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STUDY OF COHESIVE NUMBER IN HESITANT FUZZY GRAPHS

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Abstract

This study aims to determine the vertex cohesive number and edge cohesive number of hesitant fuzzy graph structures derived from Gear and Bipartite graphs. The Gear and Bipartite graphs are transformed into hesitant fuzzy graphs by assigning hesitant fuzzy membership functions to both vertices and edges. Similar hesitant fuzzy membership functions are grouped to form cohesive hesitant fuzzy structures. The vertex and edge cohesive numbers for these structures are then computed. These findings provide insights into the cohesiveness of vertices and edges within hesitant fuzzy graphs. The approach can be applied in organizational contexts, where employees are modeled as vertices. By analyzing the coordination between employees using hesitant fuzzy graph structures, it becomes possible to gain valuable insights into group dynamics and improve team efficiency. This study utilizes vertex and edge cohesive numbers in hesitant fuzzy graphs to optimize transportation networks under uncertain conditions.

2020 Mathematical Sciences Classification: 05C72.

Keywords and Phrases: Hesitant Fuzzy Vertex and Edge, Vertex and Edge Cohesive Number, Hesitant Fuzzy Gear Graph, Hesitant Fuzzy Bipartite Graph.

1 Introduction

Graph theory originated from the Königsberg bridge problem, which laid the foundation for its development into a widely applicable field across disciplines such as social networks and computer systems, Harary [6]. The introduction of fuzzy sets by Zadeh [21,1965] marked a turning point in mathematics, enabling the development of fuzzy graph theory. Rosenfeld formalized the concept of fuzzy graphs in 1973, with refinements appearing in [15, 1975]. These advancements provided a foundation for further research on fuzzy graphs, leading to significant developments in fuzzy graph structures and their applications, Zimmermann [22]. Fuzzy graph structures evolved from Rosenfeld's work were extended to specific graphs such as star, wheel, and helm graphs [1]. These structures are now widely applied in areas like computer science and social network analysis [5]. Notable advancements in this area include generalized fuzzy graph structures, which have been studied extensively [12, 16]. Hesitant fuzzy sets, introduced by Torra [19, 2010], addressed limitations in traditional fuzzy sets by allowing multiple membership values for each element, enhancing the modeling of uncertainty. This concept was further applied to hesitant fuzzy graphs, which provide a flexible framework for modeling complex systems characterized by inherent uncertainty due to Pathinathan *et al.* [14]. These graphs have been utilized in decision-making processes where relationships between vertices and edges exhibit vagueness [11, 18]. Theoretical advancements in hesitant fuzzy graphs include the study of degree, order, and size in intuitionistic fuzzy graphs [3], as well as the introduction of double-layered fuzzy graphs to capture more complex relationships [13]. Pathinathan and colleagues have also significantly contributed to the study of hesitant fuzzy graphs, exploring their properties and applications in various fields [14, 17]. Hesitant fuzzy graphs are particularly valuable in transportation problems, where mixed constraints and uncertainty are inherent. Researchers, [2, 8] have developed fuzzy approaches to optimize transportation networks and improve reliability. This paper introduces the Interval Valued Intuitionistic Trapezoidal Neutrosophic Fuzzy Graph for the Shortest Path Problem [10]. These graphs also find applications in organizational dynamics and social network analysis, providing robust solutions for modeling real-world problems [4, 7].

2 Preliminary

Definition 2.1. ([5]). *The cohesive number of a graph is defined as the maximum number of vertex-disjoint cycles in the graph. A cycle is a path of edges and vertices wherein a vertex is reachable from itself.*

Definition 2.2. ([9]). *Fuzzy Vertex and Edge in a fuzzy graph $G = (V, \mu, \sigma), \mu : V \to [0, 1]$ assigns a membership degree to each vertex, and $\sigma : V \times V \to [0, 1]$ assigns a membership degree to each edge.*

Definition 2.3. ([15]). *A fuzzy cycle is a sequence of vertices $v_1, v_2, ..., v_k$ such that v_i is connected to v_{i+1} with a non-zero membership degree, and v_k is connected back to v_1.*

Definition 2.4. ([5]). *Two cycles are vertex-disjoint if they do not share any vertices. In a fuzzy graph, this means no vertex has a positive membership degree in more than one cycle in the set of vertex-disjoint cycles.*

To calculate the cohesive number in a fuzzy graph:

1. **Identify Fuzzy Cycles**: Determine all possible fuzzy cycles in the graph. This involves finding sequences of vertices where the fuzzy membership degrees of the edges form cycles.
2. **Maximize Vertex-Disjoint Cycles**: Identify the maximum set of fuzzy cycles that do not share any vertices. This involves considering the membership degrees and ensuring no vertex is part of more than one cycle in the set.

Example 2.1. Consider a fuzzy graph with vertices $V = \{v_1, v_2, v_3, v_4\}$ and fuzzy membership degrees on edges such that:

$$\begin{aligned} \sigma(v_1, v_2) &= 0.8, \\ \sigma(v_2, v_3) &= 0.7, \\ \sigma(v_3, v_1) &= 0.9, \\ \sigma(v_3, v_4) &= 0.5, \\ \sigma(v_4, v_1) &= 0.6. \end{aligned} \tag{2.1}$$

In this fuzzy graph, one possible fuzzy cycle is $v_1 \to v_2 \to v_3 \to v_1$ with the minimum membership degree being 0.7.
Another $v_3 \to v_4 \to v_1 \to v_3$ potential fuzzy cycle could be with the minimum membership degree being 0.5.
To determine the cohesive number, we would look for the maximum number of these fuzzy cycles that do not share vertices.

Gear Graph (G_n): A Gear graph G_n is derived from a wheel graph by adding an additional vertex to the center of the cycle and connecting it to all other vertices. The Gear graph structure is as follows:
Vertices:
- G_n consists of n vertices.
- $n - 1$ vertices form a cycle (e.g., a pentagon for $n = 6$).
- 1 central vertex is connected to all the vertices of the cycle.

Edges:
- Cycle edges connect adjacent vertices in the cycle.
- Spoke edges connect the central vertex to all the vertices of the cycle.

A Gear graph G_n is defined as $G_n = (V, E)$, where
$V = \{v_1, v_2, ...v_{n-1}, v_c\}$ is the set of vertices.
Here $v_1, v_2, ...v_{n-1}, v_c$ forming a cycle C_{n-1}, and v_c is the central vertex connected to each v_i for $i = 1, 2, ..., n - 1$.
The edge set E consists of the edges forming the cycle C_{n-1} and the edges connecting v_c to each v_i.
Thus, $|E| = 2(n - 1)$.

3 Main Result

Definition in Hesitant Fuzzy Graphs

The cohesive number in a hesitant fuzzy graph is defined as the maximum number of vertex-disjoint cycles, taking into account the hesitant fuzzy relationships between vertices.

Hesitant Fuzzy Vertex and Edge: In a hesitant fuzzy graph $G = (V, \mu, \sigma), \mu : V \to H[0,1]$ and $\sigma : V \times V \to H[0,1]$, where $H[0,1]$ denotes the set of hesitant fuzzy values (i.e., finite sets of membership degrees).

Hesitant Fuzzy Cycle: A hesitant fuzzy cycle is a sequence of vertices $v_1, v_2, ..., v_k$ such that each pair of consecutive vertices $v_i$and v_{i+1} (and v_k back to v_1) is connected by an edge with non-zero hesitant fuzzy membership degrees.

Vertex-Disjoint Cycles: Two cycles are vertex-disjoint if they do not share any vertices. In a hesitant fuzzy graph, this means no vertex is part of more than one cycle in the set of vertex-disjoint cycles, considering the hesitant fuzzy values.

Example 3.1. Consider a fuzzy graph with vertices $V = \{v_1, v_2, v_3, v_4\}$ and hesitant fuzzy membership degrees on edges such that:

$$\begin{aligned} \sigma(v_1, v_2) &= \{0.7, 0.8\}, \\ \sigma(v_2, v_3) &= \{0.6, 0.7\}, \\ \sigma(v_3, v_1) &= \{0.8, 0.9\}, \\ \sigma(v_3, v_4) &= \{0.4, 0.5\}, \\ \sigma(v_4, v_1) &= \{0.5, 0.6\}. \end{aligned} \tag{3.1}$$

In this hesitant fuzzy graph, one possible hesitant fuzzy cycle is $v_1 \to v_2 \to v_3 \to v_1$ with hesitant fuzzy membership degrees.
Another possible cycle is $v_3 \to v_4 \to v_1 \to v_3$, with their respective hesitant fuzzy membership degrees.
To determine the cohesive number, we would look for the maximum number of these hesitant fuzzy cycles that do not share vertices.

Vertex Cohesive Number in Hesitant Fuzzy Graph

Vertex Cohesive Number $v_c(HG)$: These measures how strongly connected the vertices are in the hesitant fuzzy graph. A vertex cohesive number of 2, $v_c(HG) = 2$ would indicate that removing any single vertex reduces the cohesion of the graph such that it becomes less connected, or the number of connected components increases.

Edge Cohesive Number in Hesitant Fuzzy Graph
Edge Cohesive Number $e_c(HG)$This quantifies the robustness of the graph with respect to its edges. An edge cohesive number of 2, $e_c(HG) = 2$, implies that the removal of any edge similarly affects the graph's cohesion, potentially leading to a disconnected structure or increasing the number of disconnected components.

Hesitant Fuzzy Gear Graph: A hesitant fuzzy Gear graph is a fuzzy extension of the Gear graph where the edges have membership degrees represented by hesitant fuzzy sets (multiple possible values). The structure is similar to the Gear graph but includes hesitant fuzzy memberships for each edge.

A **hesitant fuzzy Gear graph** $HFG_n = (V, E, \mu)$ is defined on the Gear graph G_n where:

Hesitant Fuzzy Membership Function:
For each edge $e \in E$ a hesitant fuzzy membership function $\mu : E \to 2^{[0,1]} \setminus \{\phi\}$ is defined, where is a non-empty set of membership values, i.e., $\mu(e) = \mu_1(e), \mu_2(e), , \mu_k(e), \mu i(e) \in [0,1]$ The set $\mu(e)$ represents the degrees of membership reflecting the hesitation in the strength of the connection between the vertices connected by the edge e.

Hesitant Fuzzy Degree of a Vertex: The hesitant fuzzy degree of a vertex $v_i \in V$, denoted by $deg_h(v_i)$, is defined as the sum of the hesitant fuzzy membership degrees of all edges incident to v_i:

$$\deg_H(v_i) = \sum_{e \in E(v_i)} \mu(e), \tag{3.2}$$

where E_i is the set of edges incident to v_i.

Hesitant Fuzzy Degree of the Graph:

- The hesitant fuzzy degree of the Gear graph HFG_n denoted by $deg_h(G_n)$, is the sum of the hesitant fuzzy degrees of all its vertices:

$$\deg_H(G_n) = \sum_{v_i \in V} deg_H(v_i). \tag{3.3}$$

- **Vertices:** Same as the Gear graph.
- **Edges:**
  - Cycle edges with hesitant fuzzy memberships (e.g., edges might have memberships {0.7, 0.8}).
  - Spoke edges with hesitant fuzzy memberships connecting the central vertex to the cycle vertices (e.g., edges might have memberships {0.9, 1.0}).

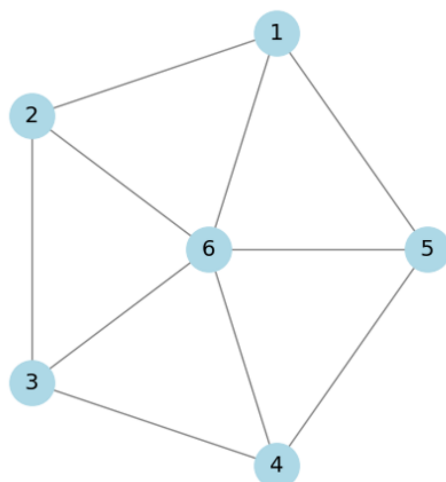

Figure 3.1: *Gear Graph* G_6

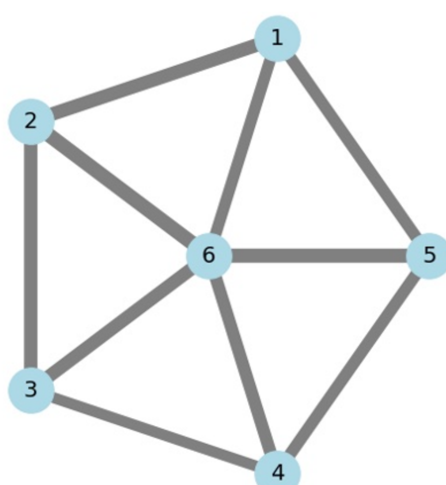

Figure 3.2: *Hesitant Fuzzy Gear Graph* G_6

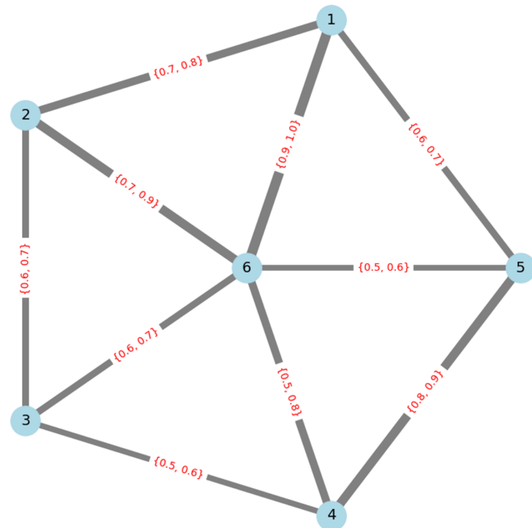

***Figure* 3.3:** *Hesitant Fuzzy Gear Graph G_6 with Membership Degrees*

Hesitant Fuzzy Bipartite Graph: This graph has the same structure as a bipartite graph but with hesitant fuzzy sets for vertices and edges. **Example of a Hesitant Fuzzy Bipartite Graph: Vertex Sets:** Let the vertex sets be the same as before:

$$\begin{aligned} V1 &= \{u_1, u_2\}, \\ V2 &= \{v_1, v_2\}. \end{aligned} \tag{3.4}$$

Hesitant Membership Function (Vertices): The hesitant membership degrees for the vertices are as follows:

$$\begin{aligned} h_\mu(u_1) &= \{0.7, 0.8\}, \\ h_\mu(u_2) &= \{0.5, 0.6\}, \\ h_\mu(v_1) &= \{0.6, 0.7\}, \\ h_\mu(v_2) &= \{0.8, 0.9\}. \end{aligned} \tag{3.5}$$

Hesitant Edge Weights (Edges): The hesitant edge weights between the vertices of and are given by:

$$\begin{aligned} h_\sigma(u_1, v_1) &= \{0.4, 0.5\}, \\ h_\sigma(u_1, v_2) &= \{0.5, 0.6\}, \\ h_\sigma(u_2, v_1) &= \{0.3, 0.4\}, \\ h_\sigma(u_2, v_2) &= \{0.6, 0.7\}. \end{aligned} \tag{3.6}$$

Calculation of Vertex and Edge Cohesive Numbers for Hesitant Fuzzy Bipartite Graph Hesitant Vertex Cohesive Number $\alpha_H(G_H)$: In the hesitant case, the vertex cohesive number $\alpha_H(G_H)$ is calculated by considering the minimum values from the intersections of hesitant membership and edge weights.

$$\alpha_H(G_H) = \sum_{u \in v_1} \sum_{v \in v_2} \min \left(h_\mu(u) \cap h_\mu(v) \cap h_\sigma(u, v) \right). \tag{3.7}$$

Calculating for each pair:

$$(u_1, v_1) : \min\{0.7, 0.8\} \cap \{0.6, 0.7\} \cap \{0.4, 0.5\} = \min(0.7) = 0.4, \tag{3.8}$$

$$(u_1, v_2) : \min\{0.7, 0.8\} \cap \{0.8, 0.9\} \cap \{0.5, 0.6\} = \min(0.7) = 0.5, \tag{3.9}$$

$$(u_2, v_1) : \min\{0.5, 0.6\} \cap \{0.6, 0.4\} \cap \{0.3, 0.4\} = \min(0.5) = 0.3, \tag{3.10}$$

$$(u_2, v_2) : \min\{0.5, 0.6\} \cap \{0.8, 0.9\} \cap \{0.6, 0.7\} = \min(0.5) = 0.6. \tag{3.11}$$

So,

$$\alpha_H(G_H) = 0.4 + 0.5 + 0.3 + 0.6 = 1.8. \tag{3.12}$$

Hesitant Edge Cohesive Number
Similarly, the edge cohesive number $\beta_H(G_H)$ is calculated by taking the product of the hesitant edge weight and the minimum hesitant membership values for connected vertices.

$$\beta_H(G_H) = \sum_{(u,v)\in E} \max\big(h_\sigma(u,v)\big) \cdot \min\big(h_\mu(u) \cap h_\mu(v)\big). \tag{3.13}$$

Calculating for each edge:

$$\text{For } (u_1, v_1) : \max\{0.4, 0.5\} \cdot \min\{0.7\} = 0.5 \times 0.6 = 0.30, \tag{3.14}$$

$$\text{For } (u_1, v_2) : \max\{0.5, 0.6\} \cdot \min\{0.7\} = 0.6 \times 0.8 = 0.48, \tag{3.15}$$

$$\text{For } (u_2, v_1) : \max\{0.3, 0.4\} \cdot \min\{0.5\} = 0.4 \times 0.5 = 0.20, \tag{3.16}$$

$$\text{For } (u_2, v_2) : \max\{0.6, 0.7\} \cdot \min\{0.5\} = 0.7 \times 0.6 = 0.42. \tag{3.17}$$

So,

$$\beta_H(G_H) = 0.30 + 0.48 + 0.20 + 0.42 = 1.40. \tag{3.18}$$

Summary

$$\textbf{Hesitant Vertex Cohesive Number}\alpha_H(G_H) = 1.8, \tag{3.19}$$

$$\textbf{Hesitant Edge Cohesive Number } \beta_H(G_H) = 1.40. \tag{3.20}$$

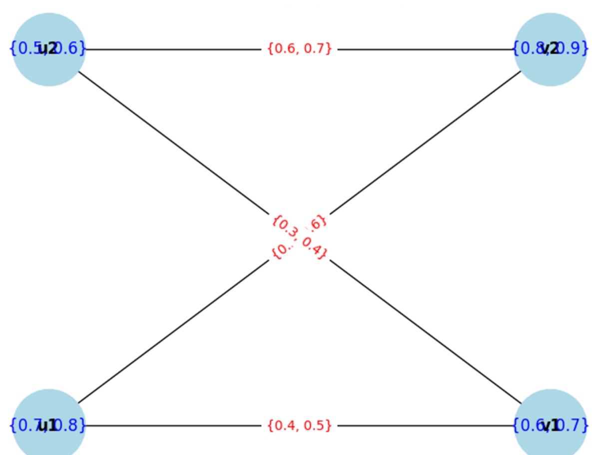

Figure 3.4: *Hesitant Fuzzy Bipartite Graph*

These measures give us an understanding of the connectivity and strength of the hesitant fuzzy bipartite graph. They provide a way to analyze the graph's structure when membership values are not certain but lie within a set of possibilities.

3.1 Comparison of Hesitant Fuzzy Cohesive Numbers with Classical Fuzzy and Intuitionistic Fuzzy Graphs

In classical fuzzy graphs, each vertex or edge is assigned a single membership value $\mu \in [0, 1]$, representing the degree of connection or association. The vertex cohesive number $\alpha(G)$ and edge cohesive number $\beta(G)$ are computed using these fixed membership degrees. This approach provides a precise measure of connectivity but fails to capture situations where the degree of membership is uncertain or varies due to multiple expert evaluations or dynamic environmental conditions.

In intuitionistic fuzzy graphs, each vertex or edge is characterized by a pair of values: the membership degree $\mu(x)$ and the non-membership degree $\nu(x)$, satisfying the condition $0 \le \mu(x) + \nu(x) \le 1$. The hesitation margin $\pi(x) = 1 - \mu(x) - \nu(x)$ expresses uncertainty indirectly. The cohesive numbers $\alpha^I(G)$ and

$\beta^I(G)$ in this context combine both membership and non-membership information, giving a more balanced but still fixed representation of uncertainty.

In hesitant fuzzy graphs, however, each vertex or edge is represented by a *set of possible membership degrees*, such as $\mu(e) = \{0.6, 0.7, 0.8\}$, instead of a single value or pair. This representation models multiple expert opinions or hesitation regarding the strength of relationships. Accordingly:

- The **hesitant vertex cohesive number** $\alpha^H(G)$ is computed using intersection and minimum operations over all possible membership degrees, providing a conservative yet realistic measure of connectivity.
- The **hesitant edge cohesive number** $\beta^H(G)$ incorporates both maximum and minimum hesitant values, reflecting the uncertainty and variability in edge strength more effectively than classical fuzzy or intuitionistic fuzzy models.

Comparison Summary:

Table 3.1: *Comparison of Cohesive Numbers across Fuzzy Graph Types*

Graph Type	Membership Representation	Captures Uncertainty	Cohesive Number Representation	Relative Robustness
Classical Fuzzy Graph	Single $\mu \in [0, 1]$	Low	$\alpha(G), \beta(G)$ based on fixed values	Limited
Intuitionistic Fuzzy Graph	Pair (μ, ν) with hesitation π	Moderate	$\alpha^I(G), \beta^I(G)$ include non-membership	Medium
Hesitant Fuzzy Graph	Set of membership values $\{\mu_1, \mu_2, \ldots, \mu_k\}$	High	$\alpha^H(G), \beta^H(G)$ based on hesitant set operations	Strong

Therefore, **hesitant fuzzy cohesive numbers** offer a richer and more flexible framework than their classical and intuitionistic counterparts. They are particularly suitable for modeling real-world systemssuch as social, transportation, or communication networkswhere uncertainty, hesitation, or multiple expert evaluations influence the assessment of connectivity and reliability.

3.2 Computational Complexity and Practical Implementation for Large-Scale Graphs

The computation of vertex and edge cohesive numbers in hesitant fuzzy graphs (HFGs) involves processing multiple possible membership degrees for each vertex and edge. Let $G = (V, E)$ denote a hesitant fuzzy graph, where each vertex $v_i \in V$ and edge $e_{ij} \in E$ are associated with hesitant membership sets $h(v_i)$ and $h(e_{ij})$, respectively. If each hesitant set contains k possible values on average, the time complexity for cohesive number computation increases significantly compared to classical fuzzy graphs.

I. Theoretical Complexity Analysis

- For a classical fuzzy graph, computing the vertex cohesive number $\alpha(G)$ requires $O(|V|^2)$ operations, as each pair of vertices is compared once based on single membership values.
- For an intuitionistic fuzzy graph, the inclusion of both membership and non-membership values increases the complexity slightly to $O(2|V|^2)$, which is still linear in the number of vertex pairs.
- For a hesitant fuzzy graph, however, each comparison involves operations over k-sized membership sets, making the overall complexity approximately $O(k^2|V|^2)$ for dense graphs, since each pairwise relation may require computing intersections or aggregations across hesitant sets.

Similarly, the computation of the edge cohesive number $\beta(G)$ for large networks with $|E|$ edges exhibits $O(k^2|E|)$ complexity. For sparse graphs, this cost remains manageable, but for dense or real-world networks (social, biological, or communication networks), the computational cost becomes significant.

II. Algorithmic and Heuristic Strategies

To address scalability challenges, several computational strategies can be adopted:

1. **Set Reduction Heuristic:** Reduce each hesitant membership set $\{\mu_1, \mu_2, \ldots, \mu_k\}$ to a representative subset (e.g., by clustering or percentile selection), which decreases the effective k without major loss of accuracy.

2. **Parallel Cohesion Computation:** Since cohesive number calculations for distinct vertices and edges are independent, they can be parallelized using multi-threading or distributed computing frameworks such as *Apache Spark GraphX* or *Hadoop Graph Processing*.
3. **Incremental Cohesion Update:** In dynamic networks, instead of recomputing the entire cohesive matrix, update only the affected regions when vertices or edges change, leading to significant time savings.
4. **Graph Sampling Techniques:** For very large networks, compute cohesive numbers on sampled subgraphs and use statistical estimators (such as Monte Carlo averaging) to approximate global cohesion metrics.
5. **Matrix-Based Computation:** Represent hesitant fuzzy membership values as interval or multi-layer adjacency matrices, enabling the use of optimized linear algebra operations and sparse matrix storage for improved efficiency.

III. Practical Considerations

For practical implementation on large-scale networks ($|V| > 10^4$), the use of sparse matrix structures and parallel processing is essential to maintain tractable computation times. Hybrid approaches that combine **set reduction** with **incremental updates** provide a balance between accuracy and scalability. Such methods ensure that hesitant fuzzy cohesive measures remain computationally feasible even in complex real-world network analysis.

4 Application

Real-Time Validation of Vertex and Edge Cohesive Numbers in Transportation Networks

Optimizing Transportation Network Reliability using Vertex and Edge Cohesive Numbers in Hesitant Fuzzy Graphs We aim to apply the concepts of vertex cohesive number and edge cohesive number of hesitant fuzzy graphs to

1. Identify critical hubs (stations) that ensure the transportation network remains connected despite uncertainties.
2. Pinpoint crucial transportation links (roads, railways, etc.) that must be maintained to ensure network reliability and smooth operation.

To validate the theoretical findings, a real-time transportation dataset is utilized, representing major city routes and traffic flow data collected over multiple intervals. Each hub (vertex) represents a transport station, and each link (edge) carries hesitant fuzzy reliability values derived from fluctuating traffic densities and maintenance schedules.

The data set includes six primary hubs and eight direct connections.

Hubs (Vertices):

$H_1, H_2, H_3, H_4, H_5, H_6$ represent major bus or train stations in the city.

Links (Edges): Each link between hubs is assigned hesitant fuzzy values representing the uncertainty in the reliability of these links, based on traffic or maintenance disruptions.

The hesitant fuzzy values for the edges are as follows:

$$\begin{aligned}
(H_1, H_2) &: \{0.8, 0.9\},\\
(H_2, H_3) &: \{0.6, 0.7\},\\
(H_3, H_4) &: \{0.5, 0.6\},\\
(H_4, H_5) &: \{0.7, 0.8\},\\
(H_5, H_6) &: \{0.5, 0.6\},\\
(H_6, H_1) &: \{0.6, 0.7\},\\
(H_1, H_3) &: \{0.7, 0.8\},\\
(H_2, H_4) &: \{0.5, 0.6\}.
\end{aligned} \tag{4.1}$$

Step 1: Vertex Cohesive Number

The **vertex cohesive number** represents the smallest group of hubs that ensure connectivity in the network. We will examine the hesitant fuzzy values to find the largest subset of vertices that forms a cohesive set.

- For each pair of hubs, we look at the reliability of the connections. We can say that a pair of hubs is cohesive if the hesitant fuzzy values between them are relatively high (e.g., ≥ 0.6).

- In this example, the hesitant fuzzy values between hubs $(H_1, H_2), (H_1, H_3), (H_4, H_5)$ and (H_6, H_1) indicate higher reliability.
- The smallest set of hubs that ensures the network remains connected is $\{H\_1, H\_6\}$ because any additional hubs are connected through at least one reliable edge. This forms a cohesive group.
- **Vertex Cohesive Number**: 2 (i.e., H_1 and H_2).

Step 2: Edge Cohesive Number

The **edge cohesive number** is the largest subset of edges that ensures the graph remains connected. Here, we need to identify the critical links between hubs that maintain the transportation networks integrity.

- Consider the edges $(H_1, H_2), (H_1, H_3)$ and (H_4, H_5)which have relatively high reliability values (hesitant fuzzy values ≥ 0.7).
- These edges form the most reliable connections and should be prioritized to ensure network cohesion. If any of these links were disrupted, the connectivity between hubs would be significantly weakened.
- **Edge Cohesive Number**: 2 (i.e., the two edges (H_1, H_2) and (H_4, H_5)).

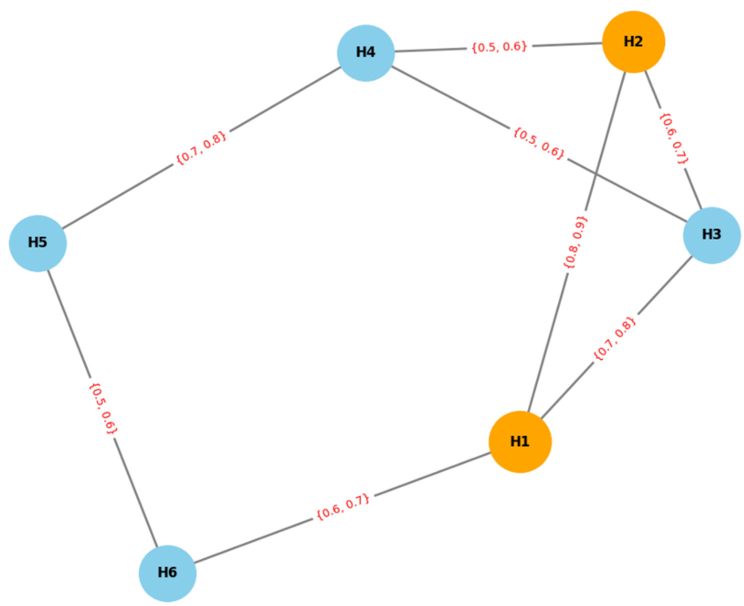

***Figure* 4.1:** *Transportation Network with Hesitant Fuzzy Values*

In this transportation network, the vertex cohesive number is 2, indicating that hubs H_1 and H_2 are the most critical nodes that must remain operational to ensure network connectivity. The edge cohesive number is also 2, signifying that the transportation links between H_1 and H_2 , and between H_4 and H_5 , are the most crucial connections to be prioritized for maintenance.

By focusing on these critical hubs and links, transportation planners can ensure the network remains reliable and functional, even under uncertain conditions such as fluctuating traffic or infrastructure maintenance.

5 Conclusion

The development of hesitant fuzzy graph theory and its subsequent advancements have significantly broadened the scope of graph theory applications. Whether in computer networks, social network analysis, or organizational studies, hesitant fuzzy graph structures offer a robust framework for understanding and analyzing complex relationships and interactions. This study demonstrates that by applying vertex and edge cohesive numbers to hesitant fuzzy graphs, transportation networks can be optimized for reliability and resilience, ensuring minimal disruption and efficient resource allocation under uncertain conditions. By utilizing vertex and edge cohesive numbers within hesitant fuzzy graph structures, this research provides a robust framework for optimizing transportation networks under uncertainty. However, the study has certain limitations. The computation of vertex cohesive number and edge cohesive number becomes complex for large-scale graphs, and the selection of hesitant membership sets may involve subjective judgment. Moreover, the present model assumes static hesitant values, whereas real-world systems often exhibit dynamic and time-varying uncertainty.

References

[1] K. R. Bhutani, Connectivity in fuzzy graphs, *Fuzzy Sets and Systems*, **33**(1) (1989), 133-138.

[2] S. M. Chen and S. C. Chang, Fuzzy transportation problems with mixed constraints using a fuzzy approach, *IEEE Transactions on Fuzzy Systems*, **19**(5) (2011), 832-842.

[3] A. N. Gani and S. Shajitha Begum, Degree, order and size in intuitionistic fuzzy graphs, International Journal of Algorithms, *Computing and Mathematics*, **3**(3) (2010), 11-15.

[4] N. Gani and V. T. Chandrasekaran, *A First Look at Fuzzy Graph Theory*, Allied Publishers Pvt. Ltd., India, 2010.

[5] P. Harinath and S. Lavanya, Fuzzy graph structures, *International Journal of Applied Engineering and Research*, **80**(10) (2015), 40165-40173.

[6] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co, New York, 1969.

[7] P. Harinath and S. Lavanya, Fuzzy graph structures and its properties, *International Journal of Mathematics Trends and Technology (IJMTT)*, **43**(9) (2016), 1-7.

[8] N. Joshi, S. S. Chauhan, and R. Raja, A new approach to solve mixed constraint transportation problem under fuzzy environment, *International Journal of Computers and Technology*, **16**(4) (2017), 6895-6902.

[9] M. Javaid, A. K. A. Khan and T. Rashid, Hesitant fuzzy graphs and their products, *Fuzzy Information and Engineering*, **12**(3) (2020), 238-252.

[10] K. Kalaiarasi and R. Divya, Shortest path on interval-valued intuitionistic trapeoidal neutrosophic fuzzy graphs with application, *Jnbha*, **53**(1) (2023), 243-252

[11] F. Karaaslan, Hesitant fuzzy graphs and their applications in decision making, *Journal of Intelligent & Fuzzy Systems*, **36**(3) (2019), 2729-2741.

[12] E. S. Kumar, Generalised graph structures, *Bulletin of Kerala Mathematics Association*, **3**(2) (2006), 67-123.

[13] T. Pathinathan and J. Jesintha Rosline, Double layered fuzzy graph, *Annals of Pure and Applied Mathematics*, **8**(1) (2014), 135-143.

[14] T. Pathinathan, J. Jon Arockiaraj and J. Jesintha Rosline, Hesitancy fuzzy graphs, *Indian Journal of Science and Technology*, **35**(8) (2015), 1-5.

[15] A. Rosenfield, Fuzzy graphs, in: L. A. Zadeh, K. S. Fu, M. Shimura (Eds.), *Fuzzy Sets and Their Application* , Academic Press, New York, (1975), 77-95.

[16] R. V. Ramakrishnan and T. Dinesh, On generalised fuzzy graph structures, *Applied Mathematical Sciences*, **5**(4) (2011), 173-180.

[17] M. S. Sunitha and S. Mathew, Fuzzy Graph Theory: A Survey, *Annals of Pure and Applied Mathematics*, **4**(1) (2013), 92-110.

[18] T. R. Sooraj, R. K. Mohanty and B. K. Tripathy, *Hesitant Fuzzy Soft Set Theory and Its Application in Decision Making*, Springer, (2017), doi:10.1007/981-10-3174-8_28.

[19] V. Torra, Hesitant fuzzy sets, *International Journal of Intelligent Systems*, **25**(6) (2010), 529-539.

[20] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, *The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea,* (2009), 1378-1382.

[21] L. A. Zadeh, Fuzzy sets. *Information and Control*, **8**(3) (1965, 338-353.

[22] H. J. Zimmermann, *Fuzzy Set Theory and Applications*, 2nd Revised edition, Kluwer Academic Publishers, 2001.