m,n\to\infty)$. Then $\{\Delta^m\kappa_1\}_{m=1}^{\infty}$ is an multiplicative Cauchy Sequence in Ω. Consider that $\{\Delta^m\kappa_1\}_{m=1}^{\infty}$ in Ω, which is multiplicative complete,multiplicative convergence to κ^*.In addition, the multiplicative Cauchy sequence $\left\{\Delta^{m+1}\kappa_1\right\}_{m=1}^{\infty}$ multiplicative converges to κ^* in Ω.
Further, $\Delta\kappa^*=\kappa^*$is obtained from the multiplicative closed graph of Δ
Thus, κ^* represents a fixed point for Δ.
The general scenario, where $\kappa_1\in\Omega_n$, for some n, may be handled by these procedures.
If $\Delta\kappa^*=\kappa^*,\Delta z^*=z^*$ in Δ, then let $\kappa^*,z^*\in\Omega_n$, for some n, so we have

$$\begin{aligned}1&\leq\Theta\left(\kappa^*,z^*\right)=\Theta\left(\Delta\kappa^*,\Delta z^*\right)\leq\Theta\left(\kappa^*,z^*\right)^{a_i}\cdot\left(\Theta\left(\Delta z^*,\kappa^*\right)\right)^{b_i}\cdot\\&\left(\Theta\left(\Delta\kappa^*,z^*\right)\right)^{c_i}\cdot\left(\Theta\left(\Delta z^*,z^*\right)\right)^{d_i}\\&\left[\frac{\Theta\left(z^*,\Delta z^*\right)\Theta\left(\kappa^*,\Delta\kappa^*\right)}{\Theta\left(\kappa^*,z^*\right)}\right]^{e_i}\cdot\left[\frac{\Theta\left(z^*,\Delta\kappa^*\right)\Theta\left(\kappa^*,\Delta\kappa^*\right)}{\Theta\left(\kappa^*,z^*\right)\Theta\left(z^*,\Delta z^*\right)}\right]^{f_i}\\&=\Theta\left(\kappa^*,z^*\right)^{a_n}\cdot\left(\Theta\left(z^*,\kappa^*\right)\right)^{b_n}\cdot\\&\left(\Theta\left(\kappa^*,z^*\right)\right)^{c_n}\cdot\left(\Theta\left(z^*,z^*\right)\right)^{d_n}\cdot\\&\left[\frac{\Theta\left(z^*,z^*\right)\Theta\left(\kappa^*,\kappa^*\right)}{\Theta\left(\kappa^*,z^*\right)}\right]^{e_n}\cdot\left[\frac{\Theta\left(z^*,\kappa^*\right)\Theta\left(\kappa^*,\kappa^*\right)}{\Theta\left(\kappa^*,z^*\right)\Theta\left(z^*,z^*\right)}\right]^{f_n}\\&\leq\Theta\left(\kappa^*,z^*\right)^{\left(a_n+b_n+c_n-e_n\right)}.\end{aligned}$$

Then, $\Theta\left(\kappa^*,z^*\right)\leq\Theta\left(\kappa^*,z^*\right)^{\left(a_n+b_n+c_n-e_n\right)^m},\forall m\in\mathbb{N}$. Since $\left(a_n+b_n+c_n-e_n\right)^m\to 0$ as $m\to\infty,\Theta\left(\kappa^*,z^*\right)=1$ and $\kappa^*=z^*$. Hence, Δ has a unique fixed point. □

Corollary 2.1. *Putting $b_i=c_i=d_i=e_i=f_i=0$ in Theorem 2.1 gives Banach-contraction [11] in the sense of multiplicative metric spaces. Suppose that (Ω,Θ) is a complete multiplicative metric space, and $\Delta:\Omega\to\Omega$ have a multiplicative closed graph. Let $\Omega_1\subseteq\Omega_2\subseteq\cdots$ be subsets of Ω such that $\Omega=\bigcup_{j=1}^{\infty}\Omega_j,\Delta\left(\Omega_i\right)\subseteq\Omega_{i+1},\forall i$, and $\Theta(\Delta\kappa,\Delta z)\leq\Theta(\kappa,z)^{a_i}$ for all $x,y\in\Omega$. for all $\kappa,z\in\Omega_i$, for all i, where $a_i\in(0,1)$ and $\sum_{n=1}^{\infty}p_1p_2\dots p_n<\infty$. Then Δ has a unique fixed point in Ω.*

Moreover, for any fixed $\kappa_1\in\Omega,\{\Delta^n\kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.2. *Putting $b_i = c_i = d_i = f_i = 0, a_i = e_i$ in Theorem 2.1 gives Kannan-contraction* [11] *in the sense of multiplicative metric spaces. Suppose that (Ω, Θ) is a complete multiplicative metric space, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \cup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq (\Theta(\Delta\kappa, \kappa)\Theta(\Delta z, z))^{a_i}$, for all $\kappa, z \in \Omega_i$, for all i, where $a_i \in (0,1)$ are real positive constants such that $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{a_i}{1-a_i}, \forall i$.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.3. *Putting $a_i = d_i = e_i = f_i = 0, b_i = c_i$ in Theorem 2.1 gives Chatterjea-contraction* [11] *in the sense of multiplicative metric spaces. Suppose that (Ω, Θ) is a complete multiplicative metric space, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq (\Theta(\Delta\kappa, z)\Theta(\Delta z, \kappa))^{b_i}, \forall \kappa, z \in \Omega_i$, for all i, where $b_i \in \left(0, \frac{1}{2}\right)$ are real positive constants such that $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{b_i}{1-b_i}$, for all i.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.4. *Putting $b_i = c_i = f_i = 0$ in Theorem 2.1 gives Kholi results* [7] *in the sense of multiplicative metric spaces as follows: Suppose that (Ω, Θ) is a complete multiplicative metric spaces, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq \Theta(\kappa, z)^{a_i} (\Theta(\Delta z, z))^{d_i} \left[\frac{\Theta(z,\Delta z)\Theta(\kappa,\Delta\kappa)}{\Theta(\kappa,z)}\right]^{e_i}$, for all $\kappa, z \in \Omega_i$, for all i, where $a_i, d_i, e_i \in (0,1)$ are real positive constants such that $a_i + d_i + e_i < 1, \forall i$, and $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{a_{n+1}+d_{n+1}}{1-e_{n+1}}, \forall i$.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.5. *Putting $d_i = e_i = f_i = 0$ in Theorem 2.1 gives Isufati results* [5] *in the sense of multiplicative metric spaces as follows: Suppose that (Ω, Θ) is a complete multiplicative metric spaces, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \cup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq \Theta(\kappa, z)^{a_i} (\Theta(\Delta z, t))^{b_i} (\Theta(\Delta\kappa, z))^{c_i}$, for all $\kappa, z \in \Omega_i$, for all i, where $a_i, b_i, c_i \in (0,1)$ are real positive constants such that $a_i + 2b_i + 2c_i < 1$, for all i, and $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{a_{n+1}+c_{n+1}}{1-c_{n+1}}, \forall i$.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.6. *Putting $b_i = c_i = d_i = f_i = 0$ in Theorem 2.1 gives Jaggi results* [6] *in the sense of multiplicative metric spaces as follows: Suppose that (Ω, Θ) is a complete multiplicative metric spaces, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq \Theta(\kappa, z)^{a_i} \left[\frac{\Theta(z,\Delta z)\Theta(\kappa,\Delta\kappa)}{\Theta(\kappa,z)}\right]^{e_i}$, for all $\kappa, z \in \Omega_i$, for all i, where $a_i, e_i \in (0,1)$ are real positive constants such that $a_i + e_i < 1$, for all i, and $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$, where $p_i = \frac{a_{n+1}}{1-e_{n+1}}, \forall i$.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Corollary 2.7. *Putting $d_i = e_i = f_i = 0, b_i = c_i$ in Theorem 2.1 gives Reich results* [12] *in the sense of multiplicative metric spaces as follows: Suppose that (Ω, Θ) is a complete multiplicative metric spaces, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq \Theta(\kappa, z)^{a_i} (\Theta(\Delta z, t)\Theta(\Delta\kappa, z))^{b_i}$, for all $\kappa, z \in \Omega_i$, for all i, where $a_i, b_i \in (0,1)$ are real positive constants such that $a_i + 2b_i < 1$, for all i, and $\sum_{n=1}^{\infty} p_1 p_2 \ldots p_n < \infty$.*
Then Δ has a unique fixed point in Ω.
Moreover, for any fixed $\kappa_1 \in \Omega, \{\Delta^n \kappa_1\}$ multiplicative converges to the unique fixed point.

Theorem 2.2. *Suppose that (Ω, Θ) is a complete multiplicative metric space, and $\Delta : \Omega \to \Omega$ have a multiplicative closed graph. Let $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be subsets of Ω such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j, \Delta(\Omega_i) \subseteq \Omega_{i+1}$, for all i, and $\Theta(\Delta\kappa, \Delta z) \leq [\Theta(t, \Delta\kappa) \cdot \Theta(z, \Delta z)]^{a_i} \cdot [\Theta(t, \Delta z) \cdot \Theta(z, \Delta\kappa)]^{b_i} \cdot [\Theta(t,\ z)]^{c_i} \cdot \left[\frac{\Theta(t,\Delta\kappa)\Theta(z,\Delta z)}{\Theta(t,z)}\right]^{d_i}$.*

$\left\{\max\left\{\Theta(t,\Delta\kappa),\Theta(z,\Delta z),\Theta(t,\Delta z),\Theta(z,\Delta\kappa),\frac{\Theta(t,\Delta\kappa)\cdot\Theta(z,\Delta z)\cdot\Theta(z,\Delta\kappa).}{\Theta(t,z)}\right\}\right\}^{e_i}$, *for all* $\kappa, z \in \Omega_i$, *for all* i, *where* $a_i, b_i, c_i, d_i, e_i \in \left(0,\frac{1}{2}\right)$ *are real positive constants such that* $2a_i + 2b_i + c_i + d_i + 3e_i < 1$, *for all* i, *and* $\sum_{n=1}^{\infty} p_1 p_2 \dots p_n < \infty$, *where* $p_i = \frac{a_i+b_i+c_i+e_i}{1-a_i-b_i-d_i-2e_i}$, *for all* i. *Then* Δ *has a unique fixed point in* Ω. *Moreover, for any fixed* $\kappa_1 \in \Omega, \{\Delta^n\kappa_1\}$ *multiplicative converges to the unique fixed point.*

Proof. Fix $\kappa_1 \in \Omega_1$, and set $\kappa_{n+1} = \Delta\kappa_n = \Delta^n\kappa_1$, for all $n = 1, 2, 3, \dots$. Then we have
$\Theta\left(\Delta^{n+1}\kappa_1, \Delta^n\kappa_1\right) \le$
$\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right).\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)\right]^{a_{n+1}}\cdot \left[\Theta\left(\Delta^n\kappa_1,\Delta^n\kappa_1\right).\Theta\left(\Delta^{n-1}\kappa_1,\Delta^{n+1}\kappa_1\right)\right]^{b_{n+1}}\cdot$
$\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right]^{c_{n+1}}\cdot\left[\frac{\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)}{\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)}\right]^{d_{n+1}}\cdot$

$$\left[\max\left\{\begin{array}{c}\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right),\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right),\Theta\left(\Delta^n\kappa_1,\Delta^n\kappa_1\right),\Theta\left(\Delta^{n-1}\kappa_1,\Delta^{n+1}\kappa_1\right)\\ \frac{\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\cdot\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)\cdot\Theta\left(\Delta^{n-1}\kappa_1,\Delta^{n+1}\kappa_1\right)}{\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)}\end{array}\right\}\right]^{e_{n+1}}.$$

Since $\Theta\left(\Delta^n\kappa_1,\Delta^n\kappa_1\right) = 1$, we get
$\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right) \le \left[\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right).\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right)\right]^{a_{n+1}}\cdot$
$\left[\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\cdot\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right]^{b_{n+1}}\cdot\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\right]^{c_{n+1}}\cdot\left[\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\right]^{d_{n+1}}\cdot$

$$\left[\max\left\{\begin{array}{c}\Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right),\Theta\left(\Delta^{n-1}\kappa_1,\Delta^n\kappa_1\right),1,\\ \Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\cdot\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right),\\ \Theta\left(\Delta^n\kappa_1,\Delta^{n+1}\kappa_1\right)\cdot\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)\cdot\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)\end{array}\right\}\right]^{e_{n+1}}.$$

Now, we get
$\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right) \le \Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{\left(\frac{a_{n+1}+b_{n+1}+c_{n+1}+e_{n+1}}{1-a_{n+1}-b_{n+1}-d_{n+1}-2e_{n+1}}\right)}$
$\Theta\left(\Delta^n\kappa_1,\Delta^{n-1}\kappa_1\right)^{p_{n+1}} \le \Theta\left(\Delta\kappa_1,\kappa_1\right)^{p_{n+1}p_np_{n-1}\dots p_2}$.
Further, for $1 \le n < m$, we have

$$\begin{aligned}\Theta\left(\Delta^m\kappa_1,\Delta^n\kappa_1\right) &\le \Theta\left(\Delta^m\kappa_1,\Delta^{m-1}\kappa_1\right)^{m-1}\Theta\left(\Delta^{m-1}\kappa_1,\Delta^{m-2}\kappa_1\right)^{m-2}\dots\Theta\left(\Delta^{n+1}\kappa_1,\Delta^n\kappa_1\right)^n\\ &\le \Theta\left(\Delta\kappa_1,\kappa_1\right)\left(\sum_{i=n}^{m-1}p_2p_3\dots p_{i+1}\right).\end{aligned}$$

Therefore, $\Theta\left(\Delta^m\kappa_1,\Delta^n\kappa_1\right) \to 1(m,n\to\infty)$. Then $\{\Delta^m\kappa_1\}_{m=1}^{\infty}$ is an multiplicative Cauchy Sequence in Ω. Let $\{\Delta^m\kappa_1\}_{m=1}^{\infty}$ multiplicative converge to κ^* in Ω, which is multiplicative complete. Also, $\left\{\Delta^{m+1}\kappa_1\right\}_{m=1}^{\infty}$ is an multiplicative Cauchy Sequence and it multiplicative converges to κ^* in Ω. Also, multiplicative closed graph of Δ gives $\Delta\kappa^* = \kappa^*$. Hence, we obtained a fixed point κ^* of Δ. These processes can be extended to the general case: $\kappa_1 \in \Omega_n$, for some n.
If $\Delta\kappa^* = \kappa^*, \Delta z^* = z^*$ in Δ, then let $\kappa^*, z^* \in \Omega_n$, for some n, so we have

$$1 \le \Theta\left(\kappa^*,z^*\right) = \Theta(\Delta\kappa^*,\Delta z^*) \le \left[\Theta(t,\Delta\kappa^*)\cdot\Theta(z^*,\Delta z^*)\right]^{a_i}$$

$$\cdot\left[\Theta(t,\Delta z^*)\cdot\Theta(z^*,\Delta\kappa^*)\right]^{b_i}\cdot\left[\Theta(t,z^*)\right]^{c_i}\cdot\left[\frac{\Theta(t,\Delta\kappa^*)\Theta(z^*,\Delta z^*)}{\Theta(t,z^*)}\right]^{d_i}.$$

$$\left\{\max\left\{\Theta(t,\Delta\kappa^*),\Theta(z^*,\Delta z^*),\Theta(t,\Delta z^*),\Theta(z^*,\Delta\kappa^*),\frac{\Theta(t,\Delta\kappa^*)\cdot\Theta(z^*,\Delta z^*)\cdot\Theta(z^*,\Delta\kappa^*).}{\Theta(t,z^*)}\right\}\right\}^{e_i}.$$

Then, $\Theta\left(\kappa^*,z^*\right) \le \Theta\left(\kappa^*,z^*\right)^{(2b_n+c_n-d_n+e_n)^m}$, for all $m \in \mathbb{N}$.
Since $(2b_n+c_n-d_n+e_n)^m \to 0$ as $m\to\infty, \Theta\left(\kappa^*,z^*\right) = 1$ and $\kappa^* = z^*$.
Hence, Δ has a unique fixed point. □

Corollary 2.8. *Putting* $b_i = c_i = d_i = e_i = 0$ *in Theorem* 2.2 *gives Kannan-contraction* [11] *in the sense of multiplicative metric spaces.*

Corollary 2.9. *Putting* $b_i = d_i = e_i = 0$ *in Theorem* 2.2 *gives Fisher-contraction* [4] *in the sense of multiplicative metric spaces.*

Corollary 2.10. *Putting* $a_i = c_i = d_i = e_i = 0$ *in Theorem* 2.2 *gives Chatterjea-contraction* [11] *in the sense of multiplicative metric spaces.*

Corollary 2.11. *Putting* $a_i = b_i = d_i = e_i = 0$ *in Theorem* 2.2 *gives Banach-contraction* [11] *in the sense of multiplicative metric spaces.*

Corollary 2.12. *Putting* $d_i = e_i = 0$ *in Theorem* 2.2 *gives Ciric-contraction* [3] *in the sense of multiplicative metric spaces.*

Corollary 2.13. *Putting* $a_i = d_i = e_i = 0$ *in Theorem* 2.2 *gives Reich-contraction* [12] *in the sense of multiplicative metric spaces.*

Corollary 2.14. *Putting* $a_i = b_i = e_i = 0$ *in Theorem* 2.2 *gives Jaggi-contraction* [6] *in the sense of multiplicative metric spaces.*

Example 2.1. Let $\Xi = \left[\frac{1}{4}, \infty\right)$ and $\Theta(\zeta, \eta) = a^{|\zeta - \eta|}$, where $\zeta, \eta \in \mathbb{R}$ and $a > 1$. Then $\Theta(\zeta, \eta)$ is referred to as a multiplicative metric space, and (Ξ, Θ) is a complete multiplicative metric. Assuming that $\Xi_n = \left[\frac{1}{4}, n\right]$, and $\xi_n = \frac{n^2}{(n+1)^2} \in \left[\frac{1}{4}, 1\right)$, for $n = 1, 2, 3, \ldots$. Then $\sum_{n=1}^{\infty} \xi_1 \xi_2 \ldots \xi_n < \infty$. Define $P : \Xi \to \Xi$ by $Pu = u^{\frac{1}{4}}$, if $u \in \Xi_n$, for $n \in \mathbb{N}$.Then all hypotheses of Theorem 2.1 and Theorem 2.2 are satisfied. Moreover, Δ has a unique fixed point 1 .

3 Conclusion

Equations like differential, integral and algebraic equations are typically solved using fixed point iteration techniques. Thus, with relation to fixed point iteration techniques, we must discover all possible fixed point results.

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