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FIXED POINT THEOREMS FOR NEW CONTRACTIONS IN RECTANGULAR *b*-METRIC SPACES

Dileep Kumar Sharma[1] **and Jayesh Tiwari**[2]
[1]Department of Mathematics, Government Polytechnic College, Mandsaur, Madhya Pradesh, India-458001
[2]Department of Computer Science, Shri Vaishnav Institute of Management and Science,
Devi Ahilya University, Indore, Madhya Pradesh, India-452009
Email: dileepk.sharmadec1972@gmail.com; jayesh_tiwari2005@yahoo.com
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Abstract

This article aims to provides existence and uniqueness of some new innovative fixed point results regarding rectangular b-metric spaces as a generalization of metric spaces, b-metric spaces and rectangular metric spaces and confirm them with some appropriate examples.
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1 Introduction

Branciari[1] first proposed the idea of a rectangular metric space in 2000. In rectangular/Branciari metric space, a different inequality-the so-called rectangular inequality replaced the triangle inequality of a metric space. The renowned Banach contraction mapping concept was further expanded upon in this paper[4] in the setting of rectangular metric spaces. Since then, other fixed point theorems for different contractions in Branciari metric space have been developed, including[3,5,7,11,12,13,17,20].

Rectangular b-metric space as an idea, was first presented in 2015 by George *et al.* [8]. Metric space, Branciari metric space and rectangular b-metric space are extensions of one another. They provided fixed point theorems and generalised the Banach Stefan contraction principle in the context of rectangular b-metric spaces. After that, a number of publications expanded the traditional Banach Stefan contraction principle in the setting of partial b-metric space, b-metric space and Branciari b-metric space [6,9,10,14,16,17,19,20].

Branciari b-metric spaces were used by Mitrovic[15] in 2017 to demonstrate an analogue of the Banach Stefan contraction principle theorem.

We expand the findings of George *et al.* [8] and Mitrovic [15] in *RbMS* by establishing a few new fixed point theorems in this study. As additional evidence for these findings, we provide some creative instances.

2 Preliminaries

Definition 2.1 ([2])**.** *Let X be a space and $s \geq 1$ be a fixed real number. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:* (bm_1) $d(x,y) = 0$ *iff* $x = y$ *for all* $x, y \in X$,
(bm_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$,
(bm_3) $d(x,z) \leq s\Big[d(x,y) + d(y,z)\Big]$ *for all* $x, y, z \in X$.
Then the pair (X, d) *is called a b-metric space.*

Definition 2.2 ([1])**.** *Let X be a space. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:*
(Rm_1) $d(x,y) = 0$ *iff* $x = y$ *for all* $x, y \in X$,
(Rm_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$,
(Rm_3) $d(x,z) \leq d(x,u) + d(u,v) + d(v,z)$ *for all distinct points* $x, z, u, v \in X$.
Then d is called a rectangular metric on X and the pair (X, d) is called a rectangular metric space (in short RMS).

Definition 2.3 ([8]). *Let X be a space. If a function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions:*

(Rbm_1) $d(x, y) = 0$ *iff* $x = y$ *for all* $x, y \in X$,

(Rbm_2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

(Rbm_3) $d(x, z) \leq s\Big[d(x, u) + d(u, v) + d(v, z)\Big]$, *for some fixed* $s \geq 1$ *and for all distinct points* $x, z, u, v \in X$.

Then d is called a rectangular b-metric on X and the pair (X, d) is called a rectangular b-metric space (in short RbMS).

Note that every metric space is a rectangular metric space (RMS) and every rectangular metric space is a rectangular b-metric space (with coefficient $s = 1$). However the converse is not necessarily true. Also every metric space is a b-metric space with coefficient $s = 1$ and every b-metric space is a rectangular b-metric space (RbMS) with coefficient s^2, but the converse is not necessarily true.

Example 2.1. Let $X = \mathbb{N}$ and define a function $d : X * X \to \mathbb{R}^+$ by

$$d(u,v) = \begin{cases} 0, \text{ if } u = v, \\ c\lambda, \text{ if } (u,v) \in \{4,5\} \text{ and } u \neq v, \\ \lambda, \text{ if } u \text{ and } v \text{ do not belong to } \{4,5\} \text{ and } u \neq v, \end{cases}$$

where $\lambda > 0$ and $c > 3$.

Hence (X, d) is a rectangular b-metric space with coefficient $s = \frac{c}{3} > 1$,

but (X, d) is not a rectangular metric space and hence not a metric space, as

$$d(4, 5) = c\lambda > 3\lambda = d(4, 3) + d(3, 2) + d(2, 5).$$

Example 2.2. Let $X = \mathbb{N}$ and define a function $d : X * X \to \mathbb{R}^+$ by

$$d(u,v) = \begin{cases} 0, \text{ if } u = v, \\ 8\lambda, \text{ if } u = 1, v = 4, \\ 3\lambda, \text{ if } (u,v) \in \{1,2,3\} \text{ and } u \neq v, \\ \lambda, otherwise, \end{cases}$$

where $\lambda > 0$ is a constant.

Then (X, d) is a rectangular b-metric space with coefficient $s = \frac{8}{7} > 1$

but (X, d) is not a rectangular metric space and hence not a metric space, as

$$d(1, 4) = 8\lambda > 7\lambda = d(1, 2) + d(2, 3) + d(3, 4).$$

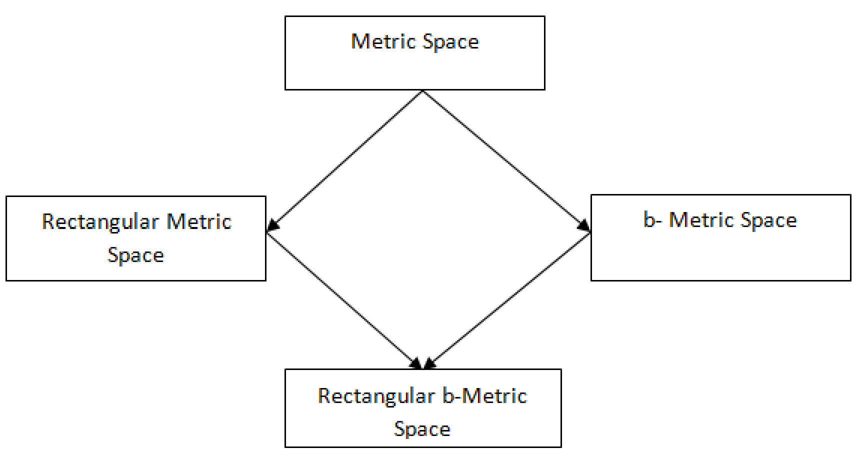

***Figure* 2.1** : *Relation between MS, b-MS, RMS and RbMS*

Definition 2.4. *Let (X, d) be a rectangular b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then,* *(a) The sequence $\{x_n\}$ is said to be rectangular b-convergent in (X, d) and converges to x, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$ i.e. $\lim_{n\to\infty} x = x$.*

(b) The sequence $\{x_n\}$ is said to be rectangular b-Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exist $m, n \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$ i.e. $\lim_{n,m\to\infty} d(x_n, x_m) = d(x, x)$ i.e. limit exist and finite.

(c) The space (X, d) is said to be a complete rectangular b-metric space if every rectangular b-Cauchy sequence in X converges to some $x \in X$.

3 Main Results

In this section we state our main results and prove existence and uniqueness of fixed points for new contractions in *RMS* and *RbMS*.

Theorem 3.1. *Let (H, d_r) be a complete rectangular b-metric space. Let T be a self map on H such that for any $\mu, \nu \in H$*

$$d_r(T\mu, T\nu) \leq a \max\Big\{d_r(\mu, T\mu), d_r(\nu, T\nu), d_r(\mu, \nu)\Big\} + b\Big\{d_r(\mu, T\mu) + d_r(\nu, T\nu)\Big\}, \tag{3.1}$$

where $a, b \in [0, \frac{1}{s})$ such that $b + s(a + b) < 1$ and $s > 1$. Then T has a unique fixed point.

Proof. Assume that $\nu_0 \in H$ be arbitrary. We establish a sequence $\{\nu_n\}$ by $\nu_{n+1} = T\nu_n \ \forall\ n \geq 0$. We prove that $\{\nu_n\}$ is a Cauchy sequence.

If $\nu_n = \nu_{n+1}$ then ν_n is obviously a fixed point of T.

Now assume that $\nu_n \neq \nu_{n+1} \ \forall\ n \geq 0$.

Setting $d_r(\nu_n, \nu_{n+1}) = d_n$, it follows from (3.1) that

$$\begin{aligned} d_r(\nu_n, \nu_{n+1}) &= d_r(T\nu_{n-1}, T\nu_n) \\ &\leq a \max\Big\{d_r(\nu_{n-1}, T\nu_{n-1}), d_r(\nu_n, T\nu_n), d_r(\nu_{n-1}, \nu_n)\Big\} \\ &\quad + b\Big\{d_r(\nu_{n-1}, T\nu_{n-1}) + d_r(\nu_n, T\nu_n)\Big\} \\ &= a \max\Big\{d_r(\nu_{n-1}, \nu_n), d_r(\nu_n, \nu_{n+1}), d_r(\nu_{n-1}, \nu_n)\Big\} \\ &\quad + b\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+1})\Big\} \\ &= a \max\Big\{d_r(\nu_{n-1}, \nu_n), d_r(\nu_n, \nu_{n+1})\Big\} \\ &\quad + b\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+1})\Big\}. \end{aligned} \tag{3.2}$$

$$\text{Let} \quad M_1 = \max\Big\{d_r(\nu_{n-1}, \nu_n), d_r(\nu_n, \nu_{n+1})\Big\}.$$

Now two cases arise.

Case-(i): If we suppose $M_1 = d_r(\nu_{n-1}, \nu_n)$.

Then from equality (3.2), we obtain

$$\begin{aligned} d_r(\nu_n, \nu_{n+1}) &\leq a\, d_r(\nu_{n-1}, \nu_n) + b\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+1})\Big\}, \\ (1 - b)d_r(\nu_n, \nu_{n+1}) &\leq (a + b)d_r(\nu_{n-1}, \nu_n), \\ d_r(\nu_n, \nu_{n+1}) &\leq \frac{(a + b)}{(1 - b)} d_r(\nu_{n-1}, \nu_n), \end{aligned}$$

$$d_n \leq k\, d_{n-1}, \ \ where \ \ k = \frac{(a + b)}{(1 - b)} \leq 1.$$

By successive use of the above inequality, we get

$$d_n \leq k^n d_0. \tag{3.3}$$

Taking limit as $n \to \infty$, we arrive at

$$\lim_{n\to\infty} d_n = 0.$$

Case-(ii): If suppose $M_1 = d_r(\nu_n, \nu_{n+1})$. Then, we obtain from inequality (3.2)

$$d_r(\nu_n, \nu_{n+1}) \leq a\, d_r(\nu_n, \nu_{n+1}) + b\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+1})\Big\},$$

$$(1-a-b)d_r\big(\nu_n,\nu_{n+1}\big) \le b\, d_r\big(\nu_{n-1},\nu_n\big),$$
$$d_r\big(\nu_n,\nu_{n+1}\big) \le \frac{b}{(1-a-b)} d_r\big(\nu_{n-1},\nu_n\big),$$
$$d_n \le k'\, d_{n-1},$$

where $k' = \frac{b}{1-a-b} \le 1$.

By successive use of the above inequality, we get

$$d_n \le (k')^n d_0. \tag{3.4}$$

Taking limit as $n \to \infty$, we arrive at

$$\lim_{n\to\infty} d_n = 0.$$

The presumption is that ν_0 is not a periodic point of T. In fact, if $\nu_0 = \nu_n$

$$d_r\big(\nu_0, T\nu_0\big) = d_r\big(\nu_n, T\nu_n\big),$$
$$d_r\big(\nu_0,\nu_1\big) = d_r\big(\nu_n,\nu_{n+1}\big).$$

This shows that $d_0 = d_n$.
Since $max\{k,k'\} = k$. Therefore from (3.3) and (3.4), we get

$$d_n \le k^n d_0.$$

Since $k < 1$, Hence the inequality mentioned above results in a contradiction.
Thus we assume that $\nu_n \ne \nu_m$ $\forall$ distinct $n,m \in \mathbb{N}$. Again using (3.1) and (3.3) for any $n \in \mathbb{N}$ we obtain.

$$\begin{aligned} d_r\big(\nu_n,\nu_{n+2}\big) = d_r\big(T\nu_{n-1},T\nu_{n+1}\big) &\le a \max\Big\{d_r\big(\nu_{n-1},T\nu_{n-1}\big), d_r\big(\nu_{n+1},T\nu_{n+1}\big), d_r\big(\nu_{n-1},\nu_{n+1}\big)\Big\} \\ &\quad + b\Big\{d_r\big(\nu_{n-1},T\nu_{n-1}\big) + d_r\big(\nu_{n+1},T\nu_{n+1}\big)\Big\} \\ &= a \max\Big\{d_r\big(\nu_{n-1},\nu_n\big), d_r\big(\nu_{n+1},\nu_{n+2}\big), d_r\big(\nu_{n-1},\nu_{n+1}\big)\Big\} \\ &\quad + b\Big\{d_r\big(\nu_{n-1},\nu_n\big) + d_r\big(\nu_{n+1},\nu_{n+2}\big)\Big\}. \end{aligned} \tag{3.5}$$

Let $M_2 = \max\Big\{d_r\big(\nu_{n-1},\nu_n\big), d_r\big(\nu_{n+1},\nu_{n+2}\big), d_r\big(\nu_{n-1},\nu_{n+1}\big)\Big\}$.

Case-(i): If $M_2 = d_r\big(\nu_{n-1},\nu_n\big)$. We obtain from inequality (3.5)

$$\begin{aligned} d_r\big(\nu_n,\nu_{n+2}\big) &\le a\, d_r\big(\nu_{n-1},\nu_n\big) + b\Big\{d_r\big(\nu_{n-1},\nu_n\big) + d_r\big(\nu_{n+1},\nu_{n+2}\big)\Big\} \\ &= (a+b)d_r\big(\nu_{n-1},\nu_n\big) + b\, d_r\big(\nu_{n+1},\nu_{n+2}\big) \\ &\le (a+b)k^{n-1}d_0 + b\, k^{n+1}d_0 \\ &= (a+b)k^{n-1}\left(1 + \frac{b}{(a+b)}\, k^2\right)d_0 \\ &= \gamma_1 k^{n-1} d_0, \end{aligned}$$

where $\gamma_1 = (a+b)\left(1 + \frac{b}{a+b}k^2\right)$.

Therefore, $d_r\big(\nu_n,\nu_{n+2}\big) \le \gamma_1 k^{n-1} d_0$.

Case-(ii): If $M_2 = d_r\big(\nu_{n+1},\nu_{n+2}\big)$. We obtain from inequality (3.5)

$$\begin{aligned} d_r\big(\nu_n,\nu_{n+2}\big) &\le a\, d_r\big(\nu_{n+1},\nu_{n+2}\big) + b\Big\{d_r\big(\nu_{n-1},\nu_n\big) + d_r\big(\nu_{n+1},\nu_{n+2}\big)\Big\} \\ &= (a+b)d_r\big(\nu_{n+1},\nu_{n+2}\big) + b\, d_r\big(\nu_{n-1},\nu_n\big) \\ &= (a+b)k^{n+1}d_0 + b\, k^{n-1}d_0 \\ &= (a+b)k^{n-1}\left(k^2 + \frac{b}{a+b}\right)d_0 \\ &= \gamma_2 k^{n-1} d_0, \end{aligned}$$

where $\gamma_2 = (a+b)\left(k^2 + \frac{b}{a+b}\right)$.

Therefore, $d_r(\nu_n, \nu_{n+2}) \leq \gamma_2 k^{n-1} d_0$.

Case-(iii): If $M_2 = d_r(\nu_{n-1}, \nu_{n+1})$. We obtain from inequality (3.5)

$$\begin{aligned}
d_r(\nu_n, \nu_{n+2}) &\leq a\, d_r(\nu_{n-1}, \nu_{n+1}) + b\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_{n+1}, \nu_{n+2})\Big\} \\
&\leq sa\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+2}) + d_r(\nu_{n+2}, \nu_{n+1})\} \\
&\quad + b\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_{n+1}, \nu_{n+2})\} \\
(1-sa) d_r(\nu_n, \nu_{n+2}) &\leq (sa+b)k^{n-1} d_0 + (sa+b)k^{n+1} d_0 \\
&= (sa+b)k^{n-1}(1+k^2) d_0 \\
d_r(\nu_n, \nu_{n+2}) &\leq \frac{(sa+b)}{(1-sa)} k^{n-1}(1+k^2) d_0 \\
&= \gamma_3 k^{n-1} d_0,
\end{aligned}$$

where $\gamma_3 = \frac{(sa+b)}{(1-sa)}(1+k^2)$.

Now, choose $\gamma = max\{\gamma_1, \gamma_2, \gamma_3\}$.
Therefore, $d_r(\nu_n, \nu_{n+2}) \leq \gamma k^{n-1} d_0$.

We take into account $d_r(\nu_n, \nu_{n+p})$ in two situations for the sequence $\{\nu_n\}$.

If $p = 2m+1$ (say odd). Then by (3.3), we obtain

$$\begin{aligned}
d_r(\nu_n, \nu_{n+2m+1}) &\leq s\Big[d_r(\nu_n, \nu_{n+1}) + d_r(\nu_{n+1}, \nu_{n+2}) + d_r(\nu_{n+2}, \nu_{n+2m+1})\Big] \\
&\leq s\big[d_n + d_{n+1}\big] + s^2\Big[d_r(\nu_{n+2}, \nu_{n+3}) + d_r(\nu_{n+3}, \nu_{n+4}) \\
&\quad + d_r(\nu_{n+4}, \nu_{n+2m+1})\Big] \\
&\leq s\big[d_n + d_{n+1}\big] + s^2\big[d_{n+2} + d_{n+3}\big] + s^3\big[d_{n+4} + d_{n+5}\big] \\
&\quad + ... + s^m d_{n+2m} \\
&\leq s\big[k^n d_0 + k^{n+1} d_0\big] + s^2\big[k^{n+2} d_0 + k^{n+3} d_0\big] + s^3\big[k^{n+4} d_0 + k^{n+5} d_0\big] \\
&\quad + ... + s^m k^{n+2m} d_0 \\
&\leq sk^n\big[1 + sk^2 + s^2k^4 + ...\big] d_0 + sk^{n+1}\big[1 + sk^2 + s^2k^4 + ... + \big] d_0 \\
&= \big[1 + sk^2 + s^2k^4 + ...\big]\big(sk^n + sk^{n+1}\big) d_0 \\
&= \frac{(1+k)}{(1-sk^2)} sk^n d_0. \qquad (3.6)
\end{aligned}$$

If $p = 2m$ (say even). Then by (3.3), we obtain

$$\begin{aligned}
d_r(\nu_n, \nu_{n+2m}) &\leq s\Big[d_r(\nu_n, \nu_{n+1}) + d_r(\nu_{n+1}, \nu_{n+2}) + d_r(\nu_{n+2}, \nu_{n+2m})\Big] \\
&\leq s\big[d_n + d_{n+1}\big] + s^2\Big[d_r(\nu_{n+1}, \nu_{n+3}) + d_r(\nu_{n+3}, \nu_{n+4}) \\
&\quad + d_r(\nu_{n+4}, \nu_{n+2m})\Big] \\
&\leq s\big[d_n + d_{n+1}\big] + s^2\big[d_{n+2} + d_{n+3}\big] + s^3\big[d_{n+4} + d_{n+5}\big] \\
&\quad + ... + s^{m-1}\big[d_{2m-4} + d_{2m-3}\big] + s^{m-1} d_r(\nu_{n+2m-2}, \nu_{n+2m}) \\
&\leq s\big[k^n d_0 + k^{n+1} d_0\big] + s^2\big[k^{n+2} d_0 + k^{n+3} d_0\big] + s^3\big[k^{n+4} d_0 + k^{n+5} d_0\big]
\end{aligned}$$

$$+ ... + s^{m-1}\left[k^{2m-4}d_0 + k^{2m-3}d_0\right] + s^{m-1}\gamma k^{n+2m-3}d_0$$
$$\le sk^n\left[1 + sk^2 + s^2k^4 + ...\right]d_0 + sk^{n+1}\left[1 + sk^2 + s^2k^4 + ...\right]d_0$$
$$+ s^{m-1}\gamma k^{n+2m-3}d_0.$$

i.e.

$$d_r\big(\nu_n, \nu_{n+2m}\big) \le \frac{1+k}{1-sk^2}sk^n d_0 + s^{m-1}\gamma k^{n+2m-3}d_0. \tag{3.7}$$

Taking the limit $n \to \infty$ of (3.6) and (3.7) we get

$$\lim_{n\to\infty} d_r\big(\nu_n, \nu_{n+p}\big) = 0 \quad \forall\ p > 0.$$

As a result in H, $\{\nu_n\}$ is a Cauchy sequence. Since (H, d_r) is complete $\exists\ \rho \in H$ such that

$$\lim_{n\to\infty} \nu_n = \rho.$$

We will now demonstrate that ρ is a fixed point of T. Assume that $T\rho \neq \rho$

$$\begin{aligned}
d_r\big(\rho, T\rho\big) &\le s\Big[d_r\big(\rho, \nu_n\big) + d_r\big(\nu_n, \nu_{n+1}\big) + d_r\big(\nu_{n+1}, T\rho\big)\Big] \\
&= s\Big[d_r\big(\rho, \nu_n\big) + d_n + d_r(T\nu_n, T\rho\big)\Big] \\
&\le s\Big[d_r\big(\rho, \nu_n\big) + d_n + a\max\Big\{d_r\big(\nu_n, T\nu_n\big), d_r(\rho, T\rho), d_r(\nu_n, \rho\big)\Big\} \\
&\quad + b\Big\{d_r\big(\nu_n, T\nu_n\big) + d_r(\rho, T\rho\big)\Big\}\Big] \\
&= s\Big[d_r\big(\rho, \nu_n\big) + d_n + a\max\Big\{d_r\big(\nu_n, \nu_{n+1}\big), d_r(\rho, T\rho), d_r(\nu_n, \rho\big)\Big\} \\
&\quad + b\Big\{d_r\big(\nu_n, \nu_{n+1}\big) + d_r(\rho, T\rho), \Big\}\Big] \\
&= sd_r\big(\rho, \nu_n\big) + s\Big[d_n + a\max\Big\{d_r\big(\nu_n, \nu_{n+1}\big), d_r(\rho, T\rho), d_r(\nu_n, \rho\big)\Big\} \\
&\quad + b\Big\{d_r\big(\nu_n, \nu_{n+1}\big) + d_r(\rho, T\rho\big)\Big\}\Big].
\end{aligned} \tag{3.8}$$

Let $M_3 = \max\Big\{d_r\big(\nu_n, \nu_{n+1}\big), d_r(\rho, T\rho), d_r(\nu_n, \rho\big)\Big\}$.

Case-(i): If $M_3 = d_r\big(\nu_n, \nu_{n+1}\big) = d_n$. We obtain from inequality (3.8)

$$d_r\big(\rho, T\rho\big) \le sd_r\big(\rho, \nu_n\big) + s\Big[d_n + a\,d_n + b\Big\{d_n + d_r(\rho, T\rho\big)\Big\}\Big].$$

$$(1 - sb)d_r\big(\rho, T\rho\big) \le sd_r\big(\rho, \nu_n\big) + s(1 + a + b)d_n.$$

$$d_r\big(\rho, T\rho\big) \le \frac{s}{(1-sb)}d_r\big(\rho, \nu_n\big) + \frac{s(1+a+b)}{(1-sb)}d_n.$$

$$d_r\big(\rho, T\rho\big) \le \frac{s}{(1-sb)}d_r\big(\rho, \nu_n\big) + \frac{s(1+a+b)}{(1-sb)}k^n d_0.$$

Taking the limit $n \to \infty$, we obtain

$$d_r\big(\rho, T\rho\big) = 0 \quad \text{i.e. } T\rho = \rho.$$

Thus ρ is a fixed point of T.

Case-(ii): If $M_3 = d_r\big(\rho, T\rho\big)$. We obtain from inequality (3.8)

$$d_r\big(\rho, T\rho\big) \le sd_r\big(\rho, \nu_n\big) + s\Big[d_n + ad_r(\rho, T\rho\big) + b\Big\{d_r\big(\nu_n, \nu_{n+1}\big) + d_r(\rho, T\rho\big)\Big\}\Big].$$

Therefore,

$$(1 - as - bs)d_r\big(\rho, T\rho\big) \le sd_r\big(\rho, \nu_n\big) + s(1 + b)d_n.$$

$$\begin{aligned}
d_r\big(\rho, T\rho\big) &\le \frac{s}{(1-as-bs)}d_r\big(\rho, \nu_n\big) + \frac{s(1+b)}{(1-as-bs)}d_n \\
&\le \frac{s}{(1-as-bs)}d_r\big(\rho, \nu_n\big) + \frac{s(1+b)}{(1-as-bs)}k^n d_0.
\end{aligned}$$

Taking the limit $n \to \infty$, we obtain

$d_r(\rho, T\rho) = 0$ i.e. $T\rho = \rho$. hence, a fixed point of T is ρ.

Case-(iii): If $M_3 = d_r(\nu_n, \rho)$. We obtain from inequality (3.8)

$$\begin{aligned} d_r(\rho, T\rho) &\leq s d_r(\rho, \nu_n) + s\Big[d_n + a d_r(\nu_n, \rho) + b\Big\{d_n + d_r(\rho, T\rho)\Big\}\Big]. \\ &\leq s(1+a) d_r(\rho, \nu_n) + s(1+b) d_n + sb d_r(\rho, T\rho). \\ (1 - sb) d_r(\rho, T\rho) &\leq s(1+a) d_r(\rho, \nu_n) + s(1+b) d_n \end{aligned}$$

Taking the limit $n \to \infty$, we obtain

$d_r(\rho, T\rho) = 0$ i.e. $T\rho = \rho$. Hence a fixed point of T is ρ.

Next, assume instead that T has two fixed points, ρ_1 and ρ_2.

$$\begin{aligned} d_r(\rho_1, \rho_2) = d_r(T\rho_1, T\rho_2) &\leq a \max\Big\{d_r(\rho_1, T\rho_1), d_r(\rho_2, T\rho_2), d_r(\rho_1, \rho_2)\Big\} \\ &\quad + b\Big\{d_r(\rho_1, T\rho_1) + d_r(\rho_2, T\rho_2)\Big\} \\ &= a \max\Big\{d_r(\rho_1, \rho_1), d_r(\rho_2, \rho_2), d_r(\rho_1, \rho_2)\Big\} \\ &\quad + b\Big\{d_r(\rho_1, \rho_1) + d_r(\rho_2, \rho_2)\Big\} \\ \text{or } \ d_r(\rho_1, \rho_2) &\leq a\, d_r(\rho_1, \rho_2). \end{aligned}$$

i.e. $(1-a) d_r(\rho_1, \rho_2) \leq 0$.

$$1 - a \neq 0 \quad \therefore \quad d_r(\rho_1, \rho_2) = 0 \Rightarrow \rho_1 = \rho_2.$$

Hence a unique fixed point of T is ρ. □

Theorem 3.2. *Let (H, d_r) be complete rectangular b-metric space. Let T be a self map on H satisfying for any $\mu, \nu \in H$ such that*

$$d_r(T\mu, T\nu) \leq a\Big\{d_r(\mu, T\mu) + d_r(\nu, T\nu)\Big\} + b\Big\{d_r(\mu, \nu) + d_r(\nu, T\nu)\Big\}, \tag{3.9}$$

where $a, b \in [0, \frac{1}{s}]$ such that $a + b \leq \frac{1}{2}$, $sa + b(2s+1) < 1$ and $s > 1$. Then T has a unique fixed point.

Proof. Let $\nu_0 \in H$ be an arbitrary point. We establish a sequence $\{\nu_n\}$ by $\nu_{n+1} = T\nu_n$ for all $n \geq 0$. Firstly we prove that $\{\nu_n\}$ is a Cauchy sequence.
If $\nu_n = \nu_{n+1}$, then we get

$$\nu_n = \nu_{n+1} = T\nu_n.$$

hence ν_n is fixed point of T.
Now assume that $\nu_n \neq \nu_{n+1}$ for all $n \geq 0$.
Setting $d_r(\nu_n, \nu_{n+1}) = d_n$, it follows from (3.9) that

$$\begin{aligned} d_r(\nu_n, \nu_{n+1}) = d_r(T\nu_{n-1}, T\nu_n) &\leq a\Big\{d_r(\nu_{n-1}, T\nu_{n-1}) + d_r(\nu_n, T\nu_n)\Big\} + b\Big\{d_r(\nu_{n-1}, \nu_n) \\ &\quad + d_r(\nu_n, T\nu_n)\Big\} \\ &= a\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+1})\Big\} + b\Big\{d_r(\nu_{n-1}, \nu_n) \\ &\quad + d_r(\nu_n, \nu_{n+1})\Big\}. \\ \big(1 - a - b\big) d_n &\leq (a+b) d_{n-1}. \\ d_n &\leq \frac{(a+b)}{\big(1-a-b\big)} d_{n-1}. \\ d_n &\leq \delta\, d_{n-1}; \qquad \text{where } \ \delta = \frac{(a+b)}{\big(1-a-b\big)} \leq 1. \end{aligned}$$

By successive use of the above inequality, we get

$$d_n \leq \delta^n d_0. \tag{3.10}$$

Taking limit as $n \to \infty$, we arrive at that

$$\lim_{n\to\infty} d_n = 0.$$

The presumption is that ν_0 is not a periodic point of T. In fact, if $\nu_0 = \nu_n$

$$\begin{aligned} d_r(\nu_0, T\nu_0) &= d_r(\nu_n, T\nu_n). \\ d_r(\nu_0, \nu_1) &= d_r(\nu_n, \nu_{n+1}). \\ \text{i.e. } d_0 &= d_n \leq \delta^n d_0. \qquad \{\text{by equation (3.10)}\} \end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$ i.e. $d_r(\nu_0, \nu_1) = 0$ or $\nu_0 = \nu_1$ so ν_0 is a fixed point of T. Thus we assume that $\nu_n \neq \nu_m$ for all distinct $n, m \in \mathbb{N}$. Again using (3.9) and (3.10) for any $n \in \mathbb{N}$ we obtain

$$\begin{aligned} d_r(\nu_n, \nu_{n+2}) = d_r(T\nu_{n-1}, T\nu_{n+1}) &\leq a\Big\{d_r(\nu_{n-1}, T\nu_{n-1}) + d_r(\nu_{n+1}, T\nu_{n+1})\Big\} \\ &\quad + b\Big\{d_r(\nu_{n-1}, \nu_{n+1}) + d_r(\nu_{n+1}, T\nu_{n+1})\Big\} \\ &= a\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_{n+1}, \nu_{n+2})\Big\} \\ &\quad + b\Big\{d_r(\nu_{n-1}, \nu_{n+1}) + d_r(\nu_{n+1}, \nu_{n+2})\Big\} \\ &\leq a\{d_{n-1} + d_{n+1}\} + bs\Big\{d_r(\nu_{n-1}, \nu_n) + d_r(\nu_n, \nu_{n+2}) \\ &\quad + d_r(\nu_{n+2}, \nu_{n+1})\Big\} + b\, d_r(\nu_{n+1}, \nu_{n+2}). \\ (1 - bs)d_r(\nu_n, \nu_{n+2}) &\leq a\{d_{n-1} + d_{n+1}\} + bs\{d_{n-1} + d_{n+1}\} + b\, d_{n+1} \\ &= (a + bs)d_{n-1} + (a + bs + b)d_{n+1} \\ &\leq (a + bs)\delta^{n-1}d_0 + (a + bs + b)\delta^{n+1}d_0 \\ d_r(\nu_n, \nu_{n+2}) &\leq \frac{(a + bs)}{1 - bs}\delta^{n-1}d_0 + \frac{(a + bs + b)}{(1 - bs)}\delta^{n+1}d_0 \\ &= \lambda\delta^{n-1}d_0 (\lambda = \frac{(a + bs)}{1 - bs} + \frac{(a + bs + b)}{(1 - bs)}\delta^2). \end{aligned}$$

Since (h, d) is $RbMS$ from condition (RbM_3) we have

$$\begin{aligned} d_r(\nu_m, \nu_n) &\leq s\Big[d_r(\nu_m, \nu_{m+p}) + d_r(\nu_{m+p}, \nu_{n+p}) + d_r(\nu_{n+p}, \nu_n)\Big] \\ d_r(\nu_m, \nu_n) &\leq s\Big[\delta^m d_r(\nu_0, \nu_p) + \delta^p d_r(\nu_m, \nu_n) + \delta^n d_r(\nu_0, \nu_p)\Big]. \\ \big(1 - s\delta^p\big)d_r(\nu_m, \nu_n) &\leq s\big(\delta^m + \delta^n\big)d_r(\nu_0, \nu_p) \\ d_r(\nu_m, \nu_n) &\leq \frac{s\big(\delta^m + \delta^n\big)}{\big(1 - s\delta^p\big)}d_r(\nu_0, \nu_p). \end{aligned} \tag{3.11}$$

Taking the limit $m, n \to \infty$ of (3.11) we obtain

$$\lim_{\infty} d_r(\nu_m, \nu_n) = 0.$$

As a result in H, $\{\nu_n\}$ is a Cauchy sequence. Since (H, d_r) is complete $\exists\, u \in H$ such that

$$\lim_{n\to\infty} \nu_n = u.$$

We now demonstrate that ρ is a fixed point of T. Assume that $T\rho \neq \rho$

$$\begin{aligned} d_r(\rho, T\rho) &\leq s\Big[d_r(\rho, \nu_n) + d_r(\nu_n, \nu_{n+1}) + d_r(\nu_{n+1}, T\rho)\Big] \\ &= s\Big[d_r(\rho, \nu_n) + d_n + d_r(T\nu_n, T\rho)\Big] \\ &\leq s\Big[d_r(\rho, \nu_n) + d_n + a\Big\{d_r(\nu_n, T\nu_n) + d_r(\rho, T\rho)\Big\} \end{aligned}$$

$$+ b\Big\{d_r(\nu_n, T\rho) + d_r(\rho, T\rho)\Big\}\Big]$$

$$= s\Big[d_r(\rho, \nu_n) + d_n + a\Big\{d_r(\nu_n, \nu_{n+1}) + d_r(\rho, T\rho)\Big\}$$

$$+ b\Big\{d_r(\nu_n, T\rho) + d_r(\rho, T\rho)\Big\}$$

$$= s\Big[d_r(\rho, \nu_n) + d_n + a\Big\{d_n + d_r(\rho, T\rho)\Big\} + b\Big\{d_r(\nu_n, T\rho) + d_r(\rho, T\rho)\Big\}$$

$$(1 - as - bs)d_r(\rho, T\rho) \leq s(1 + a)d_n + sbd_r(\nu_n, T\rho).$$

Taking the limit $n \to \infty$, we obtain from above inequality

$$(1 - as - bs)d_r(\rho, T\rho) \leq sbd_r(\rho, T\rho).$$

$$(1 - as - 2bs)d_r(\rho, T\rho) \leq 0.$$

$$d_r(\rho, T\rho) = 0. \text{ i.e. } T\rho = \rho.$$

Hence a fixed point of T is ρ.

For uniqueness Assume instead that T has two fixed points, ρ_1 and ρ_2.

$$d_r(\rho_1, \rho_2) = d_r(T\rho_1, T\rho_2) \leq a\Big\{d_r(\rho_1, T\rho_1) + d_r(\rho_2, T\rho_2)\Big\} + b\Big\{d_r(\rho_1, T\rho_2) + d_r(\rho_2, T\rho_2)\Big\}$$

$$= a\Big\{d_r(\rho_1, \rho_1) + d_r(\rho_2, \rho_2)\Big\} + b\Big\{d_r(\rho_1, \rho_2) + d_r(\rho_2, \rho_2)\Big\}.$$

$$d_r(\rho_1, \rho_2) \leq b\, d_r(\rho_1, \rho_2).$$

$$(1 - b)d_r(\rho_1, \rho_2) \leq 0.$$

Since $1 - b \neq 0$, therefore $d_r(\rho_1, \rho_2) = 0$ implies that $\rho_1 = \rho_2$.
Hence a unique fixed point of T is ρ. □

Corollary 3.1. *Let (H, d_r) be a complete rectangular b-metric space, T be a self map on H and satisfying for any $\mu, \nu \in H$ such that*

$$d_r(T\mu, T\nu) \leq a\Big[d_r(\mu, \nu) + d_r(\nu, T\nu)\Big]$$

where $a \in [0, \frac{1}{s}], as \leq 1$ and $s > 1$. Then T has a unique fixed point in H.

Example 3.1. Let $H = A \cup B$, where $A = \Big\{\frac{1}{n} : n \in \{2, 3, 4, 5\}\Big\}$ and $B = [1, 2]$. Define d_r as follows $d_r : H \times H \to [0, \infty)$ such that $d_r(\mu, \nu) = d_r(\nu, \mu)\ \forall\ \mu, \nu \in H$ and

$$\begin{cases} d_r(\frac{1}{2}, \frac{1}{3}) & = d_r(\frac{1}{4}, \frac{1}{5}) = 0.03 \\ d_r(\frac{1}{2}, \frac{1}{5}) & = d_r(\frac{1}{3}, \frac{1}{4}) = 0.02 \\ d_r(\frac{1}{2}, \frac{1}{4}) & = d_r(\frac{1}{5}, \frac{1}{3}) = 0.6 \\ d_r(\mu, \nu) & = |\mu - \nu|^2 \quad \text{otherwise.} \end{cases}$$

Here,

$$d_r(1, \frac{1}{4}) > d_r(1, \frac{1}{2}) + d_r(\frac{1}{2}, \frac{1}{3}) + d_r(\frac{1}{3}, \frac{1}{4}).$$

$$\frac{9}{16} > \frac{1}{4} + 0.03 + 0.02.$$

$$.5625 > .25 + 0.05.$$

$$.5625 > .3.$$

$$therefore, s = 1.875 > 1.$$

Since triangular inequality does not hold, therefore it is neither *RMS* nor a *MS*.

But (H, d_r) is a *RbMS* with coefficient $s = 1.875 > 1$.

Let $T : H \to H$ be defined as

$$Th = \begin{cases} \frac{1}{4} & \text{if } h \in A \\ \frac{1}{5} & \text{if } h \in B. \end{cases}$$

Then T satisfies each requirement of Theorem 3.1 and Theorem 3.2 and has a unique fixed point $\mu = \frac{1}{4}$.

Let $\mu = \frac{1}{2}$ and $\nu = 1$. Apply the condition of Theorem 3.1.

$$d_r(T_{\frac{1}{2}}, T_1\Big) \leq a \max\Big\{d_r(\frac{1}{2}, T_{\frac{1}{2}}\Big), d_r(1, T_1), d_r(\frac{1}{2}, 1\Big)\Big\} + b\Big\{d_r(\frac{1}{2}, T_{\frac{1}{2}}\Big) + d_r(1, T_1\Big)\Big\}.$$

$$d_r(\frac{1}{4}, \frac{1}{5}\Big) \leq a \max\Big\{d_r(\frac{1}{2}, \frac{1}{4}\Big), d_r(1, \frac{1}{5}\Big), d_r(\frac{1}{2}, 1\Big)\Big\} + b\Big\{d_r(\frac{1}{2}, \frac{1}{4}\Big) + d_r(1, \frac{1}{5}\Big)\Big\}.$$

$$0.03 = a \max\Big\{0.6, \frac{16}{25}, \frac{1}{4}\Big\} + b\Big\{0.6 + \frac{16}{25}\Big\}.$$

$$0.03 = .64a + 1.24b. \tag{3.12}$$

Now take $\mu = \frac{1}{3}$ and $\nu = 1$. Apply the condition of Theorem 3.1.

$$d_r(T_{\frac{1}{3}}, T_1\Big) \leq a \max\Big\{d_r(\frac{1}{3}, T_{\frac{1}{3}}\Big), d_r(1, T_1), d_r(\frac{1}{3}, 1\Big)\Big\} + b\Big\{d_r(\frac{1}{3}, T_{\frac{1}{3}}\Big) + d_r(1, T_1)\Big\}.$$

$$d_r(\frac{1}{4}, \frac{1}{5}\Big) \leq a \max\Big\{d_r(\frac{1}{3}, \frac{1}{4}\Big), d_r(1, \frac{1}{5}\Big), d_r(\frac{1}{3}, 1\Big)\Big\} + b\Big\{d_r(\frac{1}{3}, \frac{1}{4}\Big) + d_r(1, \frac{1}{5}\Big)\Big\}.$$

$$0.03 = a \max\Big\{0.02, \frac{16}{25}, \frac{4}{9}\Big\} + b\Big\{0.02 + \frac{16}{25}\Big\}.$$

$$0.03 = .64a + .66b. \tag{3.13}$$

From equation (3.12) and (3.13), we obtain $a = 0.04$ and $b = 0$

which satisfies the condition $b + s(a + b) < 1$.

We now demonstrate that T satisfies each requirement of Theorem 3.2 and has a unique fixed point $\mu = \frac{1}{4}$.

Let $\mu = \frac{1}{2}$ and $\nu = 1$ and apply the condition of Theorem 3.2.

$$d_r(T_{\frac{1}{2}}, T_1\Big) \leq a\Big\{d_r(\frac{1}{2}, T_{\frac{1}{2}}\Big) + d_r(1, T_1)\Big\} + b\Big\{d_r(\frac{1}{2}, 1\Big) + d_r(1, T_1\Big)\Big\}.$$

$$d_r(\frac{1}{4}, \frac{1}{5}\Big) = a\Big\{d_r(\frac{1}{2}, \frac{1}{4}\Big) + d_r(1, \frac{1}{5}\Big)\Big\} + b\Big\{d_r(\frac{1}{2}, 1\Big) + d_r(1, \frac{1}{5}\Big)\Big\}.$$

$$0.03 = a\Big\{0.6 + \frac{16}{25}\Big\} + b\Big\{0.25 + \frac{16}{25}\Big\}.$$

$$0.03 = 1.24a + .89b. \tag{3.14}$$

Now take $\mu = 1/3$ and $\nu = 1$

$$d_r(T_{\frac{1}{3}}, T_1\Big) \leq a\Big\{d_r(\frac{1}{3}, T_{\frac{1}{3}}\Big) + d_r(1, T_1)\Big\} + b\Big\{d_r(\frac{1}{3}, 1\Big) + d_r(1, T_1)\Big\}.$$

$$0.03 = .66a + 1.08b. \tag{3.15}$$

From equation (3.14) and (3.15) we obtain

$$a = 0.0077, b = 0.0231.$$

It fulfils the requirement $sa + b(2s + 1) < 1$.

4 Conclusion

Our fixed point theorems in this work, which expanded the findings of George *et al.* [8] and Mitrovic [15], were presented in rectangular b-metric spaces. In rectangular b-metric spaces, we developed some highly intriguing contractions. The contractions are supported by suitable instances.

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