ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

*Jñānābha*, Vol. 55(2) (2025), 58-66

# THE GRÜSS TYPE FRACTIONAL INTEGRAL INEQUALITIES OF MULTIVARIATE MITTAG-LEFFLER FUNCTION

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*(Received: February* 04, 2025*; In format: February* 12, 2025*; Revised: September* 21, 2025*; Accepted: October* 01, 2025*)*

DOI: https://doi.org/10.58250/jnanabha.2025.55208

**Abstract**

Integral inequalities are becoming more popular because they are useful in many areas. The researchers have studied integral inequalities using various methods. In this paper, we want to create new integral inequalities such as Grüss type inequalities, and other similar ones, for the fractional integral operator that uses the multivariate Mittag-Leffler function. We look at how the Riemann-Liouville integral, the Prabhakar integral, and the generalized fractional integral are related to make specific conclusions. We support our findings by giving additional details.

**2020 Mathematical Sciences Classification:** 26A33, 26D10.
**Keywords and Phrases:** Mittag-Leffler function, inequalities, extended fractional integrals.

## 1 Introduction

Fractional calculus is the area of study that looks at integrals and derivatives of any number. We can use differential equations with fractional derivatives to solve various problems in statistics, mathematics, engineering, chemistry, and biology. Riemann, Liouville, Caputo, Riesz, Hilfer, Hardmard, Erdélyi-Kober, Saigo-Maeda, and others have researched different kinds of fractional integrals. The idea of fractional conformable derivative operators was proposed by Khalil *et al.* [11]. The feature of the fractional conformable derivative operators was given by Abdeljawad and Baleanu [2]. The fractional conformable integral and derivative operator was suggested by Jarad *et al.* [9]. Atangana and Baleanu, along with other researchers, studied various types of fractional derivative operators. Researchers in fractional calculus have recently created and explored new fractional integral and derivative operators.

Studying inequalities is an important area of research in mathematical analysis. The inequality method is a helpful tool for studying special functions and approximation theory. Fractional integral inequalities are useful in numerical methods, transform theory, and tasks related to probability and statistics. Recently, many researchers have studied these inequalities, leading to various generalizations, extensions, and versions in several works [1, 3, 7, 8, 12, 17, 18, 19, 23].

Grüss [5] proved an important and recognized inequality that connects the integral of the product of two functions with the product of their individual integrals. The Grüss inequality [5] is given as follows:

Let $f_1$ and $f_2$ be two continuous functions defined on $[r, s]$ such that $m \leq f_1(t) \leq M$ and $n \leq f_2(t) \leq N$ for all $t \in [r, s]$ and some real constants $M, m, N, n \in \mathbb{R}$. Then the following inequality is true:

$$\left| \frac{1}{(s-r)} \int_r^s f_1(t) f_2(t)\, dt - \frac{1}{(s-r)^2} \int_r^s f_1(t)\, dt \int_r^s f_2(t)\, dt \right| \leq \frac{1}{4}(M-m)(N-n). \tag{1.1}$$

Here, $\frac{1}{4}$ is the best constant for making the inequality as precise as possible.

**Definition 1.1** ([10, 25]). *A function $f_1(t)$ is in the space $L_{p,r}[0,\infty)$ if the following condition holds: The norm of $f_1$ in this space, denoted as $\|f_1\|_{L_{p,r}[0,\infty)}$, is defined as:*

$$L_{p,r}[0,\infty) = \left\{ f_1 : \|f_1\|_{L_{p,r}[0,\infty)} = \left( \int_r^s |f_1(t)|^p t^r \, dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty,\ r \geq 0 \right\}. \tag{1.2}$$

*If we use (1.2) for $r = 0$, we get the following: The set of functions $f_1$ belongs to the space $L_p[0,\infty)$ if the p-norm of $f_1$ is finite. This means that:*

$$L_p[0,\infty) = \left\{ f_1 : \|f_1\|_{L_p[0,\infty)} = \left( \int_0^s |f_1(t)|^p \, dt \right)^{1/p} < \infty, \quad 1 \le p < \infty \right\}.$$

**Definition 1.2** ([16]). *Prabhakar presented the three-parameter Mittag-Leffler function in the following form:*

$$\mathcal{E}^{\gamma}_{\sigma,\zeta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\sigma n + \zeta) n!},$$

*where* $n \in \mathbb{N}$ and $\gamma, \sigma, \zeta \in \mathbb{C}$ such that $\Re(\zeta) > 0,\ \Re(\sigma) > 0,\ \Re(\gamma) > 0$.

**Definition 1.3** ([6]). *Gurjar and others created a version of the Mittag-Leffler function that can handle multiple variables, and they described it as follows:*

$$\begin{aligned} \mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta,q_j)}(z_1,\ldots,z_m) &= \mathcal{E}^{\gamma_1,\ldots,\gamma_m;\, l_1,\ldots,l_m;\, p_1,\ldots,p_m}_{\sigma_1,\ldots,\sigma_m;\, \zeta;\, q_1,\ldots,q_m}(z_1,\ldots,z_m) \\ &= \sum_{r_1,\ldots,r_m=0}^{\infty} \frac{(\gamma_1)_{p_1 r_1} \cdots (\gamma_m)_{p_m r_m}}{\Gamma\big(\zeta + \sum_{j=1}^{m} \sigma_j r_j\big)} \frac{z_1^{r_1} \cdots z_m^{r_m}}{(l_1)_{q_1 r_1} \cdots (l_m)_{q_m r_m}}. \end{aligned} \tag{1.3}$$

*Here* $\gamma_j, \sigma_j, \zeta, l_j \in \mathbb{C}$ *with*

$$\min_{1 \le j \le m} \{\Re(\zeta), \Re(\sigma_j), \Re(\gamma_j), \Re(l_j)\} > 0, \quad p_j, q_j > 0, \quad p_j < q_j + \Re(\sigma_j), \quad j = 1, 2, \ldots, m.$$

**Definition 1.4** ([13, 20]). *Let $f \in L[a,b]$. The left-sided and right-sided RiemannLiouville fractional integrals of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$(I^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a,$$

and

$$(I^{\alpha}_{a-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^a (t-x)^{\alpha-1} f(t) \, dt, \quad x < a.$$

**Definition 1.5** ([16]). *The Prabhakar type fractional integral is defined as*

$$({}_a I^{\gamma,\lambda}_{\sigma,\zeta} f)(x) = \int_a^x (x-t)^{\zeta-1} \mathcal{E}^{\gamma}_{\sigma,\zeta}\big(\lambda(x-t)^{\sigma}\big) f(t) \, dt.$$

**Definition 1.6.** *The one-sided Prabhakar type fractional integral is expressed as*

$$(I^{\gamma,\lambda}_{\sigma,\zeta} f)(x) = \int_0^x (x-t)^{\zeta-1} \mathcal{E}^{\gamma}_{\sigma,\zeta}\big(\lambda(x-t)^{\sigma}\big) f(t) \, dt. \tag{1.4}$$

**Definition 1.7** ([15, 21]). *The Prabhakar type fractional integral operator having a multivariate Mittag-Leffler function in the kernel is defined as*

$$\begin{aligned} ({}_a I^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta} f)(x) &= ({}_a I^{(\gamma_1,\ldots,\gamma_j),(\lambda_1,\ldots,\lambda_j)}_{(\sigma_1,\ldots,\sigma_j),\zeta} f)(x) \\ &= \int_a^x (x-t)^{\zeta-1} \mathcal{E}^{(\gamma_1,\ldots,\gamma_j)}_{(\sigma_1,\ldots,\sigma_j),\zeta}\big\{\lambda_1(x-t)^{\sigma_1} \cdots \lambda_j(x-t)^{\sigma_j}\big\} f(t) \, dt, \end{aligned}$$

where $\zeta, \sigma_i, \lambda_i, \gamma_i \in \mathbb{C}$ with $\Re(\zeta) > 0,\ \Re(\sigma_i) > 0,\ \Re(\gamma_i) > 0$ for $i = 1, 2, \ldots, j$.

**Definition 1.8.** *The one-sided Prabhakar type fractional integral with a multivariate Mittag-Leffler function in its kernel is defined as*

$$\begin{aligned} (I^{(\gamma)_j,(\lambda)_j}_{(\sigma)_j,\zeta} f)(x) &= (I^{(\gamma_1,\ldots,\gamma_j),(\lambda_1,\ldots,\lambda_j)}_{(\sigma_1,\ldots,\sigma_j),\zeta} f)(x) \\ &= \int_0^x (x-t)^{\zeta-1} \mathcal{E}^{(\gamma_1,\ldots,\gamma_j)}_{(\sigma_1,\ldots,\sigma_j),\zeta}\big\{\lambda_1(x-t)^{\sigma_1} \cdots \lambda_j(x-t)^{\sigma_j}\big\} f(t) \, dt, \end{aligned} \tag{1.5}$$

*where* $\zeta, \sigma_i, \lambda_i, \gamma_i \in \mathbb{C}$ *with* $\Re(\zeta) > 0,\ \Re(\sigma_i) > 0,\ \Re(\gamma_i) > 0$ for $i = 1, 2, \ldots, j$.

**Definition 1.9** [6]. *The integral operator having a multivariable Mittag-Leffler function in the kernel is defined by*

$$\begin{aligned} ({}_x I^{\gamma_j,l_j,p_j}_{\sigma_j,\zeta,q_j;\lambda_j} f)(x) &= ({}_x I^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m),\zeta,(q_1,\ldots,q_m);(\lambda_1,\ldots,\lambda_m)} f)(x) \\ &= \int_x^{\xi} (x-t)^{\zeta-1} \mathcal{E}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m),\zeta,(q_1,\ldots,q_m)}\big\{\lambda_1(x-t)^{\sigma_1} \cdots \lambda_j(x-t)^{\sigma_j}\big\} f(t) \, dt, \end{aligned} \tag{1.6}$$

*where* $\zeta, \sigma_j, \lambda_j, \gamma_j, l_j \in \mathbb{C}$,

$$\min_{1\le j\le m}\{\Re(\zeta), \Re(\sigma_j), \Re(\gamma_j), \Re(l_j)\} > 0, \quad p_j, q_j > 0, \quad p_j < q_j + \Re(\sigma_j), \quad j = 1, 2, \ldots, m.$$

**Definition 1.10.** *The one-sided integral operator of* (1.6) *having a multivariate Mittag-Leffler function in the kernel is defined by*

$$\begin{aligned}({}_xI^{\gamma_j,l_j,p_j}_{\sigma_j,\zeta,q_j;\lambda_j} f)(x) &= ({}_xI^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m),\zeta,(q_1,\ldots,q_m);(\lambda_1,\ldots,\lambda_m)} f)(x)\\ &= \int_0^x (x-t)^{\zeta-1}\mathcal{E}^{(\gamma_1,\ldots,\gamma_m),(l_1,\ldots,l_m),(p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m),\zeta,(q_1,\ldots,q_m)}\big\{\lambda_1(x-t)^{\sigma_1}\cdots\lambda_j(x-t)^{\sigma_j}\big\} f(t)\,dt.\end{aligned} \tag{1.7}$$

## 2 The Grüss Type Generalized Fractional Integral Inequality

In this section, we present the Grüss type and several other related inequalities involving multivariate Mittag-Leffler function by utilizing the generalized fractional integral operator (1.7).

**Theorem 2.1.** *Let the function* $h_1$ *be integrable on* $[0, \infty)$*. If the two functions* $\aleph_1$ *and* $\aleph_2$ *can be integrated over the range* $[0, \infty)$ *and satisfy*

$$\aleph_1(\xi) \le h_1(\xi) \le \aleph_2(\xi), \quad \xi \in [0, \infty), \tag{2.1}$$

*then for* $\xi \ge 0$ *and* $\gamma_j, l_j, p_j, \sigma_j, q_j, \eta, \zeta > 0$ *with* $j = 1, 2, \ldots, m$*, we have*

$$\begin{aligned}&I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}\aleph_2(\xi)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi) + I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}\aleph_1(\xi)\\ &\ge I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}\aleph_2(\xi)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}\aleph_1(\xi) + I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi).\end{aligned} \tag{2.2}$$

*Proof.* From (2.1), for all $\rho \ge 0$ and $\nu \ge 0$, we have

$$\big(\aleph_2(\rho) - h_1(\rho)\big)\big(h_1(\nu) - \aleph_1(\nu)\big) \ge 0,$$

or equivalently,

$$\aleph_2(\rho)h_1(\nu) + \aleph_1(\nu)h_1(\rho) \ge \aleph_2(\rho)\aleph_1(\nu) + h_1(\rho)h_1(\nu). \tag{2.3}$$

Multiplying (2.3) by

$$(\xi-\rho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\ldots,\gamma_m,l_1,\ldots,l_m,p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m,\zeta,q_1,\ldots,q_m)}\{\lambda_1(\xi-\rho)^{\sigma_1}, \ldots, \lambda_j(\xi-\rho)^{\sigma_j}\}$$

and integrating with respect to $\rho$ from 0 to $\xi$, we obtain

$$\begin{aligned}&h_1(\nu)\int_0^\xi (\xi-\rho)^{\zeta-1}\mathcal{E}_{\ldots}\,\aleph_2(\rho)\,d\rho + \aleph_1(\nu)\int_0^\xi (\xi-\rho)^{\zeta-1}\mathcal{E}_{\ldots}\,h_1(\rho)\,d\rho\\ &\ge \aleph_1(\nu)\int_0^\xi (\xi-\rho)^{\zeta-1}\mathcal{E}_{\ldots}\,\aleph_2(\rho)\,d\rho + h_1(\nu)\int_0^\xi (\xi-\rho)^{\zeta-1}\mathcal{E}_{\ldots}\,h_1(\rho)\,d\rho,\end{aligned}$$

which by the definition of the generalized fractional integral (1.7) gives

$$h_1(\nu)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}\aleph_2(\xi) + \aleph_1(\nu)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi) \ge \aleph_1(\nu)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}\aleph_2(\xi) + h_1(\nu)\, I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi). \tag{2.4}$$

Finally, multiplying (2.4) by

$$(\xi-\nu)^{\eta-1}\mathcal{E}^{(\gamma_1,\ldots,\gamma_m,l_1,\ldots,l_m,p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m,\eta,q_1,\ldots,q_m)}\{\lambda_1(\xi-\nu)^{\sigma_1}, \ldots, \lambda_j(\xi-\nu)^{\sigma_j}\}$$

and integrating with respect to $\nu$ from 0 to $\xi$, we arrive at the desired result (2.2). $\square$

**Corollary 2.1.** *Let the function* $h_1$ *be defined and integrable on* $\xi \in [0, \infty)$ *and satisfying* $m \le h_1(\xi) \le M$*,* $\xi \in [0, \infty)$*. Then for* $\xi \ge 0$ *and* $p_j, q_j, \eta, \lambda_j, \gamma_j, \sigma_j, \zeta, l_j > 0$*,* $j = 1, 2, \ldots, m$*, we have*

$$\begin{aligned}&M\,\xi^\zeta\,\mathcal{E}^{(\gamma_j;l_j;p_j)}_{(\sigma_j;\zeta+1;q_j)}\big(\lambda_1\xi^{\sigma_1}, \ldots, \lambda_j\xi^{\sigma_j}\big)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi) + m\,\xi^\eta\,\mathcal{E}^{(\gamma_j;l_j;p_j)}_{(\sigma_j;\eta+1;q_j)}\big(\lambda_1\xi^{\sigma_1}, \ldots, \lambda_j\xi^{\sigma_j}\big)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)\\ &\ge mM\,\xi^{\zeta+\eta}\,\mathcal{E}^{(\gamma_j;l_j;p_j)}_{(\sigma_j;\zeta+1;q_j)}\big(\lambda_1\xi^{\sigma_1}, \ldots, \lambda_j\xi^{\sigma_j}\big)\mathcal{E}^{(\gamma_j;l_j;p_j)}_{(\sigma_j;\eta+1;q_j)}\big(\lambda_1\xi^{\sigma_1}, \ldots, \lambda_j\xi^{\sigma_j}\big) + I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi).\end{aligned}$$

**Theorem 2.2.** *Let the two functions* $h_1$ *and* $h_2$ *be positive and integrable on* $[0, \infty)$*. Assume that* (2.1) *holds and the two functions* $Y_1$ *and* $Y_2$ *be integrable on* $[0, \infty)$ *such that*

$$Y_1(\xi) \le h_2(\xi) \le Y_2(\xi), \quad \xi \in [0, \infty). \tag{2.5}$$

*Then, for* $\xi \ge 0$ *and* $p_j, q_j, \eta, \lambda_j, \gamma_j, \sigma_j, \zeta, l_j > 0$*;* $j = 1, 2, \ldots, m$*, the following four inequalities hold:*

$$I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}\aleph_2(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2(\xi) + I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}Y_1(\xi)$$

$$\geq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_2(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_1(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi), \tag{2.6}$$

$$I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_1(\xi)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_2(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)$$
$$\geq I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_1(\xi)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_2(\xi)+I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi), \tag{2.7}$$

$$I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_2(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_2(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)$$
$$\geq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_2(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_2(\xi), \tag{2.8}$$

$$I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_1(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)$$
$$\geq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}Y_1(\xi). \tag{2.9}$$

*Proof.* To find (2.6), we use (2.1) and (2.5) for all $\rho,\upsilon\in[0,\infty)$, and we have

$$(\aleph_2(\rho)-h_1(\rho))(h_2(\upsilon)-Y_1(\upsilon))\geq 0.$$

It follows that

$$\aleph_2(\rho)h_2(\upsilon)+h_1(\rho)Y_1(\upsilon)\geq\aleph_2(\rho)Y_1(\upsilon)+h_1(\rho)h_2(\upsilon). \tag{2.10}$$

Multiplying $(\xi-\rho)^{\zeta-1}\mathcal{E}_{(\sigma_j,\zeta;q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})$ with (2.10) and integrating with respect to $\rho$ from 0 to $\xi$, we get

$$\begin{aligned}
&h_2(\upsilon)\int_0^\xi(\xi-\rho)^{\zeta-1}\mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m,p_1,\ldots,p_m)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})\aleph_2(\rho)d\rho\\
&+Y_1(\upsilon)\int_0^\xi(\xi-\rho)^{\zeta-1}\mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m,p_1,\ldots,p_m)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})h_1(\rho)d\rho\\
&\geq Y_1(\upsilon)\int_0^\xi(\xi-\rho)^{\zeta-1}\mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m,p_1,\ldots,p_m)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})\aleph_2(\rho)d\rho\\
&+h_2(\upsilon)\int_0^\xi(\xi-\rho)^{\zeta-1}\mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m,p_1,\ldots,p_m)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})h_1(\rho)d\rho.
\end{aligned}$$

In view of (1.7), this gives

$$h_2(\upsilon)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_2(\xi)+Y_1(\upsilon)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)\geq Y_1(\upsilon)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}\aleph_2(\xi)+h_2(\upsilon)I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi). \tag{2.11}$$

Again, multiplying $(\xi-\upsilon)^{\eta-1}\mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\eta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m,p_1,\ldots,p_m)}(\lambda_1(\xi-\upsilon)^{\sigma_1},\ldots,\lambda_j(\xi-\upsilon)^{\sigma_j})$ with (2.11) and integrating with respect to $\upsilon$ from 0 to $\xi$ and using (1.7), we get the required inequality (2.6).

Similarly, inequalities (2.7)–(2.9) can be proved by utilizing the following results:

$$\begin{aligned}
&(Y_2(\rho)-h_2(\rho))(h_1(\upsilon)-\aleph_1(\upsilon))\geq 0,\\
&(\aleph_2(\rho)-h_1(\rho))(h_2(\upsilon)-Y_2(\upsilon))\leq 0,\\
&(\aleph_1(\rho)-h_1(\rho))(h_2(\upsilon)-Y_1(\upsilon))\leq 0.
\end{aligned}$$

□

**Corollary 2.2.** *Let the functions $h_1$ and $h_2$ be integrable and positive on $[0,\infty)$ and satisfying*

$$m\leq h_1(\xi)\leq M \quad\text{and}\quad n\leq h_2(\xi)\leq N,\quad \xi\in[0,\infty).$$

*Then for $\xi\geq 0$ and $p_j,q_j,\eta,\lambda_j,\gamma_j,\sigma_j,\zeta,l_j>0;\ j=1,2,\ldots,m$, we have*

$$\begin{aligned}
&M\xi^\zeta\mathcal{E}_{(\sigma_j,\zeta+1,q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)\\
&\quad+n\xi^\eta\mathcal{E}_{(\sigma_j,\eta+1,q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)\\
&\geq Mn\xi^{\zeta+\eta}\mathcal{E}_{(\sigma_j,\zeta+1,q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\mathcal{E}_{(\sigma_j,\eta+1,q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\\
&\quad+I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_1(\xi)I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi),
\end{aligned}$$

$$m\xi^\eta\mathcal{E}_{(\sigma_j,\eta+1,q_j)}^{(\gamma_j,l_j,p_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)}h_2(\xi)$$

$$
\begin{aligned}
&+ N\xi^{\zeta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)\\
&\geq mN\xi^{\zeta+\eta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\eta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\\
&+ I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2(\xi),
\end{aligned}
$$

$$
\begin{aligned}
&MN\xi^{\zeta+\eta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\eta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\\
&+ I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2(\xi)\\
&\geq M\xi^{\zeta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2(\xi)\\
&+ n\xi^{\eta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\eta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi),
\end{aligned}
$$

$$
\begin{aligned}
&mn\xi^{\zeta+\eta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\eta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})\\
&+ I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2(\xi)\\
&\geq m\xi^{\zeta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\zeta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2(\xi)\\
&+ n\xi^{\eta}\mathcal{E}^{(\gamma_j,l_j,p_j)}_{(\sigma_j,\eta+1,q_j)}(\lambda_1\xi^{\sigma_1},\ldots,\lambda_j\xi^{\sigma_j})I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi).
\end{aligned}
$$

**Theorem 2.3.** *Let the functions $h_1$ and $h_2$ be positive and integrable on $[0,\infty)$. If $r_1, s_1 > 1$ are such that $1/r_1 + 1/s_1 = 1$, then for $\xi \geq 0$, we have*

$$
\begin{aligned}
&\frac{1}{r_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{r_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2^{r_1}(\xi) + \frac{1}{s_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2^{s_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{s_1}(\xi)\\
&\geq I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)h_2(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1(\xi)h_2(\xi),
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
&\frac{1}{r_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{r_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2^{s_1}(\xi) + \frac{1}{s_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2^{s_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{r_1}(\xi)\\
&\geq I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2^{s_1-1}(\xi)h_1^{r_1-1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)h_2(\xi),
\end{aligned} \tag{2.13}
$$

$$
\begin{aligned}
&\frac{1}{r_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{r_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2^{2}(\xi) + \frac{1}{s_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2^{s_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{2}(\xi)\\
&\geq I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{2/s_1}(\xi)h_2^{2/r_1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)h_2(\xi),
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
&\frac{1}{r_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{2}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_2^{s_1}(\xi) + \frac{1}{s_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2^{2}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{r_1}(\xi)\\
&\geq I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\eta,q_j)}h_1^{r_1-1}(\xi)h_2^{s_1-1}(\xi)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{2/r_1}(\xi)h_2^{2/s_1}(\xi).
\end{aligned} \tag{2.15}
$$

*Proof.* To prove (2.12), we use Young's inequality [14]:

$$
\frac{1}{r_1}u^{r_1} + \frac{1}{s_1}v^{s_1} \geq uv, \quad u,v>0, \quad \frac{1}{r_1}+\frac{1}{s_1}=1. \tag{2.16}
$$

Substituting $u = h_1(\rho)h_2(\upsilon)$ and $v = h_1(\upsilon)h_2(\rho)$ for all $\rho, \upsilon > 0$ in (2.16), we have

$$
\frac{1}{r_1}(h_1(\rho)h_2(\upsilon))^{r_1} + \frac{1}{s_1}(h_1(\upsilon)h_2(\rho))^{s_1} \geq (h_1(\rho)h_2(\upsilon))(h_1(\upsilon)h_2(\rho)). \tag{2.17}
$$

Multiplying (2.17) by $(\xi-\rho)^{\zeta-1}\mathcal{E}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m;p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}(\lambda_1(\xi-\rho)^{\sigma_1},\ldots,\lambda_j(\xi-\rho)^{\sigma_j})$ and integrating $\rho$ from 0 to $\xi$ gives

$$
\begin{aligned}
&\frac{h_2^{r_1}(\upsilon)}{r_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1^{r_1}(\xi) + \frac{h_1^{s_1}(\upsilon)}{s_1}I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_2^{s_1}(\xi)\\
&\geq h_1(\upsilon)h_2(\upsilon)I^{(\gamma_j,l_j,p_j;\lambda_j)}_{(\sigma_j,\zeta,q_j)}h_1(\xi)h_2(\xi).
\end{aligned} \tag{2.18}
$$

Multiplying (2.18) by $(\xi-\upsilon)^{\eta-1}\mathcal{E}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m;p_1,\ldots,p_m)}_{(\sigma_1,\ldots,\sigma_m,\eta;q_1,\ldots,q_m)}(\lambda_1(\xi-\upsilon)^{\sigma_1},\ldots,\lambda_j(\xi-\upsilon)^{\sigma_j})$ and integrating with respect to $\upsilon$ from 0 to $\xi$ gives inequality (2.12).

The inequalities (2.13), (2.14), and (2.15) follow by substituting in (2.16) the respective identities:

$$u = \frac{h_1(\rho)}{h_1(\upsilon)}, \quad v = \frac{h_2(\rho)}{h_2(\upsilon)}, \quad h_1(\upsilon), h_2(\upsilon) \neq 0, \tag{2.19}$$

$$u = h_1(\rho) h_2^{2/r_1}(\upsilon), \quad v = h_1^{2/s_1}(\upsilon) h_2(\rho), \tag{2.20}$$

$$u = h_1^{2/r_1}(\rho)\ h_1(\upsilon), \quad v = h_2^{2/s_1}(\rho)\ h_2(\upsilon), \tag{2.21}$$

where $p_j, q_j, \eta, \lambda_j, \gamma_j, \sigma_j, \zeta, l_j > 0$, $j = 1, 2, \ldots, m$. □

**Theorem 2.4.** *Let the functions $h_1$ and $h_2$ be positive and integrable on $[0, \infty)$. If $r_1, s_1 > 1$ be such that $1/r_1 + 1/s_1 = 1$, then for $\xi \geq 0$ and $p_j, q_j, \eta, \lambda_j, \gamma_j, \sigma_j, \zeta, l_j > 0; j = 1, 2, \ldots, m$, the following inequalities hold:*

$$\begin{aligned} &r_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2(\xi) + s_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) \\ &\quad \geq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1^{r_1}(\xi) h_2^{s_1}(\xi)\big) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1^{s_1}(\xi) h_2^{r_1}(\xi)\big), \end{aligned} \tag{2.22}$$

$$\begin{aligned} &r_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{r_1-1}(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1(\xi) h_2^{s_1}(\xi)\big) + s_1 I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{s_1-1}(\xi) I_{(\sigma_j,\xi,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1^{r_1}(\xi) h_2(\xi)\big) \\ &\quad \geq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{s_1}(\xi) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{r_1}(\xi), \end{aligned} \tag{2.23}$$

$$\begin{aligned} &r_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{2/r_1}(\xi) + s_1 I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{s_1}(\xi) I_{(\sigma_j,\xi,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{2/s_1}(\xi) \\ &\quad \geq I_{(\sigma_j,\xi,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1^{r_1}(\xi) h_2^{s_1}(\xi)\big) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_2^{s_2}(\xi) h_1^{2}(\xi)\big), \end{aligned} \tag{2.24}$$

$$\begin{aligned} &r_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{2/r_1}(\xi) h_2^{s_1}(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{r_1-1}(\xi) \\ &\quad + s_1 I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^{s_1-1}(\xi) I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{2/s_1}(\xi) h_2^{r_1}(\xi) \geq I_{(\sigma_j,\eta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{2}(\xi) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^{2}(\xi). \end{aligned} \tag{2.25}$$

*Proof.* From the arithmetic mean–geometric mean (A.M.-G.M.) inequality, we know

$$r_1 u + s_1 v \geq u^{r_1} v^{s_1}, \quad u, v > 0, \quad r_1 + s_1 = 1. \tag{2.26}$$

Substituting $u = h_1(\rho) h_2(\upsilon)$ and $v = h_1(\upsilon) h_2(\rho)$, $\forall \rho, \upsilon > 0$ in (2.26), we obtain

$$r_1 h_1(\rho) h_2(\upsilon) + s_1 h_1(\upsilon) h_2(\rho) \geq \big(h_1(\rho) h_2(\upsilon)\big)^{r_1} \big(h_1(\upsilon) h_2(\rho)\big)^{s_1}. \tag{2.27}$$

Multiplying (2.27) by $(\xi-\rho)^{\zeta-1} \mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m;p_1,\ldots,p_m)} (\lambda_1(\xi-\rho)^{\sigma_1}, \ldots, \lambda_j(\xi-\rho)^{\sigma_j})$ and integrating with respect to $\rho$ from 0 to $\xi$, we get

$$\begin{aligned} &r_1 h_2(\upsilon) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) + s_1 h_1(\upsilon) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2(\xi) \\ &\quad \geq h_2^{r_1}(\upsilon) h_1^{s_1}(\upsilon) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} \big(h_1^{r_1}(\xi) h_2^{s_1}(\xi)\big). \end{aligned} \tag{2.28}$$

Multiplying (2.28) by $(\xi-\upsilon)^{\eta-1} \mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\eta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m;p_1,\ldots,p_m)} (\lambda_1(\xi-\upsilon)^{\sigma_1}, \ldots, \lambda_j(\xi-\upsilon)^{\sigma_j})$ and integrating with respect to $\upsilon$ from 0 to $\xi$ gives inequality (2.22).

Inequalities (2.23), (2.24), and (2.25) can be obtained by replacing the identities in (2.26) as follows:

$$u = \frac{h_1(\upsilon)}{h_1(\rho)}, \quad v = \frac{h_2(\rho)}{h_2(\upsilon)}, \quad h_1(\rho), h_2(\upsilon) \neq 0, \tag{2.29}$$

$$u = h_1(\rho) h_2^{2/r_1}(\upsilon), \quad v = h_1^{2/s_1}(\upsilon) h_2(\rho), \tag{2.30}$$

$$u = \frac{h_1^{2/r_1}(\rho)}{h_2(\upsilon)}, \quad v = \frac{h_1^{2/s_1}(\upsilon)}{h_2(\rho)}, \quad h_2(\rho), h_2(\upsilon) \neq 0. \tag{2.31}$$

□

**Theorem 2.5.** *Suppose the functions $h_1$ and $h_2$ are both positive and integrable over the interval $[0, \infty)$. If $r_1, s_1 > 1$ such that*

$$\frac{1}{r_1} + \frac{1}{s_1} = 1, \tag{2.32}$$

*define*

$$\mathcal{K} = \min_{0 \leq \rho \leq \xi} \frac{h_1(\rho)}{h_2(\rho)}, \quad \mathcal{H} = \max_{0 \leq \rho \leq \xi} \frac{h_1(\rho)}{h_2(\rho)}. \tag{2.33}$$

*Then, for $\xi \geq 0$, the following inequalities hold:*

$$0 \leq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi)\, I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi) \leq \frac{(\mathcal{K}+\mathcal{H})^2}{4\mathcal{K}\mathcal{H}} \left( I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \right)^2, \tag{2.34}$$

$$\begin{aligned} 0 &\leq \sqrt{I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi)\, I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi)} - I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \\ &\leq \frac{\sqrt{\mathcal{H}}-\sqrt{\mathcal{K}}}{2\sqrt{\mathcal{K}\mathcal{H}}} \left( I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \right), \end{aligned} \tag{2.35}$$

$$\begin{aligned} 0 &\leq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi)\, I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi) - \left( I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \right)^2 \\ &\leq \frac{\mathcal{H}-\mathcal{K}}{4\mathcal{K}\mathcal{H}} \left( I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \right)^2. \end{aligned} \tag{2.36}$$

*Proof.* From (2.33), we have

$$\left( \frac{h_1(\rho)}{h_2(\rho)} - \mathcal{K} \right) \left( \mathcal{H} - \frac{h_1(\rho)}{h_2(\rho)} \right) h_2^2(\rho) \geq 0, \quad 0 \leq \rho \leq \xi. \tag{2.37}$$

This implies

$$h_1^2(\rho) + \mathcal{K}\mathcal{H} h_2^2(\rho) \leq (\mathcal{K}+\mathcal{H}) h_1(\rho) h_2(\rho). \tag{2.38}$$

Multiplying (2.38) by

$$(\xi-\rho)^{\zeta-1} \mathcal{E}_{(\sigma_1,\ldots,\sigma_m,\zeta;q_1,\ldots,q_m)}^{(\gamma_1,\ldots,\gamma_m;l_1,\ldots,l_m;p_1,\ldots,p_m)} \left( \lambda_1(\xi-\rho)^{\sigma_1}, \ldots, \lambda_j(\xi-\rho)^{\sigma_j} \right)$$

and integrating from 0 to $\xi$, using (1.7), we get

$$I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi) + \mathcal{K}\mathcal{H} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi) \leq (\mathcal{K}+\mathcal{H}) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi). \tag{2.39}$$

Since $\mathcal{K}\mathcal{H} > 0$ and

$$\left( \sqrt{I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi)} - \sqrt{\mathcal{K}\mathcal{H} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi)} \right)^2 \geq 0,$$

it follows that

$$2\sqrt{\mathcal{K}\mathcal{H} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi)} \leq I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi) + \mathcal{K}\mathcal{H} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi). \tag{2.40}$$

Squaring both sides of (2.39) and using (2.40), we obtain

$$4\mathcal{K}\mathcal{H} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi) \leq (\mathcal{K}+\mathcal{H})^2 \left( I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi) \right)^2. \tag{2.41}$$

Simplifying (2.41) gives (2.34). From (2.41), we have

$$\sqrt{I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1^2(\xi) I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_2^2(\xi)} \leq \frac{\mathcal{K}+\mathcal{H}}{2\sqrt{\mathcal{K}\mathcal{H}}} I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi). \tag{2.42}$$

Subtracting $I_{(\sigma_j,\zeta,q_j)}^{(\gamma_j,l_j,p_j;\lambda_j)} h_1(\xi) h_2(\xi)$ from (2.42) gives (2.35), and (2.36) follows directly from (2.34). □

**Theorem 2.6.** *Consider the function $h_1$, which is positive and integrable on $[0,\infty)$ and let the two functions $\aleph_1$ and $\aleph_2$ be integrable on $[0,\infty)$ such that*

$$\aleph_1(\xi) \leq h_1(\xi) \leq \aleph_2(\xi), \quad \xi \in [0,\infty). \tag{2.43}$$

*Then for $\xi \geq 0$ and $\sigma, \lambda, \gamma, l, p, q, \zeta > 0$, we have*

$$\begin{cases} I_{\sigma,\zeta,q}^{\gamma,l,p;\lambda} \aleph_2(\xi)\, h_1(\xi) + I_{\sigma,\zeta,q}^{\gamma,l,p;\lambda} h_1(\xi)\, \aleph_1(\xi) \\ \geq I_{\sigma,\zeta,q}^{\gamma,l,p;\lambda} \aleph_2(\xi)\, \aleph_1(\xi) + I_{\sigma,\zeta,q}^{\gamma,l,p;\lambda} h_1(\xi)\, h_1(\xi). \end{cases} \tag{2.44}$$

**Corollary 2.3.** *Let the condition of Theorem 2.1 be satisfied with $l_1 = \cdots = l_m = p_1 \cdots p_m = q_1 \cdots q_m = 1$, then we get the known result:*

$$\begin{aligned} & I_{(\sigma_j,\zeta)}^{\gamma(j,\lambda_j)} \aleph_2(\xi)\, I_{(\sigma_j,\eta)}^{\gamma_j,\lambda_j} h_1(\xi) + I_{(\sigma_j,\zeta)}^{\gamma(j,\lambda_j)} h_1(\xi)\, I_{(\sigma_j,\eta)}^{\gamma_j,\lambda_j} \aleph_1(\xi) \\ & \geq I_{(\sigma_j,\zeta)}^{\gamma(j,\lambda_j)} \aleph_2(\xi)\, I_{(\sigma_j,\eta)}^{\gamma_j,\lambda_j} \aleph_1(\xi) + I_{(\sigma_j,\zeta)}^{\gamma(j,\lambda_j)} h_1(\xi)\, I_{(\sigma_j,\eta)}^{\gamma_j,\lambda_j} h_1(\xi). \end{aligned} \tag{2.45}$$

*If we consider $\lambda_j = 0$, $\forall j$, then we get the known result derived by Tariboon et al.* [24]*. Several results of Theorem 2.1 can be obtained in a similar manner by choosing certain specific values for the parameters.*

## 3 Conclusion

This paper presents findings on Grüss-type inequalities and similar inequalities using the generalized fractional integral. We discussed some special situations as corollaries. These results help us better understand fractional calculus and its applications in various areas. Moreover, our results reduce to some classical results found in the work of Shao [22].

It is concluded that the results claimed in this work are general in character and provide contributions to the theory of integral inequalities and fractional calculus. Furthermore, these results are expected to lead to applications in establishing the uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations.

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