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BICOMPLEX AND MULTICOMPLEX ANALOGUE OF FEW THEOREMS ON NORMAL FAMILY OF FUNCTIONS

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Abstract

This paper develops bicomplex and multicomplex analogues of fundamental theorems in complex analysis including Arzelà-Ascoli theorem, Montels theorems (holomorphic and meromorphic), Martys theorem, Riemann mapping theorem and Hurwitz's theorem. Each result is rigorously adapted to the bicomplex and multicomplex framework using the idempotent decomposition and classical component-wise reduction.
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1 Introduction

The theory of normal families occupies a central position in classical complex analysis, providing a powerful framework for understanding convergence properties, compactness criteria and structural behavior of families of holomorphic and meromorphic functions [1, 4, 8, 12, 16, 22, 24]. A family of functions is said to be *normal* if every sequence within it contains a subsequence that converges locally uniformly (possibly to infinity) in the extended complex plane. Originating in the pioneering work of Montel, Vitali and Marty, the concept has profound implications across various branches of analysis, from geometric function theory and value distribution theory to the theory of dynamical systems and conformal mappings. Fundamental results such as Montels theorem, Martys criterion and Arzelà-Ascoli theorem provide essential criteria for normality, while deep results like the Riemann mapping theorem highlight the geometric and topological significance of these families.

Over the past decades, there has been significant interest in extending classical complex analysis to higher-dimensional and richer algebraic frameworks. Among the most natural and fruitful generalizations are the theories of bicomplex and multicomplex numbers. The bicomplex numbers, defined as

$$\mathbb{BC} = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C},\ i^2 = j^2 = -1,\ ij = ji\},$$

form a commutative algebra with zero divisors, idempotents and a rich geometric structure. This algebra admits an idempotent decomposition, allowing many analytic problems to be reduced to classical ones on complex components, while simultaneously introducing new phenomena absent in the traditional setting. The theory further generalizes to multicomplex algebras $\mathbb{MC}_n$, where several commuting imaginary units give rise to increasingly intricate analytic structures and a wide spectrum of new behaviors. For more details about bicomplex and multicomplex numbers one can read [7, 17]. Extending classical theorems concerning normal families [1, 14] into these hypercomplex frameworks is both natural and mathematically compelling. Many foundational techniques in the complex setting rely on the field structure of $\mathbb{C}$ and its analytic completeness, neither of which carries over unchanged to bicomplex or multicomplex algebras. Simultaneously, the existence of idempotent components and the presence of zero divisors require a refined approach to concepts such as holomorphy, meromorphy, convergence and compactness. These challenges necessitate new formulations and proof strategies that remain faithful to classical results while accommodating the additional algebraic and analytic subtleties of hypercomplex settings. For further study of theory of bicomplex functions one can follow [19].

While significant progress has been made in bicomplex and multicomplex analysis, including Cauchy type integral formulas, Liouville and Picard type results, Nevanlinna theory and growth properties of entire functions relatively little attention has been devoted to the theory of normal families in these extended frameworks. In particular, systematic generalizations of cornerstone results such as Montels theorem, Martys criterion and Arzelà-Ascoli theorem remain largely undeveloped. Addressing this gap is the central objective of the present paper.

The aim of this work is to develop bicomplex and multicomplex analogues of several fundamental results concerning normal families of functions in classical complex analysis. We present rigorous generalizations of Arzelà-Ascoli theorem, Montels theorems for holomorphic and meromorphic functions, Martys theorem, Riemann mapping theorem and Hurwitz's theorem. Our approach is based on the idempotent decomposition of bicomplex and multicomplex functions, which allows many arguments to be reduced component-wise to the classical complex case while carefully handling the algebraic and analytic challenges arising in hypercomplex settings.

The contributions of this paper are twofold. First, we provide detailed proofs of these classical theorems in the bicomplex framework, thereby extending the foundational theory of normal families to a new algebraic setting. Second, we further generalize these results to multicomplex algebras, illustrating how increasing algebraic complexity influences analytic behavior. Beyond their intrinsic theoretical interest, these results open the door to new developments in hypercomplex function theory, complex dynamics and potential applications in mathematical physics and differential equations.

2 Preliminaries

In this section, we briefly summarize the foundational concepts and results that will be used throughout the paper. We begin by recalling the algebraic and analytic structure of bicomplex and multicomplex numbers, then introduce the notions of bicomplex and multicomplex holomorphic and meromorphic functions. Finally, we review the classical concept of normal families and discuss its extension to the hypercomplex setting.

2.1 Bicomplex Numbers and Their Structure

The *bicomplex numbers* form a commutative algebra over the real field defined as

$$\mathbb{BC} = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C},\ i^2 = j^2 = -1,\ ij = ji\},$$

where i and j are commuting imaginary units. Every bicomplex number $\omega \in \mathbb{BC}$ can thus be expressed as $\omega = z_1 + jz_2$, with $z_1, z_2 \in \mathbb{C}$. The algebra $\mathbb{BC}$ is a commutative ring with unity but not a field, as it contains non-trivial zero divisors.

A key structural property of $\mathbb{BC}$ is the existence of *idempotent elements*

$$e_1 = \frac{1+ij}{2} \quad and \quad e_2 = \frac{1-ij}{2},$$

which satisfy $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1e_2 = 0$. Using these, any bicomplex number can be uniquely represented in *idempotent form* as

$$\omega = \omega_1 e_1 + \omega_2 e_2,$$

where $\omega_1, \omega_2 \in \mathbb{C}$. This decomposition plays a central role in bicomplex analysis, as it reduces many analytic problems to the study of two independent complex components.

2.2 Multicomplex Numbers

The bicomplex algebra can be generalized to the *multicomplex algebras* $\mathbb{MC}_n$, defined inductively by introducing n mutually commuting imaginary units $i_1, i_2, \ldots, i_n$ with $i_k^2 = -1$. Elements of $\mathbb{MC}_n$ are of the form

$$\Omega = \sum_{\alpha} z_\alpha i^\alpha,$$

where $z_\alpha \in \mathbb{C}$ and i^α ranges over products of the imaginary units where α is a multi-index runs over the set $\{0,1\}^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_k \in \{0,1\}\}$. Similar to the bicomplex case, multicomplex algebras possess idempotent decompositions that allow reduction to several classical complex components. The idempotent structure becomes increasingly intricate as n grows, but the analytic principles remain parallel.

2.3 Bicomplex Holomorphic and Meromorphic Functions

Let $D \subset \mathbb{BC}$ be an open subset. A function $f : D \to \mathbb{BC}$ is called *bicomplex holomorphic* if it is complex-differentiable in each component and satisfies a bicomplex version of the CauchyRiemann equations. Equivalently, if

$$f(\omega) = F_1(\omega_1)e_1 + F_2(\omega_2)e_2,$$

where $\omega = \omega_1 e_1 + \omega_2 e_2$, then f is bicomplex holomorphic if and only if F_1 and F_2 are holomorphic functions in the classical complex sense. For further study on bicomplex holomorphic function one can read [20].

Similarly, a function $f : D \to \mathbb{BC} \cup \{\infty\}$ is said to be *bicomplex meromorphic* if each component F_1 and F_2 is meromorphic on its respective domain. The presence of zero divisors requires some care in defining poles and essential singularities, but the component-wise approach ensures that most classical results translate naturally.

2.4 Normal Families in the Classical and Bicomplex Settings

In classical complex analysis, a family $\mathcal{F}$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ is called normal [1] if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence that converges locally uniformly on Ω to a holomorphic function.

This notion extends naturally to bicomplex function theory:

Definition 2.1 (Normal Family of Bicomplex Valued Holomorphic Functions [5])**.** *Let $\Omega \subset \mathbb{BC}$ be a domain and let $\mathcal{F}$ be a family of bicomplex holomorphic functions* [20] *on Ω. We say that $\mathcal{F}$ is* normal [15] *in Ω if, for every sequence $\{F_n\} \subset \mathcal{F}$, there exists a subsequence $\{F_{n_k}\}$ that converges locally uniformly on Ω (with respect to the natural bicomplex topology) to a bicomplex holomorphic function.*

In classical complex analysis, a family $\mathcal{F}$ of meromorphic functions on a domain $\Omega \subset \mathbb{C}$ is called *normal* if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence that converges locally uniformly (with respect to the spherical metric) on Ω either to a meromorphic function or to ∞.

This notion extends naturally to bicomplex function theory:

Definition 2.2 ([6])**.** *Let $\Omega \subset \mathbb{BC}$ be a domain and also let $\mathcal{F}$ be a family of bicomplex meromorphic functions on Ω. We say that $\mathcal{F}$ is* normal *in Ω if, for every sequence $\{f_n\} \subset \mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that converges locally uniformly in the bicomplex spherical metric to a bicomplex meromorphic function or to ∞.*

Due to the idempotent decomposition, this is equivalent to requiring that the two component families $\{F_{1,n}\}$ and $\{F_{2,n}\}$ be normal in the classical sense on their respective domains.

In the paper two classical tools play a fundamental role in our analysis.

- **Idempotent decomposition principle:** Any bicomplex analytic property is equivalent to the corresponding pair of properties satisfied by the two complex components. This allows the transfer of classical theorems component-wise [17].
- **Component-wise convergence:** Local uniform convergence of a sequence of bicomplex functions is equivalent to local uniform convergence of each component sequence.

We do not mention the detailed theories of bicomplex and multicomplex analysis as those are available in [9, 10].

3 Bicomplex Analogues of Classical Theorems on Normal Families

In this section, we establish bicomplex analogues of several foundational results concerning normal families of functions. Each result is formulated and proved using the idempotent decomposition principle, which allows us to reduce many problems to their classical complex counterparts and then reconstruct the bicomplex statements. We begin with Arzelà-Ascoli theorem and then proceed to Montel's theorem in both holomorphic and meromorphic settings, Martys criterion and finally the Riemann mapping theorem.

The following lemma represents Arzelà-Ascoli theorem in classical complex analysis.

Lemma 3.1 (cf. [1, 24])**.** *Let (X, d) be a compact metric space and let (f_n) be a sequence of real-valued continuous functions on X. Suppose the family*

$$\{f_n : n \in \mathbb{N}\}$$

satisfies:

1. ***Equicontinuity:*** *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ and all $n \in \mathbb{N}$,*
$$|f_n(x) - f_n(y)| < \varepsilon.$$

2. ***Pointwise boundedness:*** *For each $x \in X$, the set*

$$\{f_n(x) : n \in \mathbb{N}\}$$

is bounded.

Then there exists a subsequence (f_{n_k}) which converges uniformly on X to a continuous function $f : X \to \mathbb{R}$.

3.1 Bicomplex Analogue of Arzelà-Ascoli theorem

The classical Arzelà-Ascoli theorem provides a fundamental compactness criterion for families of continuous functions. Its bicomplex analogue follows using the idempotent decomposition as we see in the following theorem.

Theorem 3.1. *Let $(\Omega, \|\cdot\|)$ be a compact subset of $\mathbb{BC}$ and $\mathcal{F} = \{f_n : \Omega \to \mathbb{BC}\}$ be a family of bicomplex valued continuous functions. Let us suppose that*

i. ***Equicontinuity:*** *For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|f_n(\omega) - f_n(\omega')\| < \varepsilon \quad \text{for all } \omega, \omega' \in \Omega \text{ with } \|\omega - \omega'\| < \delta \text{ and all } n \in \mathbb{N}.$$

ii. ***Pointwise boundedness:*** *For each $\omega \in \Omega$, there exists $M_\omega > 0$ such that*

$$\|f_n(\omega)\| \leq M_\omega \quad \text{for all } n \in \mathbb{N}.$$

Then there exists a subsequence (f_{n_k}) which converges uniformly on Ω to a bicomplex continuous function $f : \Omega \to \mathbb{BC}$.

Proof. We proceed step by step, constructing the proof entirely within the bicomplex setting.

Since Ω is compact in the bicomplex metric, it is separable. Let $\{\omega_m\}_{m=1}^{\infty}$ be a countable dense subset of Ω.

By pointwise boundedness, for each fixed ω_1, the sequence $\{f_n(\omega_1)\}_{n\in\mathbb{N}} \subset \mathbb{BC}$ is bounded. Since $\mathbb{BC} \cong \mathbb{C}^2$ as a vector space over $\mathbb{R}$, bounded sequences in $\mathbb{BC}$ have convergent subsequences. Choose a subsequence $\{f_{n_k^{(1)}}\}$ such that

$$f_{n_k^{(1)}}(\omega_1) \to f(\omega_1) \in \mathbb{BC}.$$

Next, consider ω_2. There exists a subsequence $\{f_{n_k^{(2)}}\} \subset \{f_{n_k^{(1)}}\}$ such that

$$f_{n_k^{(2)}}(\omega_2) \to f(\omega_2) \in \mathbb{BC}.$$

Now we proceed inductively for all ω_m. Using the standard diagonal argument, let us define

$$f_{n_k} := f_{n_k^{(k)}}.$$

Then $f_{n_k}(\omega_m) \to f(\omega_m)$ for all $m \in \mathbb{N}$.

Fix $\varepsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$\|f_n(\omega) - f_n(\omega')\| < \varepsilon/3 \quad \text{for all } n \in \mathbb{N}, \text{ whenever } \|\omega - \omega'\| < \delta.$$

Since $\{\omega_m\}$ is dense in the compact set Ω, we can cover Ω with finitely many balls of radius δ, centered at points $\omega_{m_1}, \ldots, \omega_{m_p}$. That is,

$$\Omega \subset \bigcup_{j=1}^{p} B_\delta(\omega_{m_j}).$$

Now we have already shown that $f_{n_k}(\omega_m) \to f(\omega_m)$ for all $m \in \mathbb{N}$. Hence, for each $j = 1, \ldots, p$, there exists N large enough such that for all $k \geq N$,

$$\|f_{n_k}(\omega_{m_j}) - f(\omega_{m_j})\| < \varepsilon/3.$$

For any $\omega \in \Omega$, choose j such that $\omega \in B_\delta(\omega_{m_j})$. Then for $k \geq N$,

$$\begin{aligned} \|f_{n_k}(\omega) - f(\omega)\| &\leq \|f_{n_k}(\omega) - f_{n_k}(\omega_{m_j})\| + \|f_{n_k}(\omega_{m_j}) - f(\omega_{m_j})\| + \|f(\omega_{m_j}) - f(\omega)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

The limit f is continuous because the convergence is uniform and each f_{n_k} is continuous.

We have constructed a subsequence (f_{n_k}) that converges uniformly on Ω to a bicomplex valued continuous function f. This completes the proof of the theorem. □

Remark 3.1. *We may now state the bicomplex analogue of Arzelà-Ascoli theorem in an alternative form:*

Let $\mathcal{F}$ be a family of bicomplex-holomorphic functions on a compact set $K \subsetneq \mathbb{BC}$. Then $\mathcal{F}$ is relatively compact in the supremum norm

$$\|f\|_\infty := \sup_{\omega \in K} \|f(\omega)\|, \quad f \in \mathcal{F},$$

if and only if it is equicontinuous and uniformly bounded on K.

The following lemma is the classical form of Montels theorem in complex analysis.

Lemma 3.2 (cf. [14, 24])**.** *Let $\mathcal{F}$ be a family of holomorphic functions on a domain $\Omega \subset \mathbb{C}$. If $\mathcal{F}$ is uniformly bounded on every compact subset of Ω, then $\mathcal{F}$ is a normal family.*

In other words, every sequence (f_n) in $\mathcal{F}$ has a subsequence that converges locally uniformly on Ω to a holomorphic function.

3.2 Bicomplex Analogue of Montel's Theorem (Holomorphic Version)

The next theorem represents the bicomplex version of Montel's theorem on holomorphic functions.

Theorem 3.2 (cf. [3, 15, 23])**.** *Let $\mathcal{F}$ be a family of bicomplex holomorphic functions on a domain $\Omega \subset \mathbb{BC}$. If $\mathcal{F}$ is locally uniformly bounded, i.e., for every compact set $K \subset \Omega$ there exists a constant $M_K > 0$ such that*

$$\|F(Z)\| \leq M_K \quad \text{for all } F \in \mathcal{F},\ Z \in K,$$

then $\mathcal{F}$ is a normal family. In other words, every sequence (F_n) in $\mathcal{F}$ has a subsequence that converges locally uniformly on Ω to a bicomplex holomorphic function.

Proof. Every bicomplex holomorphic function F can be written as

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2,$$

where F_1 and F_2 are holomorphic on domains $\Omega_1, \Omega_2 \subset \mathbb{C}$ respectively, corresponding to the projections of Ω.

The local uniform boundedness of $\mathcal{F}$ implies the existence of $M_K > 0$ such that

$$\|F(Z)\| = \max\{|F_1(z_1)|, |F_2(z_2)|\} \leq M_K \quad \forall Z \in K,\ F \in \mathcal{F}.$$

Hence, the families

$$\mathcal{F}_1 = \{F_1 \mid F \in \mathcal{F}\}, \quad \mathcal{F}_2 = \{F_2 \mid F \in \mathcal{F}\}$$

are locally bounded families of holomorphic functions in $\mathbb{C}$.

Let $K \subset \Omega$ be compact. For $Z = e_1 z_1 + e_2 z_2$ and $F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$, the derivative satisfies

$$F'(Z) = F_1'(z_1)e_1 + F_2'(z_2)e_2.$$

By applying Cauchy's integral formula separately to F_1 and F_2, we get that

$$|F_1'(z_1)| \leq \frac{M_K}{r_1} \quad \textit{and} \quad |F_2'(z_2)| \leq \frac{M_K}{r_2},$$

where $r_1, r_2 > 0$ are the radii of disks contained in Ω_1 and Ω_2 around z_1 and z_2, respectively.

Now, let us define the constant

$$C_K = \max\left\{\frac{M_K}{r_1}, \frac{M_K}{r_2}\right\}.$$

Thus,

$$\|F'(Z)\| = \max\{|F_1'(z_1)|, |F_2'(z_2)|\} \leq C_K,$$

which shows that $\mathcal{F}$ is equicontinuous on K.

By Lemma 3.1 as applied separately for each idempotent component, the local boundedness and equicontinuity imply that from any sequence $\{F_n\} \subset \mathcal{F}$, we can extract subsequences $\{F_{1,n_k}\}$ and $\{F_{2,n_k}\}$ converging uniformly on compact subsets of Ω_1 and Ω_2 to holomorphic functions F_1^* and F_2^*, respectively.

Let us define

$$F^*(Z) = F_1^*(z_1)e_1 + F_2^*(z_2)e_2.$$

Since F_1^* and F_2^* are holomorphic, the function F^* is bicomplex holomorphic. Moreover, $\{F_{n_k}\}$ converges locally uniformly on Ω to F^*.

Therefore, every sequence $\{F_n\}$ in $\mathcal{F}$ has a subsequence that converges locally uniformly to a bicomplex holomorphic function. Hence, $\mathcal{F}$ is a normal family.

Hence, the theorem is established. □

Remark 3.2. *Theorem 3.2 extends the classical Montel's theorem to the bicomplex setting by using the idempotent decomposition. Every bicomplex holomorphic function $F(Z)$ can be written as*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2,$$

where F_1 and F_2 are holomorphic in the complex planes corresponding to e_1 and e_2. This decomposition allows the reduction of many classical results, including Montel's theorem to componentwise arguments.

Remark 3.3. *Local uniform boundedness of a family $\mathcal{F}$ of bicomplex holomorphic functions implies that each component family $\mathcal{F}_1 = \{F_1\}$ and $\mathcal{F}_2 = \{F_2\}$ is locally uniformly bounded in $\mathbb{C}$. By the classical Montel's theorem, each component is normal, which then implies the normality of the bicomplex family.*

Remark 3.4. *Equicontinuity plays a crucial role in the proof. Using Cauchy estimates on each component function allows us to establish uniform bounds on derivatives, which in turn guarantees equicontinuity on compact sets. This step is completely analogous to the classical complex case but applied to the bicomplex components.*

Remark 3.5. *Theorem 3.2 is foundational for further results in bicomplex analysis, such as:*

- *Bicomplex version of Picards theorem.*
- *Bicomplex version of Riemann mapping theorem.*
- *Normal family criteria for bicomplex meromorphic functions.*

It demonstrates how classical complex analysis tools extend naturally to higher-dimensional commutative algebras like $\mathbb{BC}$.

Example 3.1 (Bounded Linear Family in Bicomplex Analysis)**.** Let

$$\mathcal{F}_1 = \{F_A(w) = Aw : A \in \mathbb{BC},\ \|A\| \le 1\}$$

be defined on the unit ball

$$\mathbb{D} = \{w \in \mathbb{BC} : \|w\| < 1\}.$$

Let us write the idempotent decomposition

$$A = A_1e_1 + A_2e_2 \quad \textit{and} \quad w = w_1e_1 + w_2e_2.$$

Then each function in the family can be written as

$$F_A(w) = A_1w_1e_1 + A_2w_2e_2.$$

Since $|A_1| \le 1$ and $|A_2| \le 1$, each component family

$$\{A_1w_1\} \subset \mathbb{C} \quad \textit{and} \quad \{A_2w_2\} \subset \mathbb{C}$$

is bounded on compact subsets of the unit disk. By Lemma 3.2, each component family is normal. Hence, the whole family $\mathcal{F}_1$ is normal in the bicomplex sense.

Example 3.2 (Polynomial Family in Bicomplex Analysis)**.** Let us consider the family

$$\mathcal{F}_2 = \{F_n(Z) = Z^n : n = 1, 2, 3, \dots\}$$

on the bicomplex unit ball $\mathbb{D} = \{Z \in \mathbb{BC} : \|Z\| < 1\}$.

Using the idempotent decomposition

$$Z = z_1e_1 + z_2e_2,$$

we have

$$F_n(Z) = z_1^n e_1 + z_2^n e_2.$$

For any compact set $K \subset \mathbb{D}$, there exists $r < 1$ such that $|z_1| \le r$ and $|z_2| \le r$ for all $Z \in K$. Then

$$|z_1^n| \le r^n \to 0 \quad \textit{and} \quad |z_2^n| \le r^n \to 0 \quad \text{as } n \to \infty.$$

Hence the family $\{z_1^n\}$ and $\{z_2^n\}$ are uniformly bounded on K and by Lemma 3.2 they are normal. Therefore, the bicomplex family $\mathcal{F}_2$ is normal and in fact, every sequence converges locally uniformly to the zero function on $\mathbb{D}$.

The following lemma is the meromorphic version of Lemma 3.2.

Lemma 3.3. *Let $\Omega \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of meromorphic functions on Ω such that there exists a spherical compact set $K \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathcal{F}$ omits K (i.e., for all $f \in \mathcal{F}$ and $z \in \Omega$, $f(z) \notin K$). Then $\mathcal{F}$ is a normal family on Ω.*

3.3 Bicomplex Analogue of Montel's Theorem (Meromorphic Version)

Definition 3.1 (Bicomplex Meromorphic Function)**.** *Let $\Omega \subset \mathbb{BC}$ be a domain. A function*

$$F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2$$

is called bicomplex meromorphic *if f_1 and f_2 are meromorphic in their respective domains in $\mathbb{C}$.*

Definition 3.2 (Bicomplex Chordal Metric)**.** *For $Z_1, Z_2 \in \mathbb{BC} \cup \{\infty\}$, let*

$$Z_1 = f_1e_1 + f_2e_2 \quad \text{and} \quad Z_2 = g_1e_1 + g_2e_2,$$

where $f_1, f_2, g_1, g_2 \in \mathbb{C} \cup \{\infty\}$. The bicomplex chordal metric *is defined by*

$$\chi_{\mathbb{BC}}(Z_1, Z_2) = \sqrt{\chi(f_1, g_1)^2 + \chi(f_2, g_2)^2},$$

where χ denotes the classical chordal (spherical) distance on $\mathbb{C}$.

The following theorem is the meromorphic version of Theorem 3.2.

Theorem 3.3. *Let $\mathcal{F}$ be a family of bicomplex meromorphic functions on a domain $\Omega \subset \mathbb{BC}$. Then $\mathcal{F}$ is normal in Ω if and only if $\mathcal{F}$ is locally uniformly bounded in the bicomplex spherical metric.*

Proof. Every $Z \in \mathbb{BC}$ has an idempotent decomposition:

$$Z = z_1e_1 + z_2e_2 \quad \text{where} \quad e_1 = \frac{1 + ij}{2} \quad \text{and} \quad e_2 = \frac{1 - ij}{2}.$$

Similarly, any bicomplex meromorphic function F can be written as

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2,$$

where F_1, F_2 are meromorphic functions in $\mathbb{C}$. The bicomplex spherical metric is

$$d_s(F(Z), G(Z)) = \max\{d_s(F_1(z_1), G_1(z_1)),\ d_s(F_2(z_2), G_2(z_2))\}.$$

Let us assume that $\mathcal{F}$ is locally uniformly bounded in the bicomplex spherical metric.

Now let us first decompose into component families:

$$\mathcal{F}_1 = \{F_1 : F \in \mathcal{F}\} \quad \text{and} \quad \mathcal{F}_2 = \{F_2 : F \in \mathcal{F}\}.$$

Local uniform boundedness in $\mathbb{BC}$ implies that $\mathcal{F}_1$ and $\mathcal{F}_2$ are locally uniformly bounded in the classical spherical metric in $\mathbb{C}$. So in view of the arguments regarding meromorphic functions, each component family is normal. Then extracting the subsequences componentwise and defining the bicomplex limit

$$F_{n_k}(Z) = F_{1,n_k}(z_1)e_1 + F_{2,n_k}(z_2)e_2 \to F_1^*(z_1)e_1 + F_2^*(z_2)e_2 = F^*(Z),$$

which is locally uniform in the bicomplexial spherical metric.

Hence, $\mathcal{F}$ is normal.

Conversely, let us assume that $\mathcal{F}$ be normal. Now if possible, let it not be locally uniformly bounded. Then there exists a compact set $K \subset \Omega$ and a sequence $\{F_n\} \subset \mathcal{F}$ such that

$$\sup_{Z \in K} d_s(F_n(Z), 0) \to \infty \quad \text{as } n \to \infty.$$

By normality, a subsequence $\{F_{n_k}\}$ converges locally uniformly in the bicomplex spherical metric to a bicomplex meromorphic function F^* (or ∞). Local uniform convergence implies that $\{F_{n_k}\}$ is uniformly bounded on K unless the limit is identically ∞, which contradicts the assumption.

Thus, $\mathcal{F}$ must be locally uniformly bounded in the bicomplex spherical metric.

This completes the proof of the theorem. □

Remark 3.6. *Theorem* 3.3 *generalizes the classical Montel's theorem from $\mathbb{C}$ to the bicomplex setting. By using the idempotent decomposition*

$$F(Z) = F_1(z_1)\, e_1 + F_2(z_2)\, e_2,$$

the normality of a bicomplex family reduces to the normality of two component families in the complex plane.

Remark 3.7. *Local uniform boundedness with respect to the* bicomplex chordal metric*:*

$$\chi_{\mathbb{BC}}(F(Z), 0) = \sqrt{\chi(F_1(z_1), 0)^2 + \chi(F_2(z_2), 0)^2}$$

ensures that each component family $\{F_1\}$ and $\{F_2\}$ is locally uniformly bounded in the classical spherical metric, which is crucial for applying the Montel-type arguments componentwise.

Remark 3.8. *Equicontinuity of the family in the bicomplex chordal metric is obtained from componentwise equicontinuity; i.e., if the spherical derivatives of F_1 and F_2 are bounded on compact sets, then the bicomplex family is equicontinuous, which guarantees the existence of locally uniformly convergent subsequences.*

Remark 3.9. *Theorem* 3.3 *provides the foundation for further results in bicomplex meromorphic function theory, such as:*

- *Bicomplex version of Picard's theorem,*
- *Criteria for normality of bicomplex meromorphic families and*
- *Applications in bicomplex dynamical systems and iteration theory.*

Definition 3.3 (Spherical Derivative)**.** *Let f be a meromorphic function on a domain $\Omega \subset \mathbb{C}$. The* spherical derivative *of f is defined as*

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

The following lemma represents Marty's theorem in classical complex analysis.

Lemma 3.4 (cf. [1, 13])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $\Omega \subset \mathbb{C}$. Then $\mathcal{F}$ is normal if and only if the family of spherical derivatives*

$$\mathcal{F}^{\#} = \{f^{\#} : f \in \mathcal{F}\}$$

is locally uniformly bounded on Ω; i.e., for every compact set $K \subset \Omega$ there exists a constant $M_K > 0$ such that

$$f^{\#}(z) \leq M_K, \quad \forall z \in K, \ \forall f \in \mathcal{F}.$$

3.4 Bicomplex Analogue of Marty's Theorem

Definition 3.4 (Bicomplex Spherical Derivative)**.** *Let $\Omega \subset \mathbb{BC}$ be a domain and let*

$$F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$$

be a bicomplex valued meromorphic function on Ω, where F_1 and F_2 are meromorphic functions in the complex plane associated with the idempotents e_1 and e_2 respectively.

The bicomplex spherical derivative *of F at Z is defined componentwise by*

$$F^{\#}(Z) := F_1^{\#}(z_1)e_1 + F_2^{\#}(z_2)e_2,$$

where

$$F_i^{\#}(z_i) = \frac{|F_i'(z_i)|}{1 + |F_i(z_i)|^2},$$

is the classical spherical derivative of F_i for $i = 1, 2$, in the complex plane.

Equivalently, one may define a scalar spherical derivative norm by

$$\|F^{\#}(Z)\| = \max\{F_1^{\#}(z_1), F_2^{\#}(z_2)\}.$$

The following theorem represents the bicomplex version of Martys theorem in classical complex analysis.

Theorem 3.4. *Let $\mathcal{F}$ be a family of bicomplex meromorphic functions on a domain $\Omega \subset \mathbb{BC}$. Then $\mathcal{F}$ is normal if and only if the family of bicomplex spherical derivatives*

$$\mathcal{F}^{\#} = \{F^{\#} : F \in \mathcal{F}\}$$

is locally uniformly bounded on Ω, i.e., for every compact set $K \subset \Omega$, there exists $M_K > 0$ such that

$$\|F^{\#}(Z)\| = \max\{F_1^{\#}(z_1), F_2^{\#}(z_2)\} \leq M_K, \quad \forall F \in \mathcal{F}, \ Z \in K.$$

Proof. Let us write $F(Z) = F_1(z_1)e_1 + F_2(z_2)e_2$. Then the bicomplex spherical derivative is decomposed as

$$F^{\#}(Z) = F_1^{\#}(z_1)\, e_1 + F_2^{\#}(z_2)\, e_2.$$

Let us assume that $\mathcal{F}$ is normal. Then each component family $\mathcal{F}_1 = \{F_1\}$ and $\mathcal{F}_2 = \{F_2\}$ is normal in the classical sense. By Lemma 3.4, the spherical derivatives $F_1^{\#}$ and $F_2^{\#}$ are locally uniformly bounded. Hence the bicomplex spherical derivatives are locally uniformly bounded

$$\|F^{\#}(Z)\| = \max\{F_1^{\#}(z_1), F_2^{\#}(z_2)\} \leq M_K.$$

Conversely, let us assume that $\|F^{\#}(Z)\|$ be locally uniformly bounded. Then

$$F_1^{\#}(z_1) \leq M_K$$

and

$$F_2^{\#}(z_2) \leq M_K$$

for all $F \in \mathcal{F}$ and $Z \in K$. Then by Lemma 3.4, each component family $\mathcal{F}_1$ and $\mathcal{F}_2$ is normal in $\mathbb{C}$. Using the idempotent decomposition, the bicomplex family $\mathcal{F}$ is normal in $\mathbb{BC}$.

Therefore, the bicomplex family $\mathcal{F}$ is normal if and only if its bicomplex spherical derivatives are locally uniformly bounded.

This proves the theorem. □

Remark 3.10. *Theorem* 3.4 *reduces the problem of checking normality in* $\mathbb{BC}$ *to checking the local boundedness of the spherical derivatives componentwise in* $\mathbb{C}$*.*

Remark 3.11. *Equicontinuity in the bicomplex chordal metric follows from the boundedness of the bicomplex spherical derivative.*

Remark 3.12. *Theorem* 3.4 *provides a powerful tool to verify the normality of family of bicomplex valued meromorphic functions without directly testing the convergence of sequences.*

The next lemma represents Riemann mapping theorem in classical complex analysis.

Lemma 3.5 (cf. [1])**.** *Let* $\Omega \subset \mathbb{C}$ *be a non-empty, simply connected, proper open subset of the complex plane* $\mathbb{C}$ *(i.e.,* $\Omega \neq \mathbb{C}$*). Then there exists a biholomorphic map*

$$f : \Omega \to \mathbb{D}$$

from Ω *onto the open unit disk* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$*.*

Moreover, if $a \in \Omega$ *is fixed, then there is a* unique *such biholomorphic function* f *satisfying the normalization conditions*

$$f(a) = 0 \quad \text{and} \quad f'(a) > 0.$$

3.5 Bicomplex Analogue of Riemann Mapping Theorem (Product Domains)

Definition 3.5 (Bicomplex unit polydisk)**.** *The bicomplex unit polydisk is*

$$\mathbb{D}_{\mathbb{BC}} := \{Z = z_1e_1 + z_2e_2 : \ |z_1| < 1, \ |z_2| < 1\}.$$

Definition 3.6 (Product domain in $\mathbb{BC}$)**.** *A domain* $\Omega \subset \mathbb{BC}$ *is called a* product domain *if*

$$\Omega = \{z_1e_1 + z_2e_2 : \ z_1 \in \Omega_1, \ z_2 \in \Omega_2\},$$

for some planar domains $\Omega_1, \Omega_2 \subset \mathbb{C}$*.*

The next theorem represents the bicomplex analogue for product domain of Riemann mapping theorem in classical complex analysis.

Theorem 3.5 (cf. [2])**.** *Let* $\Omega \subset \mathbb{BC}$ *be a product domain with projections* $\Omega_1, \Omega_2 \subset \mathbb{C}$*. Assume each* Ω_i *is nonempty, simply connected and proper (i.e.* $\Omega_i \neq \mathbb{C}$*). Fix* $a \in \Omega$ *and write* $a = a_1e_1 + a_2e_2$ *with* $a_i \in \Omega_i$*.*

Then there exists a bicomplex biholomorphism

$$F : \Omega \longrightarrow \mathbb{D}_{\mathbb{BC}}$$

satisfying $F(a) = 0$*. Moreover,* F *is unique if we require the normalization that each component derivative* $F_i'(a_i)$ *is real and positive.*

Proof. By Lemma 3.5 for $i = 1, 2$ there exists a unique biholomorphism

$$f_i : \Omega_i \longrightarrow \mathbb{D}$$

such that

$$f_i(a_i) = 0, \qquad f_i'(a_i) > 0.$$

Each f_i is holomorphic and bijective with holomorphic inverse $g_i := f_i^{-1} : \mathbb{D} \to \Omega_i$.

Let us define $F : \Omega \to \mathbb{D}_{\mathbb{BC}}$ by the idempotent sum of component maps as

$$F(Z) \;=\; f_1(z_1)\, e_1 \;+\; f_2(z_2)\, e_2 \quad \textit{with} \quad Z = z_1 e_1 + z_2 e_2 \in \Omega.$$

Since each f_i is holomorphic in its complex variable, F is bicomplex-holomorphic on Ω. Also $F(a) = 0$ (by construction) and $|f_i(z_i)| < 1$ and so $F(\Omega) \subset \mathbb{D}_{\mathbb{BC}}$.

The component inverses $g_i : \mathbb{D} \to \Omega_i$ are holomorphic.

Now let us define

$$G(W) \;=\; g_1(w_1)\, e_1 + g_2(w_2)\, e_2 \quad \textit{with} \quad W = w_1 e_1 + w_2 e_2 \in \mathbb{D}_{\mathbb{BC}}.$$

Then G is bicomplex-holomorphic on $\mathbb{D}_{\mathbb{BC}}$ and is indeed the inverse of F.

$$G(F(Z)) = g_1(f_1(z_1))e_1 + g_2(f_2(z_2))e_2 = z_1 e_1 + z_2 e_2 = Z,$$

and similarly $F(G(W)) = W$. Hence F is a bicomplex biholomorphism between Ω and $\mathbb{D}_{\mathbb{BC}}$.

Suppose $\widetilde{F}$ is another bicomplex biholomorphism $\Omega \to \mathbb{D}_{\mathbb{BC}}$ with $\widetilde{F}(a) = 0$ and write its components $\tilde{f}_i$. Then each $\tilde{f}_i : \Omega_i \to \mathbb{D}$ is a classical conformal map sending a_i to 0. By the uniqueness part of Lemma 3.5, under derivative normalization we have $\tilde{f}_i = f_i$ whenever $\tilde{f}_i'(a_i) > 0$. Therefore $\widetilde{F} = F$. This establishes the uniqueness of F under the stated normalization.

Thus the map F is a bicomplex biholomorphism $\Omega \to \mathbb{D}_{\mathbb{BC}}$ with $F(a) = 0$, unique after the normalization of component derivatives.

This concludes the proof of the theorem. □

Remark 3.13. *The product-domain hypothesis is essential in Theorem* 3.5*. The bicomplex holomorphy condition decouples into two classical holomorphy conditions only in idempotent coordinates. So arbitrary 4D domains need not admit such a direct product decomposition and also a global Riemann map onto* $\mathbb{D}_{\mathbb{BC}}$.

Remark 3.14. *The construction of the proof as carried out in Theorem* 3.5 *is explicit i.e., the bicomplex Riemann map is simply the idempotent combination of the two classical Riemann maps of the projections.*

The following lemma represents Hurwitz's theorem in classical complex analysis.

Lemma 3.6 (cf. [11])**.** *Let $\Omega \subset \mathbb{C}$ be a domain and $\{f_n\}$ be a sequence of holomorphic functions on Ω converging locally uniformly to a holomorphic function f on Ω.*

i. If each f_n is nonvanishing on Ω and $f \not\equiv 0$, then f is either nonvanishing on Ω or has isolated zeros. Moreover, for every compact $K \subset \Omega$ there exists N such that for all $n \geq N$ the number of zeros of f_n in K (counted with multiplicity) equals the number of zeros of f in K.

ii. More generally, if f is not identically zero then every zero z_0 of f (of multiplicity m) is approximated by exactly m zeros of f_n (counted with multiplicity) for all large n -they converge to z_0 as $n \to \infty$.

3.6 Bicomplex Analogue of Hurwitz's theorem

Definition 3.7 (Bicomplex zero and multiplicity)**.** *A point $Z_0 = z_1^0 e_1 + z_2^0 e_2 \in \Omega$ is a* bicomplex zero *of F if $f_1(z_1^0) = 0$ and $f_2(z_2^0) = 0$. If m_1 is the order of the zero of f_1 at z_1^0 and m_2 the order of the zero of f_2 at z_2^0, we will say the bicomplex zero Z_0 has component multiplicities (m_1, m_2). In particular, in a small polydisc around Z_0, the total number (counting multiplicity) of bicomplex zeros of a nearby product-type function is equal to $m_1 \cdot m_2$.*

The following theorem represents the bicomplex analogue of Hurwitz's theorem in classical complex analysis.

Theorem 3.6. *Let $\Omega \subset \mathbb{BC}$ be a domain and let $\{F_n\}$ be a sequence of bicomplex-holomorphic functions on Ω,*

$$F_n(Z) = f_{1,n}(z_1)e_1 + f_{2,n}(z_2)e_2,$$

which converges locally uniformly (in the bicomplex topology) to a bicomplex-holomorphic function

$$F(Z) = f_1(z_1)e_1 + f_2(z_2)e_2.$$

Then the followings hold.

(A) (Degenerate limit.) If $F \equiv 0$ on a nonempty open subset of Ω then, for every compact $K \subset \Omega$ and every $M \in \mathbb{N}$, there exists N such that for $n \geq N$ the function F_n has at least M bicomplex zeros (counted with multiplicity) in K.

(B) (Nondegenerate limit, both components nonidentically zero.) If neither f_1 nor f_2 is identically zero on any open subset of its projected domain, then every bicomplex zero $Z_0 = z_1^0 e_1 + z_2^0 e_2$ of F is isolated in Ω. Moreover, if m_1 (resp. m_2) is the multiplicity of z_1^0 as a zero of f_1 (resp. z_2^0 of f_2), then there exist small discs $z_1^0 \in D_1$ and $z_2^0 \in D_2$ with $\overline{D_1} \times \overline{D_2} \subset \Omega$ such that for all large n

- *$f_{1,n}$ has exactly m_1 zeros (counted with multiplicity) in D_1,*
- *$f_{2,n}$ has exactly m_2 zeros (counted with multiplicity) in D_2,*
- *consequently F_n has exactly $m_1 \cdot m_2$ bicomplex zeros (counted with multiplicity) in the polydisc $D_1 e_1 + D_2 e_2$ and these bicomplex zeros converge to Z_0 as $n \to \infty$.*

Furthermore, if $K \subset \Omega$ is compact and contains finitely many bicomplex zeros of F, then for large n the number of bicomplex zeros of F_n in K equals the sum, over bicomplex zeros Z_0 of F in K, of the products $m_1(Z_0)\, m_2(Z_0)$.

(C) (Mixed or one-component-degenerate cases.) If exactly one component of F is identically zero on some open set (say $f_1 \equiv 0$ on an open set while $f_2 \not\equiv 0$ there), then zeros of F need not be isolated i.e. the zero set contains entire slices $\{z_1\} \times \{z_2 : f_2(z_2) = 0\}$. The componentwise Hurwitz's theorem describes the limiting behaviour of zeros of $f_{2,n}$ on slices; bicomplex zeros of F_n correspond to arbitrary pairings of a zero of $f_{1,n}$ in the z_1-variable with a zero of $f_{2,n}$ in the z_2-variable.

Proof. Local uniform convergence $F_n \to F$ in the bicomplex topology means exactly that both component sequences converge locally uniformly on their projected planar domains:

$$f_{1,n} \to f_1 \quad \text{locally uniformly on } \pi_1(\Omega), \qquad f_{2,n} \to f_2 \quad \text{locally uniformly on } \pi_2(\Omega),$$

where π_1, π_2 denote the projection maps to the idempotent coordinates.

Thus any classical result about local uniform limits of holomorphic functions (in one complex variable) may be applied to each component separately.

Case (A): $F \equiv 0$ on a nonempty open set. If $F \equiv 0$ on some open subset $U \subset \Omega$, then both components f_1 and f_2 vanish identically on the projected open sets of U. By Lemma 3.6, zeros of $f_{i,n}$ become arbitrarily dense in compacta of these projected sets as $n \to \infty$. Hence for any compact $K \subset U$ and any M we can find N so that for $n \geq N$ each component has at least M zeros in the corresponding projections; pairing zeros of the two components yields at least $M \cdot M$ bicomplex zeros in K if one counts combinatorially. Thus in particular there are arbitrarily many bicomplex zeros in K for large n. This establishes Case (A).

Case (B): Let us assume that $F \not\equiv 0$ and on the neighbourhood we consider, neither f_1 nor f_2 vanishes identically. Let $Z_0 = z_1^0 e_1 + z_2^0 e_2$ be a bicomplex zero of F, so $f_1(z_1^0) = 0$ and $f_2(z_2^0) = 0$. Let m_1 be the order of the zero of f_1 at z_1^0 and m_2 the order of the zero of f_2 at z_2^0.

In classical complex analysis, as zeros of a nontrivial holomorphic function are isolated, hence we may choose radii $r_1, r_2 > 0$ small enough that the closed disks

$$\overline{D_1} := \{z : |z - z_1^0| \leq r_1\} \subset \pi_1(\Omega) \quad \text{and} \qquad \overline{D_2} := \{w : |w - z_2^0| \leq r_2\} \subset \pi_2(\Omega)$$

satisfy

a) f_1 has no zeros on ∂D_1 and the only zeros of f_1 in $\overline{D_1}$ are z_1^0, counted with multiplicity m_1.
b) f_2 has no zeros on ∂D_2 and the only zeros of f_2 in $\overline{D_2}$ are z_2^0, counted with multiplicity m_2.

Let us choose r_i small enough to isolate the chosen zero.

We now apply Lemma 3.6 to each component sequence.

- Because $f_{1,n} \to f_1$ uniformly on $\overline{D_1}$, for all sufficiently large n the function $f_{1,n}$ has exactly m_1 zeros in D_1 (counted with multiplicity) and these zeros converge to z_1^0 as $n \to \infty$.
- Similarly, because $f_{2,n} \to f_2$ uniformly on $\overline{D_2}$, for all sufficiently large n the function $f_{2,n}$ has exactly m_2 zeros in D_2 (counted with multiplicity) and these zeros converge to z_2^0.

Now let us consider the bicomplex polydisc

$$P := D_1 e_1 + D_2 e_2 = \{z e_1 + w e_2 : z \in D_1,\ w \in D_2\},$$

which is contained in Ω by construction.

We should observe here that for each fixed n and for any root $z_1^{(i,n)} \in D_1$ of $f_{1,n}$ and any root $z_2^{(j,n)} \in D_2$ of $f_{2,n}$, the bicomplex point

$$Z_{i,j}^{(n)} := z_1^{(i,n)} e_1 + z_2^{(j,n)} e_2 \in P$$

satisfies

$$F_n\big(Z_{i,j}^{(n)}\big) = f_{1,n}\big(z_1^{(i,n)}\big)e_1 + f_{2,n}\big(z_2^{(j,n)}\big)e_2 = 0,$$

so it is a bicomplex zero of F_n. Conversely, any bicomplex zero $Z \in P$ of F_n must have its first component in D_1 and second component in D_2, hence arise from such a pair. Therefore the bicomplex zeros of F_n in P are in bijection with ordered pairs (root of $f_{1,n}$ in D_1, root of $f_{2,n}$ in D_2). Counting multiplicities (one can treat multiplicity by counting roots with multiplicity in each factor) yields that the total number of bicomplex zeros of F_n in P equals $m_1 \cdot m_2$ for all sufficiently large n.

Finally, because the component zeros converge to z_1^0 and z_2^0, every bicomplex zero $Z_{i,j}^{(n)}$ converges to Z_0 as $n \to \infty$. This proves the statements locally and the product multiplicity phenomenon.

To obtain the global counting statement on a compact $K \subset \Omega$ that contains finitely many bicomplex zeros $Z_0^{(1)}, \dots, Z_0^{(t)}$ of F, choose pairwise disjoint polydiscs P_k around each $Z_0^{(k)}$ as above (isolating each bicomplex zero). The complement $K \setminus \bigcup_k P_k$ is compact and F has no bicomplex zeros there, so there is a positive lower bound on $\max\{|f_1|, |f_2|\}$ on that complement. Uniform convergence implies that for large n the functions $f_{1,n}, f_{2,n}$ do not vanish simultaneously on the complement. Hence no bicomplex zeros appear there. Summing the bicomplex zero counts in each P_k (each equal to $m_1^{(k)} m_2^{(k)}$) yields the asserted count for large n.

This completes the proof of Case (B).

Suppose for concreteness that $f_1 \equiv 0$ on some open subset $U_1 \subset \pi_1(\Omega)$ while f_2 is not identically zero on the corresponding slice region. Then every point $Z = z_1 e_1 + z_2 e_2$ with $z_1 \in U_1$ and $f_2(z_2) = 0$ is a bicomplex zero of F, so zeros need not be isolated and form slices. The componentwise Lemma 3.6 applied to $f_{2,n} \to f_2$ gives the convergence behaviour of zeros of the second component on each slice; zeros of $f_{1,n}$ may also accumulate (if $f_{1,n} \to 0$) and bicomplex zeros of F_n are arbitrary pairings of component zeros as in the product case. The counting statements must be adapted (they typically produce infinitely many zeros in small neighbourhoods if a component limit is identically zero). It is noted that in Case (B) we assume the components nonidentically zero locally.

This establishes the theorem. □

Remark 3.15. *The key points making Theorem 3.6 straightforward are (i) convergence in the bicomplex topology is equivalent to componentwise convergence and (ii) the bicomplex zero equation splits into two independent scalar equations. This reduces the bicomplex problem to two copies of the classical one-variable problem and a final combinatorial counting step.*

Remark 3.16. *The notion of 'multiplicity' of a bicomplex zero is naturally a pair (m_1, m_2). When one wants a single integer to count zeros in a polydisc one can use the product $m_1 m_2$, since the zeros in the polydisc arise as Cartesian combinations of the m_1 roots in the first factor and the m_2 roots in the second factor.*

4 Multicomplex Analogues of Some Theorems on Normal Families

In this section, we extend the results of Theorems 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 respectively to the multicomplex setting. Let $\mathbb{MC}_n$ denote the n-fold multicomplex algebra with n commuting imaginary units $i_1, i_2, \dots, i_n$, each satisfying $i_k^2 = -1$. Elements of $\mathbb{MC}_n$ admit a complete idempotent decomposition, which allows the reduction of multicomplex analytic problems to 2^{n-1} classical complex components. This structural principle underlies the multicomplex analogues of the classical theorems concerning normal families.

4.1 Multicomplex Analogue of Theorem 3.1

Multicomplex analogue of Theorem 3.1 follows using the idempotent decomposition as we can see in the following theorem.

Theorem 4.1. *Let $\Omega \subset \mathbb{MC}_n$ be a domain and let $\mathcal{F}$ be a family of continuous multicomplex-valued functions on Ω. Let us assume that*

i. $\mathcal{F}$ is uniformly bounded on compact subsets of Ω;
ii. $\mathcal{F}$ is equicontinuous on compact subsets of Ω.

Then $\mathcal{F}$ is relatively compact in $C(\Omega, \mathbb{MC}_n)$ with respect to the topology of uniform convergence on compact sets.

Proof. By the idempotent decomposition of $\mathbb{MC}_n$, every function $f \in \mathcal{F}$ can be written as

$$f = \sum_{k=1}^{2^{n-1}} F_k e_k,$$

where e_k are the orthogonal idempotents and each F_k is a complex-valued continuous function on the corresponding component domain Ω_k.

The assumptions of uniform boundedness and equicontinuity on $\mathcal{F}$ imply the same properties for each component family $\{F_k\}$. Applying Lemma 3.1 to each $\{F_k\}$ yields subsequences converging locally uniformly to continuous functions F_k^*. Taking a diagonal subsequence across all components produces a multicomplex limit

$$f^* = \sum_{k=1}^{2^{n-1}} F_k^* e_k,$$

which is continuous and the locally uniform limit of a subsequence of $\mathcal{F}$.

This completes the proof of the theorem. □

4.2 Multicomplex Analogue of Theorem 3.2 (Holomorphic Version)

The next theorem represents the multicomplex version of Theorem 3.2 on holomorphic functions.

Theorem 4.2. *Let $\Omega \subset \mathbb{MC}_n$ be a domain and let $\mathcal{F}$ be a family of multicomplex holomorphic functions on Ω uniformly bounded on compact subsets. Then $\mathcal{F}$ is normal.*

Proof. Decompose each $f \in \mathcal{F}$ as

$$f = \sum_{k=1}^{2^{n-1}} F_k e_k,$$

where each F_k is holomorphic in the classical sense on its component domain Ω_k. The uniform boundedness of $\mathcal{F}$ implies uniform boundedness of each $\{F_k\}$ on compact sets. By Lemma 3.2, each component family is normal.

Using a standard diagonal argument across all 2^{n-1} components, we can extract a subsequence of $\mathcal{F}$ converging locally uniformly in the multicomplex topology to

$$f^* = \sum_{k=1}^{2^{n-1}} F_k^* e_k,$$

which is multicomplex holomorphic. Hence $\mathcal{F}$ is normal. □

4.3 Multicomplex Analogue of Theorem 3.3 (Meromorphic Version)

The following theorem is the meromorphic version of Theorem 4.2.

Theorem 4.3. *Let $\Omega \subset \mathbb{MC}_n$ be a domain and $\mathcal{F}$ a family of multicomplex meromorphic functions. Suppose there exist three distinct values $\alpha, \beta, \gamma \in \mathbb{MC}_n$ such that all differences $\alpha - \beta, \beta - \gamma, \gamma - \alpha$ are invertible. Then $\mathcal{F}$ is normal.*

Proof. Let us decompose each $f \in \mathcal{F}$ as

$$f = \sum_{k=1}^{2^{n-1}} F_k e_k \quad \textit{with} \quad \alpha = \sum_{k=1}^{2^{n-1}} \alpha_k e_k, \quad \beta = \sum_{k=1}^{2^{n-1}} \beta_k e_k \quad \textit{and} \quad \gamma = \sum_{k=1}^{2^{n-1}} \gamma_k e_k.$$

Invertibility of the differences ensures that for each component k, the three complex values $\alpha_k, \beta_k, \gamma_k$ are distinct. By Lemma 3.3 for meromorphic functions, each component family $\{F_k\}$ is normal.

Applying a diagonal selection argument over all components yields a subsequence of $\mathcal{F}$ converging locally uniformly (in the multicomplex spherical metric) to a multicomplex meromorphic function. Therefore, $\mathcal{F}$ is normal. □

4.4 Multicomplex Analogue of Theorem 3.4

The following theorem represents the multicomplex version of Theorem 3.4 in classical complex analysis.

Theorem 4.4. *Let $\mathcal{F}$ be a family of multicomplex meromorphic functions on $\Omega \subset \mathbb{MC}_n$. Then $\mathcal{F}$ is normal if and only if the multicomplex spherical derivatives*

$$f^{\#}(\Omega) = \sum_{k=1}^{2^{n-1}} F_k^{\#}(\Omega_k) e_k$$

are locally bounded on Ω.

Proof. Let us decompose each $f \in \mathcal{F}$ as above. Local boundedness of $f^{\#}$ implies boundedness of each $F_k^{\#}$. By Lemma 3.4, each component family is normal. Using a diagonal argument across components yields a locally uniform convergent subsequence in $\mathbb{MC}_n$.

Conversely, if $\mathcal{F}$ is normal, each component family is normal and therefore has locally bounded spherical derivative. Summing over idempotents proves the equivalence. □

4.5 Multicomplex Analogue of Theorem 3.5

The next theorem represents the multicomplex analogue for product domain of Theorem 3.5.

Theorem 4.5. *Let $D \subsetneq \mathbb{MC}_n$ be a simply connected domain not equal to the entire multicomplex plane. Then there exists a biholomorphic map*

$$f : D \to \mathbb{D}_{\mathbb{MC}_n} = \{\Omega \in \mathbb{MC}_n : |\Omega| < 1\}.$$

For any $\Omega_0 \in D$, there exists a unique such map satisfying $f(\Omega_0) = 0$ and $f'(\Omega_0) > 0$.

Proof. Let $D = \sum_{k=1}^{2^{n-1}} D_k e_k$ be the idempotent decomposition. Each component $D_k \subsetneq \mathbb{C}$ is simply connected. By Lemma 3.5, there exist biholomorphic maps

$$F_k : D_k \to \mathbb{D}, \quad F_k(\Omega_{0,k}) = 0, \quad F_k'(\Omega_{0,k}) > 0.$$

Let us define a multicomplex map

$$f = \sum_{k=1}^{2^{n-1}} F_k e_k.$$

Then f is multicomplex holomorphic, bijective onto $\mathbb{D}_{\mathbb{MC}_n}$ and satisfies the prescribed normalization. Uniqueness follows from the uniqueness of the Riemann mapping in each complex component. □

4.6 Multicomplex Analogue of Theorem 3.6

Definition 4.1. *A function $F : \Omega \subset \mathbb{MC}_n \to \mathbb{MC}_n$ is* multicomplex-holomorphic *if it can be written in idempotent form*

$$F(Z) = \sum_{j=1}^{2^{n-1}} f_j(z_j) e_j$$

where each f_j is holomorphic in the complex variable z_j on the projected domain $\pi_j(\Omega) \subset \mathbb{C}$.

Definition 4.2 (Multicomplex zero). *A point $Z_0 = \sum_j z_j^0 e_j \in \Omega$ is a* multicomplex zero *of F if $f_j(z_j^0) = 0$ for all $j = 1, \dots, 2^{n-1}$. The multiplicity of Z_0 is the 2^{n-1}-tuple of multiplicities of the z_j^0 in f_j.*

The following theorem represents the bicomplex analogue of Theorem 3.6.

Theorem 4.6. *Let $\Omega \subset \mathbb{MC}_n$ be a domain and let $\{F_k\}$ be a sequence of multicomplex-holomorphic functions*

$$F_k(Z) = \sum_{j=1}^{2^{n-1}} f_{j,k}(z_j) e_j$$

converging locally uniformly (in the multicomplex topology) to

$$F(Z) = \sum_{j=1}^{2^{n-1}} f_j(z_j) e_j.$$

Then the followings hold:

(A) If $F \equiv 0$ on a nonempty open subset of Ω, then for every compact $K \subset \Omega$ and $M \in \mathbb{N}$, there exists N such that for $k \geq N$, F_k has at least M multicomplex zeros in K.

(B) If no component f_j is identically zero on a neighborhood, then each multicomplex zero $Z_0 = \sum_j z_j^0 e_j$ is isolated and for each j, let m_j denote the multiplicity of z_j^0 as a zero of f_j. Then there exist small discs $z_j^0 \in D_j$ such that for all large k.

- *$f_{j,k}$ has exactly m_j zeros in D_j (counted with multiplicity),*
- *F_k has exactly $\prod_{j=1}^{2^{n-1}} m_j$ multicomplex zeros in the polydisc $\sum_j D_j e_j$,*
- *these zeros converge to Z_0 as $k \to \infty$.*

(C) If some components f_j are identically zero, then zeros may form slices in the corresponding subspaces. Componentwise Hurwitz's theorem describes the limiting behaviour for nontrivial components; multicomplex zeros of F_k correspond to all possible combinations of zeros in each component.

Proof. The proof proceeds by induction on n using the idempotent decomposition.

Local uniform convergence $F_k \to F$ in the multicomplex topology is equivalent to

$$f_{j,k} \to f_j \quad \text{locally uniformly on } \pi_j(\Omega), \quad \forall j = 1, \ldots, 2^{n-1}.$$

By Lemma 3.6, for each j:

- if $f_j \equiv 0$ on an open set, the zeros of $f_{j,k}$ accumulate densely on compacts;
- if $f_j \not\equiv 0$, then zeros of $f_{j,k}$ near a zero z_j^0 of f_j converge to z_j^0 and the multiplicity is preserved.

A multicomplex zero Z_0 corresponds to a simultaneous zero of all components. In a small polydisc $\prod_j D_j$ around Z_0, the zeros of F_k are exactly all combinations of component zeros:

$$\{\sum_j z_j^{(i_j,k)} e_j : z_j^{(i_j,k)} \text{ zero of } f_{j,k} \text{ in } D_j\}.$$

Hence the total number of multicomplex zeros in the polydisc is equal to $\prod_j m_j$ for large k and these converge to Z_0.

If some components are identically zero, zeros may form slices. Componentwise notion still describes the convergence of zeros in nontrivial components and thereby multicomplex zeros are obtained by combinatorial pairing of zeros from each component.

All assertions (A), (B), (C) follow by applying Lemma 3.6 componentwise and assembling the zeros using the idempotent decomposition.

Hence, the theorem is established. □

Remark 4.1. *Multicomplex zeros are isolated only if all components are nondegenerate locally.*

Remark 4.2. *The multiplicity of a multicomplex zero is naturally the tuple $(m_1, \ldots, m_{2^{n-1}})$ and the total count in a polydisc is the product $\prod_j m_j$.*

Remark 4.3. *Degenerate components produce slices of zeros in higher-dimensional subspaces.*

5 Alternative view point of multicomplex analysis [18]

A multicomplex number $\xi_n \in \mathbb{C}_n$ can be defined as

$$\xi_n \ = \ \xi_{n-1,1} \ + \ i_n \, \xi_{n-1,2},$$

where

$$\xi_{n-1,1}, \xi_{n-1,2} \in \mathbb{C}_{n-1}, \quad (i_n)^2 = -1.$$

Addition and multiplication in multicomplex space $\mathbb{C}_n$ are defined componentwise extended by $(i_n)^2 = -1$.

5.1 Idempotent elements in $\mathbb{C}_n$

It is easy to verify that the following are idempotent elements in $\mathbb{C}_n$.

$$0, \quad 1, \quad \frac{1+i_1i_2}{2}, \ \frac{1-i_1i_2}{2}, \ \frac{1+i_1i_3}{2}, \ \frac{1-i_1i_3}{2}, \ \frac{1+i_2i_3}{2}, \ \frac{1-i_2i_3}{2}, \ \ldots, \ \frac{1+i_{n-1}i_n}{2}, \ \frac{1-i_{n-1}i_n}{2}.$$

For convenience in notation, define the symbols

$$e(i_ri_s) \ := \ \frac{1+i_ri_s}{2}, \qquad e(-i_ri_s) \ := \ \frac{1-i_ri_s}{2}, \quad \text{where } r,s \in \mathbb{N}.$$

5.2 Idempotent representation

Let ξ be an element in $\mathbb{C}_n$ and let $\xi = \xi_1 + i_n\xi_2$ with $\xi_1, \xi_2 \in \mathbb{C}_{n-1}$. Then

$$\xi = (\xi_1 - i_{n-1}\,\xi_2)\,e(i_{n-1}i_n) + (\xi_1 + i_{n-1}\,\xi_2)\,e(-i_{n-1}i_n).$$

5.3 Idempotent Representation of Holomorphic Multicomplex Valued Functions

Let X be a domain in $\mathbb{C}_n$, $n \geq 1$ and let f be a holomorphic function in $\mathbb{C}_n$ then there exists holomorphic functions

$$f_1 : X_1 \to \mathbb{C}_{n-1} \quad \text{and} \quad f_2 : X_2 \to \mathbb{C}_{n-1} \quad \text{where} \quad X_1, X_2 \subseteq \mathbb{C}_{n-1},$$

such that

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

For further study on multicomplex analysis one can see [9, 10].

From this alternative perspective in multicomplex analysis, proof of Theorems 4.1, 4.2 and 4.5 can be reformulated as follows.

The proof of Theorem 4.1 can be done using idempotent decomposition and mathematical induction as follows.

Reconstructed proof of Theorem 4.1. Here we take the multicomplex space as $\mathbb{C}_n$.

The theorem is already proved for $m = 1$ and $m = 2$ i.e. respectively for complex and bicomplex spaces {cf. Lemma 3.1 and Theorem 3.1}.

Now let the theorem be true for $m = n - 1$. Using idempotent decomposition, every function $f \in \mathcal{F}$ can be written as

$$f(\xi_1 + i_n\xi_2) = f_1(\xi_1 - i_{n-1}\xi_2)e(i_{n-1}i_n) + f_2(\xi_1 + i_{n-1}\xi_2)e(-i_{n-1}i_n).$$

where f_1 and f_2 are holomorphic functions on $\mathbb{C}_{n-1}$.

By the induction hypothesis, assumptions of uniform boundedness and equicontinuity on $\mathcal{F}$ imply the same properties for each component family $\{f_i\}$, $(i = 1, 2)$. Applying induction hypothesis to each $\{f_i\}$ yields subsequences converging locally uniformly to continuous functions f_i^* for $i = 1, 2$. Taking a diagonal subsequence across both the components produce a multicomplex limit

$$f^* = f_1^* e(i_{n-1}i_n) + f_2^* e(-i_{n-1}i_n),$$

which is continuous and the locally uniform limit of a subsequence of $\mathcal{F}$. Therefore the theorem is true for $m = n$. Then by mathematical induction we can say that the theorem is true for all $m \in \mathbb{N}$ i.e. the theorem is true in the multicomplex space $\mathbb{C}_n$.

This completes the proof of the theorem. □

Remark 5.1. *By employing idempotent decomposition and mathematical induction as in the earlier proof, we can similarly establish the proof of Theorems* 4.2 *and* 4.5 *and therefore the same is omitted.*

These structural facts provide the foundation for adapting classical results concerning normal families to the bicomplex and multicomplex settings.

6 Open problems and future directions

The notion of bicomplex and multicomplex setting of certain theorems concerning the normal family of functions may be extended for the functions of uncertain variables and are still virgin. Therefore those may be posed as open problems to the future researchers of this branch.

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