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LI'S RELATION INVOLVING THE NUMBER OF REPRESENTATIONS OF AN INTEGER AS A SUM OF SQUARES AND APPLICATION IN DIFFERENCE EQUATIONS

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Abstract

The formulae for the sum of inverses of odd divisors of an integer n are deduced by various authors in the literature. Here, in this paper, we exhibit that their expressions are related by certain identity of Li involving the number of representations of n as a sum of squares.
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1 Introduction

The formulae for the sum of inverses of odd divisors of an integer n are deduced by Jha [4, 14] and Glaisher [9,11]. Here, in this paper, we exhibit that their expressions are related by certain identity of Li involving the number of representations of n as a sum of squares.

Jha [4, 14] obtained the following expression for the sum of inverses of odd divisors of a positive integer n :

$$2(-1)^n D(n) := 2(-1)^n \sum_{odd d|n} \frac{1}{d} = \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} r_j(n), \tag{1.1}$$

where $r_k(n)$ is the number of representations of n as a sum of k squares [1, 12, 19]. On the other hand, we have the Glaisher's result [9, 11]:

$$2(-1)^n D(n) = \sum_{k=1}^{n} \frac{(-1)^k}{k} R_k(n), \tag{1.2}$$

where $R_k(n)$ is the number of representations of n as a sum of k nonvanishing squares [12].
Li [16] deduced the relation:

$$R_k(n) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} r_j(n), \quad n \geq 1, \tag{1.3}$$

In Sec. 2 we exhibit that (1.2) and (1.3) imply (1.1). Again, we obtain the inversion of (1.3). In Sec. 3 we show that $D(n)$ has connection with the Jacobi theta functions.

2 Glaisher, Jha and Li formulae

From (1.1), (1.2) and (1.3):

$$2(-1)^n \sum_{odd d|n} \frac{1}{d} = \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} r_j(n) = \sum_{j=1}^{n} (-1)^j r_j(n) \sum_{k=j}^{n} \frac{1}{k} \binom{k}{j},$$

where we used the property [22]:

$$\sum_{k=j}^{n} \frac{1}{k} \binom{k}{j} = \frac{1}{j} \binom{n}{j}. \tag{2.1}$$

We know [15, 24] that $r_k(n)$ is a polynomial of degree n, in k with the structure:

$$r_k(n) = \sum_{l=1}^{n} a(n,l) k^l, \tag{2.2}$$

such that:

$$a(n,n) = \frac{2^n}{n!}, \quad n \geq 0; \quad a(n,1) = 2(-1)^{n-1} \sum_{\text{odd } d \ln \frac{1}{d}}, \quad n \geq 1,$$

$$a(n,n-1) = -\frac{2^{n-1}}{(n-2)!}, \quad n \geq 2; \quad a(n,n-2) = \frac{2^{n-3}(3n-1)}{3(n-3)!}, \quad n \geq 3 \tag{2.3}$$

$$a(n,n-3) = \frac{2^{n-4}(n+2)(3-n)}{3(n-4)!}, \quad n \geq 4, \quad \text{etc.}$$

For example [12, 15, 26]:

$$r_k(1) = 2k, \quad r_k(2) = 2k(k-1), \quad r_k(3) = \frac{4}{3}k(k-1)(k-2),$$

$$r_k(4) = \frac{2}{3}k[3(2k-1) + k(k-1)(k-5)], \quad r_k(5) = \frac{4}{15}k(k-1)[3(2k-3) + k(k-4)(k-5)],$$

$$r_k(6) = \frac{4}{45}k(k-1)(k-2)[45 + (k-3)(k-4)(k-5)], \tag{2.4}$$

$$r_k(7) = \frac{8}{315}k(k-1)(k-2)(k-3)\left(k^3 - 15k^2 + 74k - 15\right), \dots .$$

The application of (2.2) in (1.3) gives the interesting expression:

$$R_k(n) = k! \sum_{l=1}^{n} a(n,l) S_l^{[k]}, \tag{2.5}$$

where participate the Stirling numbers of the second kind [5, 13, 17, 22]. From (2.3) and (2.5) it is possible to deduce results which are in harmony with Grosswald [12], in fact:

$$R_1(n) = r_1(n) = \sum_{l=1}^{n} a(n,l), \quad R_k(1) = 2k! S_1^{[k]} = \begin{cases} 2, & k=1 \\ 0, & k \geq 2 \end{cases}, \quad R_k(n) = 0, \quad k \geq n+1,$$

$$R_k(2) = 2k!\left(S_2^{[k]} - S_1^{[k]}\right) = \begin{cases} 0, & k=1 \\ 4, & k=2, \\ 0, & k \geq 3 \end{cases}, \quad R_{m-1}(m) = 0, \quad m \geq 2, \quad R_{n-2}(n) = 0, \quad n \geq 3, \tag{2.6}$$

$$R_k(3) = \frac{4}{3}k!\left(2S_1^{[k]} - 3S_2^{[k]} + S_3^{[k]}\right) = \begin{cases} 8, & k=3 \\ 0, & k \neq 3 \end{cases}, \quad R_{m-3}(m) = 2^{m-3}(m-3), \quad m \geq 4,$$

$$R_n(n) = n! \sum_{l=1}^{n} a(n,l) S_l^{[n]} = n! a(n,n) = 2^n, \quad n \geq 1, \dots$$

Remark 2.1. *The relation* (1.3) *is a binomial transform* [3] *whose inversion is immediate:*

$$r_k(n) = \sum_{j=1}^{m} \binom{k}{j} R_j(n) \quad \text{with} \quad m = \begin{cases} k, & 1 \leq k \leq n \\ n, & k \geq n+1 \end{cases}. \tag{2.7}$$

Remark 2.2. *It is easy to obtain the corresponding inversion of* (2.5)*, in fact:*

$$a(n,l) = \sum_{j=l}^{n} \frac{1}{j!} R_j(n) S_j^{(l)}, \tag{2.8}$$

with the presence of the Stirling numbers of the first kind [18, 22].

Remark 2.3. *The application of* (1.3) *in* (2.8) *gives the inversion of* (2.2)*:*

$$a(n,l) = \sum_{j=0}^{n} b(n,l,j) r_j(n), \quad b(n,l,j) = \frac{1}{j!} \sum_{t=0}^{n-j} \frac{(-1)^t}{t!} S_{t+j}^{(l)}. \tag{2.9}$$

The quantities $a(n,k)$ *can be written in terms of the partial Bell polynomials* [6, 8, 15, 20].
Hence, our work shows that Jha's and Glaisher's expressions are related through Li's identity.

3 Sum of inverses of odd divisors of a positive integer

Now we consider the arithmetic function:

$$A(n) := (-1)^n n D(n) = (-1)^{n-1} n \sum_{odd d in} \frac{1}{d}, \tag{3.1}$$

then is easy to obtain the values $A(1) = 1, A(2) = -2, A(3) = 4, A(4) = -4$, etc., thus it appears the sequence:

$$1, -2, 4, -4, 6, -8, 8, -8, 13, -12, 12, -16, 14, -16, 24, -16, 18, -26, 20, -24, 32, -24, 24, -32, 31, -28, \dots . \tag{3.2}$$

We visited the On-line Encyclopedia of Integer Sequences [27] and we find that (3.2) is the sequence A186690, therefore:

$$-\frac{1}{8} \frac{\vartheta_3''(0,q)}{\vartheta_3(0,q)} = \sum_{n=1}^{\infty} A(n) q^n = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} j q^j}{1 - q^{2j}}, \tag{3.3}$$

that is, the quantities (3.1) are connected to the Jacobi theta functions [2, 7, 23, 28].

On the other hand, Prof. Michael Somos [21] indicates the following relation between sequences:

$$A186690(n) = (-1)^{n-1} A002131(n), \tag{3.4}$$

hence:

$$A(n) = \begin{cases} -\left(\sigma(n) - \sigma\left(\frac{n}{2}\right)\right), & n \text{ is even} \\ \sigma(n), & n \text{ is odd} \end{cases} \tag{3.5}$$

involving the sum of divisors function $\sigma(n)$[10, 25, 26].

4 Application of $R_k(n)$ in solving difference equations

Consider that for $k \neq n, Y(h,k;n) = R_k(n) h^n \ \forall k \geq 1, n \geq 0$ and when $k = n$, there exists

$$Y(h,n;n) = Y(h;n) = R_n(n) h^n = 2^n h^n \ \forall n \geq 0. \tag{4.1}$$

Since by (2.5) we have

$$R_k(n) = k! \sum_{l=1}^{n} a(n,l) S_l^{[k]}, R_k(n) = 0 \ \ \forall k \geq n+1. \tag{4.2}$$

Then (4.2) satisfies the result

$$R_k(0) = 0 \ \ \forall k \geq 1. \tag{4.3}$$

Theorem 4.1. *If* $k \neq n, n \geq 0, k \geq 1$ *and* $h > 0$*, then* $Y(h,k;n)$ *of* (4.1) *satisfies a difference equation*

$$Y(h,k;n+1) - \frac{1}{h} Y(h,k;n) = 0. \tag{4.4}$$

Proof. Multiplying by h^n in left hand sides of difference equation (4.4) and then summing up $n = 0$ to $n = \infty$, we get

$$\sum_{n=0}^{\infty} Y(h,k;n+1) h^n - \sum_{n=0}^{\infty} Y(h,k;n) h^{n-1}. \tag{4.5}$$

Applying (4.1), for $k \neq n, Y(h,k;n) = R_k(n)h^n \ \forall k \geq 1, n \geq 0$, in equation (4.5) we find

$$\left(Y(h,k;1) + Y(h,k;2)h + Y(h,k;3)h^2 + \cdots\right) - \left(Y(h,k;0)\frac{1}{h} + Y(h,k;1) + Y(h,k;2)h + \cdots\right)$$

$$\left(R_k(1)h + R_k(2)h^3 + R_k(3)h^5 + \cdots\right) - \left(R_k(0)\frac{1}{h} + R_k(1)h + R_k(2)h^3 + \cdots\right). \tag{4.6}$$

Then in the relation (4.6) using the result (4.3) $\forall k \geq 1$, we get right hand sides of difference equation (4.4).

Hence, $Y(h,k;n) = R_k(n)h^n$ is a solution of difference equation (4.4) $\forall k \geq 1, n \geq 0$ and $k \neq n$. □

Theorem 4.2. *If $n \geq 0$ and $h > 0$, then $Y(h;n)$ given in* (4.1) *satisfies a difference equation*

$$Y(h;n+1) - \frac{1}{2h}Y(h;n) + \frac{1}{2^{n+1}h^{n+1}} = 0. \tag{4.7}$$

Proof. Multiplying by $2^n h^n$ in left hand sides of difference equation (4.7) and then summing up $n = 0$ to $n = \infty$, we get

$$\sum_{n=0}^{\infty} Y(h;n+1)2^n h^n - \sum_{n=0}^{\infty} Y(h;n)2^{n-1}h^{n-1} + \frac{1}{2h}. \tag{4.8}$$

Now, applying (4.1) for $Y(h;n) = 2^n h^n \ \forall k \geq 1, n \geq 0$, in equation (4.8), we find

$$\left(Y(h;1) + Y(h;2)2h + Y(h;3)2^2h^2 + \cdots\right) - \left(Y(h;0)\frac{1}{2h} + Y(h;1) + Y(h;2)2h + \cdots\right) + \frac{1}{2h}. \tag{4.9}$$

Since in formula (4.1), $Y(h;n) = R_n(n)h^n = 2^n h^n \ \forall n \geq 0$, which for $n = 0$ gives $Y(h;0) = 1$, therefore in (4.9) putting $Y(h;0) = 1$, we get zero, therefore right hand side of equation (4.7) is satisfied.

Hence, $Y(h;n) = R_n(n)h^n = 2^n h^n$ is the solution of equation (4.7). □

5 Conclusion

In this paper, we deduce the formulae for the sum of inverses of odd divisors of an integer n by Jha [4, 14] and Glaisher [9,11]. Again, we exhibit the expressions related by certain identity of Li involving the number of representations of n as a sum of squares. In the Section 4, we search some of the difference equations satisfying the function given by

$$Y(h,k;n) = \begin{cases} R_k(n)h^n \ \forall k \geq 1, n \geq 0 \text{ and } k \neq n; \\ R_n(n)h^n \ \forall n \geq 0 \text{ when } k = n. \end{cases} \tag{5.1}$$

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