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SOME POLYNOMIALS AND RESULTS ON DEFICIENT DEGREE OF VERTEX IN STACKED BOOK GRAPH

Ravi Ratn Gaur and Pravin Garg
Department of Mathematics
University of Rajasthan, Jaipur,Rajasthan, India-302004
Email: ravig2016uor@gmail.com,garg.pravin@gmail.com

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Abstract

The degree $d(v)$ of a vertex v in a graph $\mathcal{G}$ has several fundamental applications in graph theory. A new version of the degree of a vertex v in the graph $\mathcal{G}$ is defined as $d_f(v) = \Delta(\mathcal{G}) - d(v)$, where $\Delta(\mathcal{G}) = max\{d(v)|v \in \mathcal{V}(\mathcal{G})\}$. Further, we found some results on this new degree of a vertex in the graph $\mathcal{G}$ called "deficient degree". The main object of this article is to study the new version of the polynomial for the stacked book graph as deficient degree vertex polynomial, nbd deficient degree vertex polynomial, deficient M-polynomial, and nbd deficient M-polynomial.

2020 Mathematical Sciences Classification: 05C07, 05C09, 05C31.

Keywords and Phrases: Deficient degree, vertex polynomial, neighborhood vertex polynomial, M-polynomial and neighborhood M-polynomial, stacked book graph.

1 Introduction

In graph theory, several graph polynomials have been extensively explored. Among them, the chromatic polynomial enumerates various ways to color a graph. The polynomial that lists matchings is known as a matching polynomial. Many algebraic polynomials are utilized to make the calculation of topological indices smoother.

The graph can be expressed algebraically with the vertex polynomial. The M-polynomial is a very useful technique for calculating degree-based topological indices. The role of the nbd M-polynomial in the nbd degree sum-based indices is similar to that of the M-polynomial in the degree-based indices.

Throughout this article, we consider a simple connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. $d(v)$ is the number of edges which are incident at v for $v \in \mathcal{V}$. The deficient degree of the vertex v in the graph $\mathcal{G}$ is defined as $d_f(v) = \Delta(\mathcal{G}) - d(v)$, where $\Delta(\mathcal{G}) = max\{d(v)|v \in \mathcal{G}\}$.

In 2012, Devaraj & Sukumaran [4] introduced the vertex degree polynomial of a graph $\mathcal{G}$ which is defined as

$$V(\mathcal{G}; x) = \sum_{i=0}^{\Delta(\mathcal{G})} | \mathcal{V}_i | x^i, \tag{1.1}$$

where $\mathcal{V}_i$ is the collection of the vertices of degree i in the graph $\mathcal{G}$. The roots of the vertex polynomial are called vertex polynomial roots. They established the relation between the vertex polynomial of the disjoint graphs. They had constructed some graphs namely leaf graph, max to max graph, steam graph, *etc.* Anto & Hawkins [1] proved some results on the vertex polynomial of the splitting graph and the degree splitting graph in 2019. Kattimani & Sridhara [9] discussed the vertex polynomial of the middle graph, the line graph, and the total graph of a complete graph, star graph, path graph, cycle graph, and wheel graph, *etc.*, in 2020.

In 2015, Deutsch & Klavžar [3] introduced the M-polynomial of a graph $\mathcal{G}$ defined as,

$$M(\mathcal{G}) = \sum_{i \le j} \mu_{i,j} x^i y^j, \tag{1.2}$$

where $\mu_{i,j}$ is the number of all the edges uv such that $d(u) = i$ and $d(v) = j$.

Riaz *et al.* [14] derived the degree-based topological invariants using the M-polynomial of three general classes of the convex polytopes in 2018. Basavanagoud & Barangi [2] discussed some topological invariants of the cactus chain using the M-polynomial in 2019. Verma *et al.* [16] proved properties of Bismuth triiodide using M-polynomial and results are interpreted graphically in 2019. Shanmukha *et al.* [15] calculated various topological indices for anti-cancer drugs and proposed several new indices in 2020 to enhance the understanding of their physical and chemical properties. Gaur *et al.* [8] obtained some topological indices of the Hanoi graph and the generalized wheel graph using their M-polynomial in 2020. Gaur *et al.* [7] identified of insect's order by estimating M-polynomial and topological indices in 2022.

In 2019, Verma *et al.* [16] introduced the nbd M-polynomial of a graph $\mathcal{G}$ defined as,

$$NM(\mathcal{G};x,y) = \sum_{\omega \leq i \leq j \leq \Omega} n_{i,j}(\mathcal{G})x^i y^j,$$

where $\omega = min\{\eta(v) : v \in \mathcal{V}\}$, $\Omega\{= max\{\eta(v) : v \in \mathcal{V}\}$, and $n_{i,j}$ is the number of all the edges uv in the graph $\mathcal{G}$ such that $\eta(u) = i$ and $\eta(v) = j$ and $\eta(u)$ denotes the sum of the degree of all those vertices of the graph $\mathcal{G}$ which are adjacent to the vertex u.

In 2021, Mondal *et al.* [13] computed the general form of the nbd M-polynomial for the structure of face-centered cubic lattice and the crystallographic structure of cuprous oxide Cu_2O and graphical representation of the results has also been created in order to visualize them. In 2022, Gaur & Garg [6] obtained some topological indices that are based on the degree and nbd degree of vertices of molecular graphs of PF-00835231 and PF-07304814 drugs which are used to combat COVID-19.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple connected graph and $\mathcal{V}_i^*$ be the collection of the vertices of the deficient degree i in the graph $\mathcal{G}$. Then, the deficient degree vertex polynomial of the graph $\mathcal{G}$ is defined as

$$D_f D(\mathcal{G};x) = \sum_{i=0}^{\Theta(\mathcal{G})} |\mathcal{V}_i^*| x^i, \tag{1.3}$$

where $\Theta(\mathcal{G}) = max\{d_f(u) : u \in \mathcal{V}\}$.

The deficient M-polynomial of the graph $\mathcal{G}$ is defined as

$$D_f M(\mathcal{G};x,y) = \sum_{\theta \leq i \leq j \leq \Theta} m^*_{i,j}(\mathcal{G})x^i y^j, \tag{1.4}$$

where $\theta = min\{d_f(u) : u \in \mathcal{V}\}$, $\Theta = max\{d_f(v) : v \in \mathcal{V}\}$. $m^*_{i,j}$ is the number of edges of the graph $B_{m,n}$ that have end vertices of deficient degrees i and j.

Let $nd_f(u)$ be the nbd deficient degree (sum of the deficient degree of all those vertices which are to adjacent to the vertex u) of vertex u in the graph $\mathcal{G}$. Then, the nbd deficient degree vertex polynomial of the ladder graph $\mathcal{G}$ is

$$ND_f D(\mathcal{G};x) = \sum_{u \in \mathcal{V}(\mathcal{G})} |\mathcal{V}_{nd_f(u)}| x^{nd_f(u)}. \tag{1.5}$$

The nbd deficient M-polynomial of the graph $\mathcal{G}$ is defined as

$$ND_f M(\mathcal{G};x,y) = \sum_{\sigma \leq i \leq j \leq \Psi} n^*_{i,j}(\mathcal{G})x^i y^j, \tag{1.6}$$

where $\sigma = min\{nd_f(u) : u \in \mathcal{V}\}$, $\Psi = max\{nd_f(v) : v \in \mathcal{V}\}$. $n^*_{i,j}$ is the number of edges of the graph $\mathcal{G}$ which have end vertices of nbd deficient degree i and j.

Recall, a path with n vertices is known as a path graph and it is denoted by P_n. In this graph P_n, there are two sets of vertices. One of them is the set of two vertices of deficient degree 1 and the other is the set of $(n-2)$ vertices of deficient degree 0. Some examples of the path graph are given in Figure 1.1.

A complete bipartite graph $K_{1,m}$ is called a star graph of order $(m+1)$, and it is denoted by S_{m+1}. There are two sets of vertices in the star graph S_{m+1}. One of them contains m vertices of deficient degree m and the other contains one vertex of deficient degree 0. Some examples of the star graph are given in Figure 1.2.

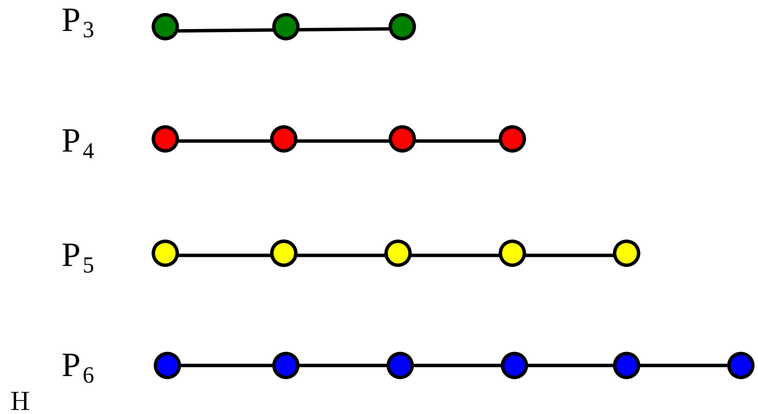

***Figure* 1.1:** *Path graphs* P_n

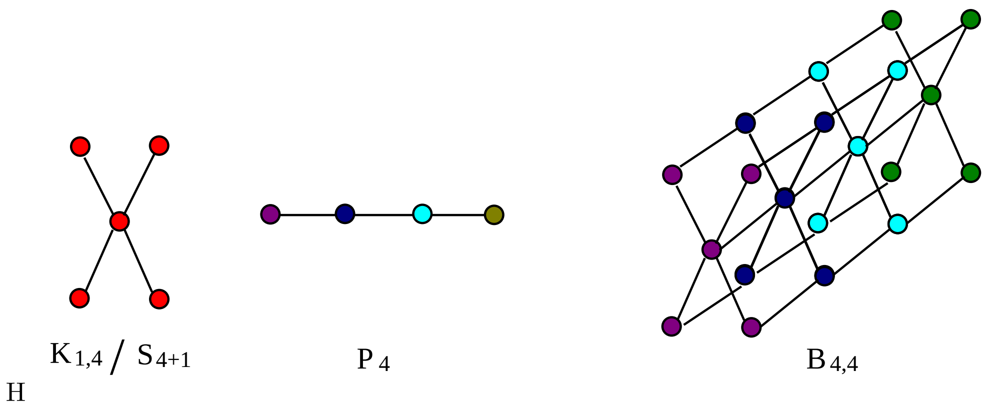

***Figure* 1.3:** $S_{4+1} \times P_4 \; = \; B_{4,4}$

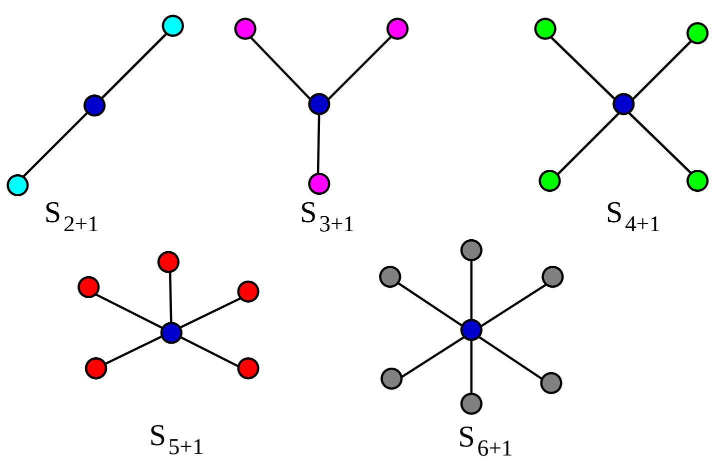

***Figure* 1.2:** *Star graphs* S_{m+1}

The cartesian product of the star graph of order $m+1$ and the path graph of order n, *i.e.* $S_{m+1} \times P_n$, is called a stacked book graph of order $n(m+1)$ and size $(mn+(m+1)(n-1))$. It is denoted by $B_{m,n}$. the stacked book graph $B_{4,4}$ is given in Figure 1.3. Being a cartesian product of two unit-distance graphs, it is also called a *unit-distance graph*. It is generalization of the book graph $B_{m,2}$ which is defined by the cartesian product of the star graph S_{m+1} and the path graph P_2. Some examples of the stacked book graph are given in Figure 1.4.

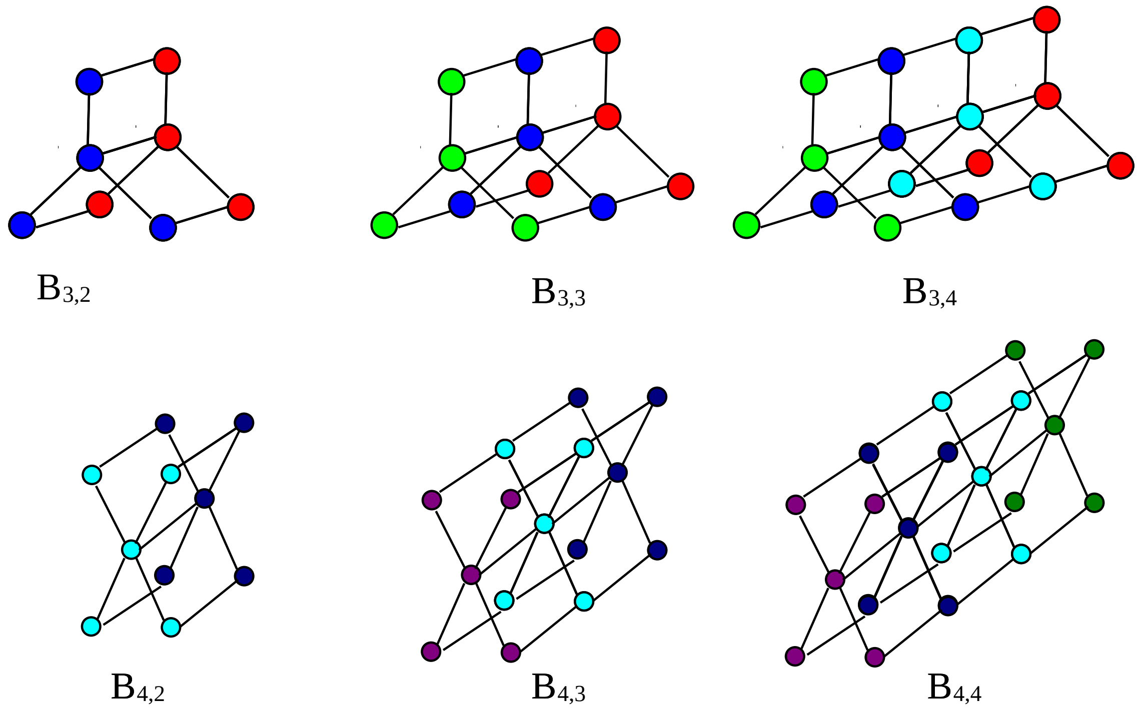

Figure 1.4: *Examples of stacked book graph*

In Table 1.1, some cases of the stacked book graph are summarized.

Table 1.1: *Some special cases of stacked book graph* $B_{m,n}$

(m, n)	$S_{m+1} \times P_n$	Stacked book graph $B_{m,n}$	Graphs
$(1, 1)$	$S_2 \times P_1$	$B_{1,1}$	Path graph
$(1, 2)$	$S_2 \times P_2$	$B_{1,2}$	Square graph
$(1, 3)$	$S_2 \times P_3$	$B_{1,3}$	Domino graph
$(3, 1)$	$S_4 \times P_1$	$B_{3,1}$	Claw graph
$(1, n)$	$S_2 \times P_n$	$B_{1,n}$	Ladder graph
$(2, n)$	$S_3 \times P_n$	$B_{2,n}$	Grid graph
$(m, 1)$	$S_{m+1} \times P_1$	$B_{m,1}$	Star graph
$(m, 2)$	$S_{m+1} \times P_2$	$B_{m,2}$	Book graph

Khalid *et al.* [11] analyzed the graph-theoretic invariants of the book graph and the stacked book graph in 2019. For the book graph and the stacked book graph, they established the generalised formulae for various topological invariants and computed the results mathematically.

In 2020, Ma *et al.* [12] defined the uniform subdivision of a stacked book graph by cycles of varying lengths and the edge covering of the stacked book graph and discussed applications of graph labeling in data science and communication networks. During this year, Khalaf *et al.* [10] obtained M-polynomial of the book graph. They have discussed the graphical representation of the topological invariants of the book graph with the help of M-polynomial.

Gaur & Garg [5] introduced the deficient degree of a vertex in a graph and the deficient degree based M-polynomial of a graph in 2022. They have obtained some deficient topological indices using the M-polynomial with respect to the deficient degree of the vertex of the identified graph $\mathcal{G}$.

Some deficient topological indices with a formula using the deficient degree M-polynomial of graph $\mathcal{G}$ are shown in Table 1.2.

***Table* 1.2:** *Deficient degree based topological indices*

Topological Index	**Formula**	**Derivation through M- polynomial $\mathcal{F}(\mu,\nu)$**
Deficient I^{st} Zagreb	$\sum_{xy\in\mathcal{E}}\{d_f(x)+d_f(y)\}$	$[(D_\mu+D_\nu)\mathcal{F}(\mu,\nu)]_{\mu=\nu=1}$
Deficient II^{nd} Zagreb	$\sum_{xy\in\mathcal{E}}\{d_f(x)d_f(y)\}$	$[(D_\mu D_\nu)\mathcal{F}(\mu,\nu)]_{\mu=\nu=1}$
Deficient redefined III^{rd} Zagreb	$\sum_{xy\in\mathcal{E}}\{d_f(x)d_f(y)\left(d_f(x)+d_f(y)\right)\}$	$[D_\mu D_\nu(D_\mu+D_\nu)\mathcal{F}(\mu,\nu)]_{\mu=\nu=1}$
Deficient inverse sum indeg	$\sum_{xy\in\mathcal{E}}\left\{\dfrac{d_f(x)d_f(y)}{d_f(x)+d_f(y)}\right\}$	$[S_\mu J D_\mu D_\nu\mathcal{F}(\mu,\nu)]_{\mu=1}$
Deficient forgotten	$\sum_{xy\in\mathcal{E}}\{\ d_f^2(x)+d_f^2(y)\}$	$[(D_\mu^2+D_\nu^2)\mathcal{F}(\mu,\nu)]_{\mu=1}$
Deficient augmented Zagreb	$\sum_{xy\in\mathcal{E}}\left\{\dfrac{d_f(x)d_f(y)}{d_f(x)+d_f(y)-2}\right\}^3$	$\left[S_\mu^3 Q_{-2} J D_\nu^3 D_\mu^3\mathcal{F}(\mu,\nu)\right]_{\mu=1}$

In Table 1.2, some operators are defined as follows:
$D_\mu\mathcal{F}(\mu,\nu) = \mu\dfrac{\partial}{\partial\mu}\mathcal{F}(\mu,\nu)$, $D_\nu\mathcal{F}(\mu,\nu) = \nu\dfrac{\partial}{\partial\nu}\mathcal{F}(\mu,\nu)$, $J\mathcal{F}(\mu,\nu) = \mathcal{F}(\mu,\mu)$, $S_\mu\mathcal{F}(\mu,\nu) = \int_0^\mu\dfrac{\mathcal{F}(t,\nu)}{t}dt$, and $Q^\alpha\mathcal{F}(\mu,\nu)=\mu^\alpha\mathcal{F}(\mu,\nu)$.

2 Properties of Deficient Degree

In this section, we proved some results on the deficient degree of the vertex in a graph $\mathcal{G}$.

Theorem 2.1. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a simple connected graph. If n and e are the order and the size of the graph $\mathcal{G}$, respectively, then*

$$\sum_{v\in\mathcal{V}}d_f(v)=n\Delta(\mathcal{G})-2e.$$

Proof. By definition of the deficient degree,

$$d_f(v)=\Delta(\mathcal{G})-d(v)$$

$$\Rightarrow \sum_{v\in\mathcal{V}} d_f(v) = \sum_{v\in\mathcal{V}} \Delta(\mathcal{G}) - \sum_{v\in\mathcal{V}} d(v)$$
$$\Rightarrow \sum_{v\in\mathcal{V}} d_f(v) = n\Delta(\mathcal{G}) - 2e.$$

□

Proposition 2.1. *If the order of a graph $\mathcal{G}$ is even. Then, the sum of the deficient degree of all the vertices of the graph $\mathcal{G}$ is also even.*

Proof. It can be proved easily by Theorem 2.1,

$$\sum_{v\in\mathcal{V}} d_f(v) = n\Delta(\mathcal{G}) - 2e. \tag{2.1}$$

Since the order of the graph $\mathcal{G}$ is even. Therefore, the right hand side of Equation (2.1) is also even. As aresult, the sum of the deficient degree of all the vertices of the graph $\mathcal{G}$ is even. □

Theorem 2.2. *If $\Delta(\mathcal{G})$ is even, then the number of odd (even) deficient vertices is equal to the number of odd (even) vertices in the graph $\mathcal{G}$.*

Proof. We assume that $\Delta(\mathcal{G})$ is even. By definition of the deficient degree,

$$\begin{aligned} d_f(v) &= \Delta(\mathcal{G}) - d(v) \\ &\Rightarrow d_f(v) + d(v) = \Delta(\mathcal{G}) \\ &\Rightarrow d_f(v) + d(v) = \text{Even number.} \end{aligned} \tag{2.2}$$

The right hand side of Equation (2.2) is an even number. Then, the both term on the left hand side must be even or odd, to make the sum an even number. Hence, the number of odd (or even) deficient vertices is equal to the number of odd (or even) vertices in the graph $\mathcal{G}$. □

Proposition 2.2. *If $\Delta(\mathcal{G})$ is odd, then the number of odd (even) deficient vertices is equal to the number of even (odd) vertices in the graph $\mathcal{G}$.*

Proof. Proof is in similar fashion as of proof of the Theorem 2.2. □

Example 2.1. Consider a path graph P_n of order n and size $(n-1)$. Then,

$$\sum_{v\in\mathcal{V}(P_n)} d_f(v) = 2 \textit{ and } n\Delta(\mathcal{G}) - 2e = 2n - 2(n-1) = 2.$$

Example 2.2. Consider a complete bipartite graph $K_{m,n}$ of order $(m+n)$ and size mn. Then,

$$\sum_{v\in\mathcal{V}(K_{m,n})} d_f(v) = n^2 - nm \quad \text{and}$$
$$n\Delta(\mathcal{G}) - 2e = n(m+n) - 2mn = n^2 - nm.$$

Theorem 2.3. *A graph $\mathcal{G}$ is an Euler graph if and only if the deficient degree of all the vertices in $\mathcal{G}$ and the largest degree $\Delta(\mathcal{G})$ of vertices in $\mathcal{G}$ are even.*

Proof. Let the graph $\mathcal{G}$ be an Euler graph. Therefore, all the vertices of $\mathcal{G}$ are of even degree. Clearly, $\Delta(\mathcal{G})$ is also even.

Now, we shall show that the deficient degrees of all the vertices in $\mathcal{G}$ are even. By definition of the deficient degree,

$$d_f(v) = \Delta(\mathcal{G}) - d(v). \tag{2.3}$$

The right-hand side of Equation (2.3) is even. Hence, the deficient degrees of all the vertices in $\mathcal{G}$ are even. The converse part is easy to prove. □

Theorem 2.4. *Let $\mathcal{G}$ be a simple connected graph of order n and $\Delta(\mathcal{G})$ is the largest degree of a vertex in $\mathcal{G}$. Then, the number of vertices of odd deficient degree in $\mathcal{G}$ is odd if and only if n and $\Delta(\mathcal{G})$ both are odd.*

Proof. Let the number of vertices of odd deficient degree in $\mathcal{G}$ is odd. Then, the sum of all odd deficient degree is odd.
i.e.,

$$\sum_{O} d_f(v) = odd$$

Now,

$$\sum_{v \in \mathcal{V}} d_f(v) = \sum_{E} d_f(v) + \sum_{O} d_f(v), \tag{2.4}$$

where E and O denote set of all the vertices of deficient degree even and odd, respectively.

The first sum on the right hand side of Equation (2.4) is even, because each term in this sum is even. The second sum on the right hand side is odd, because the number of vertices of odd deficient degree in $\mathcal{G}$ is odd.

$$\begin{aligned}
&\Rightarrow \sum_{v \in \mathcal{V}} (\Delta(\mathcal{G}) - d(v)) = even + odd \\
&\Rightarrow \sum_{v \in \mathcal{V}} \Delta(\mathcal{G}) - \sum_{v \in \mathcal{V}} d(v) = even + odd \\
&\Rightarrow n\Delta(\mathcal{G}) - 2e = even + odd \\
&\Rightarrow n\Delta(\mathcal{G}) = 2e + even + odd. \qquad (2.5)
\end{aligned}$$

The sum on the left hand side of Equation (2.5) must be odd.

$$\Rightarrow n\Delta(\mathcal{G}) = \text{an odd number}. \tag{2.6}$$

Clearly from Equation (2.6), n and $\Delta(\mathcal{G})$ both are odd, to make the product an odd number.

Conversely, let n and $\Delta(\mathcal{G})$ both be odd. Again Equation (2.4),

$$\begin{aligned}
&\sum_{v \in \mathcal{V}} d_f(v) = \sum_{E} d_f(v) + \sum_{O} d_f(v) \\
&\Rightarrow \sum_{v \in \mathcal{V}} (\Delta(\mathcal{G}) - d(v)) = \sum_{E} d_f(v) + \sum_{O} d_f(v) \qquad (\because d_f(v) = \Delta(\mathcal{G}) - d(v)) \\
&\Rightarrow \sum_{v \in \mathcal{V}} \Delta(\mathcal{G}) - \sum_{v \in \mathcal{V}} d(v) = \sum_{E} d_f(v) + \sum_{O} d_f(v).
\end{aligned}$$

By Handshaking lemma,

$$n\Delta(\mathcal{G}) - 2e = \sum_{E} d_f(v) + \sum_{O} d_f(v), \tag{2.7}$$

where e is the total no. of edges in the graph $\mathcal{G}$. The first term on the left-hand side of Equation (2.7) is odd, because n and $\Delta(\mathcal{G})$ both are odd. The second term is even. The first sum on the right hand side is also even, because each term in this sum is even. Hence, the second sum on the right-hand side must also be odd. *i.e.*,

$$\sum_{O} d_f(v) = \text{an odd number}. \tag{2.8}$$

Since each term $d_f(v)$ in Equation (2.8) is odd, the total number of terms in the sum must be odd, to make the sum an odd number.

□

Theorem 2.5. *If a graph $\mathcal{G}$ has exactly two vertices of odd deficient degree and $\Delta(\mathcal{G})$ is even, then there exists a path joining these two vertices.*

Proof. Let u and v be only two vertices of odd deficient degree and $\Delta(\mathcal{G})$ is even. Then, by Theorem 2.2, u and v are of odd degree in the graph $\mathcal{G}$. Therefore, u and v must belong to same component in graph $\mathcal{G}$. Hence, there exist a path joining u and v. □

3 Deficient Polynomials of Stacked Book Graph

In this section, we derive the deficient degree vertex polynomial, the nbd deficient degree vertex polynomial, the deficient M-polynomial, and the nbd deficient M-polynomial for the stacked book graph.

Recall, a stacked book graph is the cartesian product of the star graph of order $(m+1)$ and the path graph of order n. *i.e.*, $S_{m+1} \times P_n$ and it is denoted by $B_{m,n}$. It is of the size $(mn + (m+1)(n-1))$ and the order $n(m+1)$.

Theorem 3.1. *Let $B_{m,n}$ be a stacked book graph. Then, the deficient degree vertex polynomial of the graph $B_{m,n}$ is*

$D_f D(B_{m,n}; x) =$

$$\begin{cases} 4x + (2n-4), & m = 1, n \geq 3, \\ mx^{m-1} + 1, & m \geq 1, n = 1, \\ 2mx^{m-1} + 2, & m \geq 1, n = 2, \\ 2mx^m + m(n-2)x^{m-1} + 2x + n - 2, & m \geq 2, n \geq 3. \end{cases}$$

Proof. The set of vertices $\mathcal{V}(B_{m,n})$, contains $(m+1)n$ vertices. From Equation (1.3), the deficient degree vertex polynomial of the stacked book graph $B_{m,n}$ is

$$D_f D(B_{m,n}; x) = \sum_{u \in \mathcal{V}(B_{m,n})} |\mathcal{V}^*_{d_f(u)}| x^{d_f(u)}. \tag{3.1}$$

We considered the four cases of the stacked book graph $B_{m,n}$ as follows:

Case(i) $m = 1$, $n \geq 3$.

The set of vertices of the graph $B_{1,n}$ contains $2n$ vertices. There are two types of subsets of the set of vertices of the graph $B_{1,n}$. Let $\mathcal{V}^*_0$ and $\mathcal{V}^*_1$ be the sets of vertices of the deficient degrees 0 and 1, respectively. Then,

$$|\mathcal{V}^*_0| = 2n - 4 \text{ and } |\mathcal{V}^*_1| = 4.$$

Using the Equation (3.1), the deficient degree vertex polynomial of the ladder graph $B_{1,n} (n \geq 3)$ is

$$\begin{aligned} D_f D(B_{1,n}; x) &= \sum_{u \in \mathcal{V}} |\mathcal{V}^*_{d_f(u)}| x^{d_f(u)} \\ &= |\mathcal{V}^*_1| x^1 + |\mathcal{V}^*_0| x^0 \\ &= 4x + (2n-4). \end{aligned}$$

Case(ii) $m \geq 1$, $n = 1$.

The set of vertices of graph $B_{m,1}$ contains $(m+1)$ vertices. There are two types of subsets of the set of vertices of the graph $B_{m,1}$. Let $\mathcal{V}^*_0$ and $\mathcal{V}^*_{m-1}$ be the sets of vertices of deficient degrees 0 and deficient degrees $m-1$, respectively. Then,

$$|\mathcal{V}^*_0| = 1 \text{ and } |\mathcal{V}^*_{m-1}| = m.$$

The deficient degree vertex polynomial of the graph $B_{m,1} (m \geq 1)$ is

$$\begin{aligned} D_f D(B_{m,1}; x) &= \sum_{u \in \mathcal{V}(B_{m,1})} |\mathcal{V}^*_{d_f(u)}| x^{d_f(u)} \\ &= |\mathcal{V}^*_{m-1}| x^{m-1} + |\mathcal{V}^*_0| x^0 \\ &= mx^{m-1} + 1. \end{aligned}$$

Case(iii) $m \geq 1$, $n = 2$.

The set of vertices of the graph $B_{m,2}$ contains $2(m+1)$ vertices. There are two types of subsets of the set of vertices of the graph $B_{m,2}$. Let $\mathcal{V}^*_0$ and $\mathcal{V}^*_{m-1}$ be the sets of vertices of deficient degree 0 and deficient degree $m-1$, respectively. Then,

$$|\mathcal{V}^*_0| = 2 \text{ and } |\mathcal{V}^*_{m-1}| = 2m.$$

The deficient degree vertex polynomial of the graph $B_{m,2}(m \geq 1)$ is

$$\begin{aligned} D_f D(B_{m,2}; x) &= \sum_{u \in \mathcal{V}(B_{m,2})} |\mathcal{V}^*_{d_f(u)}| x^{d_f(u)} \\ &= |\mathcal{V}^*_{m-1}| x^{m-1} + |\mathcal{V}^*_0| x^0 \\ &= 2mx^{m-1} + 2. \end{aligned}$$

Case(*iv*) $m \geq 2$, $n \geq 3$.

The set of vertices of the graph $B_{m,n}$ contains $(m+1)n$ vertices. There are four types of subsets of the set of vertices of the graph $B_{m,n}$. Let $\mathcal{V}^*_0, \mathcal{V}^*_1, \mathcal{V}^*_{m-1}$, and $\mathcal{V}^*_m$ be the sets of vertices of the deficient degrees 0, 1, $m-1$, and m, respectively. Then,

$$|\mathcal{V}^*_0| = n-2, \; |\mathcal{V}^*_1| = 2, \; |\mathcal{V}^*_{m-1}| = m(n-2), \text{ and } |\mathcal{V}^*_m| = 2m.$$

The deficient degree vertex polynomial of the graph $B_{m,n}(m \geq 2, n \geq 3)$ is

$$\begin{aligned} D_f D(B_{m,n}; x) &= \sum_{u \in \mathcal{V}(B_{m,n})} |\mathcal{V}^*_{d_f(u)}| x^{d_f(u)} \\ &= |\mathcal{V}^*_m| x^m + |\mathcal{V}^*_{m-1}| x^{m-1} + |\mathcal{V}^*_1| x^1 + |\mathcal{V}^*_0| x^0 \\ &= 2mx^m + m(n-2)x^{m-1} + 2x + n - 2. \end{aligned}$$

□

Theorem 3.2. *The nbd deficient degree vertex polynomial of the ladder graph $B_{1,n}$ is*

$$ND_f D(B_{1,n}; x) = \begin{cases} 2n, & m = 1, n = 1, 2, \\ 2x^2 + 4x, & m = 1, n = 3, \\ 8x + 2(n-4), & m = 1, n \geq 4. \end{cases}$$

Proof. In the graph $B_{1,n}$, the set of vertices $\mathcal{V}(B_{1,n})$ contains $2n$ vertices . Let $|\mathcal{V}_{nd_f(u)}|$ be the number of the vertices of the nbd deficient degree $nd_f(u)$. Then, from Equation (1.5), the nbd deficient degree vertex polynomial of the ladder graph $B_{1,n}$ is

$$ND_f D(B_{1,n}; x) = \sum_{u \in \mathcal{V}(B_{1,n})} |\mathcal{V}_{nd_f(u)}| x^{nd_f(u)}.$$

There are three cases of ladder graph $B_{1,n}$ as follows:

Case(*i*) $m = 1$, $n = 1,2$.

In both graphs $B_{1,1}$ and $B_{1,2}$, all vertices have nbd deficient degree zero. Let $\mathcal{V}_0$ be the set of all these vertices. Then,

$$|\mathcal{V}_0| = 2n.$$

The nbd deficient degree vertex polynomial of the graph $B_{1,n}(n = 1, 2)$ is

$$\begin{aligned} ND_f D(B_{1,n}; x) &= \sum_{u \in \mathcal{V}(B_{1,n})} |\mathcal{V}_{nd_f(u)}| x^{nd_f(u)} \\ &= |\mathcal{V}_0| x^0 \\ &= 2n. \end{aligned}$$

Case(*ii*) $m = 1$, $n = 3$.

In the graph $B_{1,3}$, the set of vertices $\mathcal{V}(B_{1,3})$ contains 6 vertices. There are two types of sets of vertices. The first of them is $\mathcal{V}_1$, the set of vertices of deficient degree 1 and the other is $\mathcal{V}_2$, the set of vertices of deficient degree 2. Then,

$$|\mathcal{V}_1| = 4 \text{ and } |\mathcal{V}_2| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{1,3}$ is

$$\begin{aligned} ND_fD(B_{1,3};x) &= \sum_{u\in\mathcal{V}(B_{1,3})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)} \\ &= |\mathcal{V}_2|x^2 + |\mathcal{V}_1|x^1 \\ &= 2x^2 + 4x. \end{aligned}$$

Case(*iii*) $m = 1,\ n \geq 4$.

There are two types of subsets of the set of vertices of the graph $B_{1,n}$, *i.e.*, $\mathcal{V}_0$ and $\mathcal{V}_1$ are the set of vertices of the nbd deficient degrees 0 and 1, respectively. Then,

$$|\mathcal{V}_0| = 2(n-4) \text{ and } |\mathcal{V}_1| = 8.$$

The nbd deficient degree vertex polynomial of the graph $B_{1,n}(n \geq 4)$ is

$$\begin{aligned} ND_fD(B_{1,n};x) &= \sum_{u\in V} |\mathcal{V}_{d_f(u)}|x^{d_f(u)} \\ &= |\mathcal{V}_1|x^1 + |\mathcal{V}_0|x^0 \\ &= 8x + 2(n-4). \end{aligned}$$

□

Theorem 3.3. *The nbd deficient degree vertex polynomial of a graph $B_{2,n}$ is*

$$ND_fD(B_{2,n};x) = \begin{cases} n(x^2 + 2x^{n-1}), & m = 2, n = 1, 2, \\ 5x^4 + 4x^2, & m = 2, n = 3, \\ 2x^4 + 6x^3 + (3n-8)x^2, & m = 2, n \geq 4. \end{cases}$$

Proof. The set of vertices $\mathcal{V}(B_{2,n})$ contains $3n$ vertices. Let $|\mathcal{V}_{nd_f(u)}|$ be the number of vertices of the nbd deficient degree $nd_f(u)$, then the nbd deficient degree vertex polynomial of the graph $B_{2,n}$ is

$$ND_fD(B_{2,n};x) = \sum_{u\in\mathcal{V}(B_{2,n})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}.$$

We considered three cases of the graph $B_{2,n}$ as follows:

Case(*i*) $m = 2,\ n = 1{,}2$.

There are two sets $\mathcal{V}_{n-1}$ and $\mathcal{V}_2$ of the vertices of the nbd deficient degrees $n-1$ and 2 of the graph $B_{2,n}(n = 1, 2)$, respectively. Then,

$$|\mathcal{V}_{n-1}| = 2n \text{ and } |\mathcal{V}_2| = n.$$

The nbd deficient degree vertex polynomial of the graph $B_{2,n}(n = 1, 2)$ is

$$\begin{aligned} ND_fD(B_{2,n};x) &= \sum_{u\in\mathcal{V}(B_{2,n})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)} \\ &= |\mathcal{V}_2|x^2 + |\mathcal{V}_{n-1}|x^{n-1} \\ &= nx^2 + 2nx^{n-1} \\ &= n(x^2 + 2x^{n-1}). \end{aligned}$$

Case(*ii*) $m = 2,\ n = 3$.

The set of vertices of the graph $B_{2,3}$ contains 9 vertices. In this case, there are two sets of vertices of the graph $B_{2,3}$. The first of them is $\mathcal{V}_2$, the set of vertices of deficient degree 2 and the other is $\mathcal{V}_4$, the set of the vertices of deficient degree 4. Then,

$$|\mathcal{V}_2| = 4 \text{ and } |\mathcal{V}_4| = 5.$$

The nbd deficient degree vertex polynomial of the graph $B_{2,3}$ is

$$\begin{aligned} ND_fD(B_{2,3};x) &= \sum_{u\in\mathcal{V}(B_{2,3})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)} \\ &= |\mathcal{V}_4|x^4 + |\mathcal{V}_2|x^2 \\ &= 5x^4 + 4x^2. \end{aligned}$$

Case(iii) $m = 2$, $n \geq 4$.

There are three types of subsets of the set of the vertices of graph $B_{2,n}(n \geq 4)$. Let $\mathcal{V}_2, \mathcal{V}_3$ and $\mathcal{V}_4$ be the set of vertices of the nbd deficient degrees 2,3, and 4, respectively, in the graph $B_{2,n}(n \geq 4)$. Then,

$$|\mathcal{V}_2| = 3n - 8,\ |\mathcal{V}_3| = 6,\ \text{and } |\mathcal{V}_4| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{2,n}(n \geq 4)$ is

$$\begin{aligned} ND_fD(B_{2,n};x) &= \sum_{u\in\mathcal{V}(B_{2,n})} |V_{d_f(u)}|x^{d_f(u)} \\ &= |\mathcal{V}_4|x^4 + |\mathcal{V}_3|x^3 + |\mathcal{V}_2|x^2 \\ &= 2x^4 + 6x^3 + (3n-8)x^2. \end{aligned}$$

□

Theorem 3.4. *The nbd deficient degree vertex polynomial of a graph $B_{3,n}$ is*

$$ND_fD(B_{3,n};x) = \begin{cases} x^6 + 3, & m = 3, n = 1, \\ 2x^6 + 6x^2, & m = 3, n = 2, \\ 2x^9 + x^8 + 3x^6 + 6x^3, & m = 3, n = 3, \\ 2x^9 + 2x^7 + (n-4)x^6 + 6x^5 + 3(n-4)x^4 + 6x^3, & m = 3, n \geq 4. \end{cases}$$

Proof. The set of vertices $\mathcal{V}(B_{3,n})$ contains $3n$ vertices. Let $|\mathcal{V}_{nd_f(u)}|$ be the number of the vertices of the nbd deficient degree $nd_f(u)$. Then, the nbd deficient degree vertex polynomial of the graph $B_{3,n}$ is

$$ND_fD(B_{3,n};x) = \sum_{u\in\mathcal{V}(B_{3,n})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}.$$

There are four cases of the graph $B_{3,n}$ as follows:

Case(i) $m = 3$, $n = 1$.

There are two sets $\mathcal{V}_0$ and $\mathcal{V}_6$ of the vertices of nbd deficient degrees 0 and 6, respectively in the graph $B_{3,1}$. Then,

$$|\mathcal{V}_0| = 3 \text{ and } |\mathcal{V}_6| = 1.$$

The nbd deficient degree vertex polynomial of the graph $B_{3,1}$ is

$$\begin{aligned} ND_fD(B_{3,1};x) &= \sum_{u\in\mathcal{V}(B_{3,1})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)} \\ &= |\mathcal{V}_6|x^6 + |\mathcal{V}_0|x^0 \\ &= x^6 + 3x^0 \\ &= x^6 + 3. \end{aligned}$$

Case(ii) $m = 3$, $n = 2$.

There are two sets $\mathcal{V}_2$ and $\mathcal{V}_6$ of the vertices of the nbd deficient degrees 2 and 6, respectively. Then,

$$|\mathcal{V}_2| = 6 \text{ and } |\mathcal{V}_6| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{3,2}$ is

$$ND_fD(B_{3,2};x) = \sum_{u\in\mathcal{V}(B_{3,2})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}$$

$$= |\mathcal{V}_6|x^6 + |\mathcal{V}_2|x^2$$
$$= 2x^6 + 6x^2.$$

Case(*iii*) $m = 3$, $n = 3$.

The set of vertices $\mathcal{V}(B_{3,3})$ contains 12 vertices. There are four sets of vertices $\mathcal{V}_3, \mathcal{V}_6, \mathcal{V}_8$, and $\mathcal{V}_9$ of nbd deficient degrees 3, 6, 8, and 9, respectively. Then,

$$|\mathcal{V}_3| = 6, |\mathcal{V}_6| = 3, |\mathcal{V}_8| = 1, \text{ and } |\mathcal{V}_9| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{3,3}$ is

$$ND_fD(B_{3,3}; x) = \sum_{u\in\mathcal{V}(B_{3,3})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}$$
$$= |\mathcal{V}_9|x^9 + |\mathcal{V}_8|x^8 + |\mathcal{V}_6|x^6 + |\mathcal{V}_3|x^3$$
$$= 2x^9 + x^8 + 3x^6 + 6x^3.$$

Case(*iv*) $m = 3$, $n \geq 4$.

There are six types of subset of $\mathcal{V}(B_{3,n})$. Let $\mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7$, and $\mathcal{V}_9$ be the set of vertices of nbd deficient degrees 3, 4, 5, 6, 7, and 9, respectively. Then,

$$|\mathcal{V}_3| = 6, |\mathcal{V}_4| = 3(n-4), |\mathcal{V}_5| = 6, |\mathcal{V}_6| = n-4, |\mathcal{V}_7| = 2,$$
$$\text{and } |\mathcal{V}_9| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{3,n}(n \geq 4)$ is

$$ND_fD(B_{3,n}; x) = \sum_{u\in\mathcal{V}(B_{3,n})} |\mathcal{V}_{d_f(u)}|x^{d_f(u)}$$
$$= |\mathcal{V}_9|x^9 + |\mathcal{V}_7|x^7 + |\mathcal{V}_6|x^6 + |\mathcal{V}_5|x^5 + |\mathcal{V}_4|x^4 + |\mathcal{V}_3|x^3$$
$$= 2x^9 + 2x^7 + (n-4)x^6 + 6x^5 + 3(n-4)x^4 + 6x^3.$$

□

Theorem 3.5. *The nbd deficient degree vertex polynomial of the stacked book graph $B_{m,n}$ is* $ND_fD(B_{m,n}; x) =$

$$\begin{cases} x^{m(m-1)} + m, & m \geq 4, n = 1, \\ 2x^{m(m-1)} + 2mx^{m-1}, & m \geq 4, n = 2, \\ 2x^{m^2} + x^{mm-2} + mx^{2m} + 2mx^m, & m \geq 4, n = 3, \\ 2x^{m^2} + 2x^{m^2-m+1} + (n-4)x^{m^2-m} & \\ +2mx^{2m-1} + m(n-4)x^{2m-2} + 2mx^m, & m \geq 4, n \geq 4. \end{cases}$$

Proof. The set of vertices $\mathcal{V}(B_{m,n})$ contains $(m+1)n$ vertices. Let $|\mathcal{V}_{nd_f(u)}|$ be the number of vertices of the nbd deficient degree $nd_f(u)$. Then, the nbd deficient degree vertex polynomial of the graph $B_{m,n}$ is

$$ND_fD(B_{m,n}; x) = \sum_{u\in\mathcal{V}(B_{m,n})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}.$$

We considered the four cases of the graph $B_{m,n}$ as follows:

Case(*i*) $m \geq 4$, $n = 1$.

There are two sets $\mathcal{V}_0$ and $\mathcal{V}_{m(m-1)}$ of the vertices of the nbd deficient degrees 0 and $m(m-1)$, respectively. Then,

$$|\mathcal{V}_{m(m-1)}| = 1 \text{ and } |\mathcal{V}_0| = m.$$

The nbd deficient degree vertex polynomial of the graph $B_{m,1}(m \geq 4)$ is

$$ND_fD(B_{m,1}; x) = \sum_{u\in\mathcal{V}(B_{m,1})} |\mathcal{V}_{nd_f(u)}|x^{nd_f(u)}$$
$$= |\mathcal{V}_{m(m-1)}|x^{m(m-1)} + |\mathcal{V}_0|x^0$$

$$= x^{m(m-1)} + mx^0$$
$$= x^{m(m-1)} + m.$$

Case(*ii*) $m \geq 4$, $n = 2$.

There are two sets $\mathcal{V}_{m(m-1)}$ and $\mathcal{V}_{m-1}$ of vertices of nbd deficient degrees $m(m-1)$ and $m-1$, respectively. Then,

$$|\mathcal{V}_{m(m-1)}| = 2 \text{ and } |\mathcal{V}_{m-1}| = 2m.$$

The nbd deficient degree vertex polynomial of the graph $B_{m,2}(m \geq 4)$ is

$$\begin{aligned} ND_fD(B_{m,2};x) &= \sum_{u \in \mathcal{V}(B_{m,2})} |\mathcal{V}_{nd_f(u)}| x^{nd_f(u)} \\ &= |\mathcal{V}_{m(m-1)}| x^{m(m-1)} + |\mathcal{V}_{m-1}| x^m \\ &= 2x^{m(m-1)} + 2mx^{m-1}. \end{aligned}$$

Case(*iii*) $m \geq 4$, $n = 3$.

The set of vertices $\mathcal{V}(B_{m,3})$ contains $3(m+1)$ vertices. In this case, we get four types of sets of vertices. Let $\mathcal{V}_m, \mathcal{V}_{2m}, \mathcal{V}_{m^2-m+2}$, and $\mathcal{V}_{m^2}$ be the set of vertices of nbd deficient degrees m, $2m$, m^2-m+2, and m^2, respectively. Then,

$$|\mathcal{V}_m| = 2m, |\mathcal{V}_{2m}| = m, |\mathcal{V}_{m^2-m+2}| = 1, \text{ and } |\mathcal{V}_{m^2}| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{m,3}(m \geq 4)$ is

$$\begin{aligned} ND_fD(B_{m,3};x) &= \sum_{u \in \mathcal{V}(B_{m,3})} |\mathcal{V}_{nd_f(u)}| x^{nd_f(u)} \\ &= |\mathcal{V}_{m^2}| x^{m^2} + |\mathcal{V}_{m^2-m+2}| x^{m^2-m+2} + |\mathcal{V}_{2m}| x^{2m} + |\mathcal{V}_m| x^m \\ &= 2x^{m^2} + x^{m^2-m+2} + mx^{2m} + 2mx^m. \end{aligned}$$

Case(*iv*) $m \geq 4$, $n \geq 4$.

There are six types of subsets of $\mathcal{V}(B_{m,n})$. Let $\mathcal{V}_m, \mathcal{V}_{2m-2}, \mathcal{V}_{2m-1}, \mathcal{V}_{m^2-m}, \mathcal{V}_{m^2-m+1}$, and $\mathcal{V}_{m^2}$ be the set of vertices of nbd deficient degrees m, $2m-2$, $2m-1$, m^2-m, m^2-m+1, and m^2, respectively. Then,

$$|\mathcal{V}_m| = 2m, |\mathcal{V}_{2m-2}| = m(n-4), |\mathcal{V}_{2m-1}| = 2m,$$
$$|\mathcal{V}_{m^2-m}| = n-4, |\mathcal{V}_{m^2-m+1}| = 2, \text{ and } |\mathcal{V}_{m^2}| = 2.$$

The nbd deficient degree vertex polynomial of the graph $B_{m,n}(m \geq 4, n \geq 4)$ is

$$\begin{aligned} &ND_fD(B_{m,n};x) \\ &= \sum_{u \in \mathcal{V}(B_{m,n})} |\mathcal{V}_{d_f(u)}| x^{d_f(u)} \\ &= |\mathcal{V}_{m^2}| x^{m^2} + |\mathcal{V}_{m^2-m+1}| x^{m^2-m+1} + |\mathcal{V}_{m^2-m}| x^{m^2-m} + |\mathcal{V}_{2m-1}| x^{2m-1} + |\mathcal{V}_{2m-2}| x^{2m-2} + |\mathcal{V}_m| x^m \\ &= 2x^{m^2} + 2x^{m^2-m+1} + (n-4)x^{m^2-m} + 2mx^{2m-1} + m(n-4)x^{2m-2} + 2mx^m. \end{aligned}$$

□

Theorem 3.6. *Let $B_{m,n}$ be a stacked book graph. The deficient M-polynomial of the graph $B_{m,n}$ is*

$$D_fM(B_{m,n};x,y) = \begin{cases} my^{m-1}, & m \geq 1, n = 1, \\ 1 + 2my^{m-1} + mx^{m-1}y^{m-1}, & m \geq 1, n = 2, \\ (n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m & \\ +m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m, & m \geq 1, n \geq 3. \end{cases}$$

Proof. The stacked book graph $B_{m,n}$ has $(m+1)n$ vertices and $(2mn+n-m-1)$ edges. Let $d_f(u)$ be the deficient degree of a vertex u in the graph $B_{m,n}$. $m^*_{i,j}$ is the number of edges of the graph $B_{m,n}$ that have end vertices of deficient degrees i and j. Then, from Equation (1.4), the deficient M-polynomial of the graph $B_{m,n}$ is

$$D_f M(B_{m,n};x,y) = \sum_{\theta \le i \le j \le \Theta} m^*_{i,j}(B_{m,n}) x^i y^j,$$

where $\theta = min\{d_f(u) : u \in \mathcal{V}\}$, $\Theta = max\{d_f(v) : v \in \mathcal{V}\}$.

There are following three cases as follows:

Case(*i*) $m \ge 1$, $n = 1$.

The graph $B_{m,1}$ has m edges with $\theta = 0$ and $\Theta = m-1$, *i.e.*, $m^*_{0,m-1} = m$. Now, we get deficient M-polynomial of the graph $B_{m,1}$ by

$$\begin{aligned} D_f M(B_{m,1};x,y) &= \sum_{0 \le i \le j \le m-1} m^*_{i,j}(B_{m,1}) x^i y^j \\ &= m^*_{0,m-1} x^0 y^{m-1} \\ &= m y^{m-1}. \end{aligned}$$

Case(*ii*) $m \ge 1$, $n = 2$.

The graph $B_{m,2}$ has $3m+1$ edges with $\theta = 0$ and $\Theta = m-1$. There are three types of edges of the deficient degree of end vertices $(0,0); (0,m-1); (m-1,m-1)$. Then,

$$m^*_{0,0} = 1, m^*_{0,m-1} = 2m, \text{ and } m^*_{m-1,m-1} = m.$$

Now, we get the deficient M-polynomial of the graph $B_{m,2}$ by

$$\begin{aligned} D_f M(B_{m,2};x,y) &= \sum_{0 \le i \le j \le m-1} m^*_{i,j}(B_{m,2}) x^i y^j \\ &= m^*_{0,0} x^0 y^0 + m^*_{0,m-1} x^0 y^{m-1} + m^*_{m-1,m-1} x^{m-1} y^{m-1} \\ &= 1 + 2m y^{m-1} + m x^{m-1} y^{m-1}. \end{aligned}$$

Case(*iii*) $m \ge 1$, $n \ge 3$.

In this case, the graph $B_{m,n}$ has $2mn+n-m-1$ edges with $\theta = 0$ and $\Theta = m$. There are six types of edges of the deficient degree of end vertices $(0,0); (0,1); (0,m-1); (1,m); (m-1,m-1); (m-1,m)$. Then,

$$\begin{aligned} &m^*_{0,0} = n-3,\ m^*_{0,1} = 2,\ m^*_{0,m-1} = m(n-2),\ m^*_{1,m} = 2m, \\ &m^*_{m-1,m-1} = m(n-3), \text{ and } m^*_{m-1,m} = 2m. \end{aligned}$$

Now, we get deficient M-polynomial of the graph $B_{m,n}$ by

$$\begin{aligned} &D_f M(B_{m,n};x,y) \\ &= \sum_{0 \le i \le j \le m} m^*_{i,j}(B_{m,n}) x^i y^j \\ &= m^*_{0,0} x^0 y^0 + m^*_{0,1} x^0 y^1 + m^*_{0,m-1} x^0 y^{m-1} + m^*_{1,m} x^1 y^m + m^*_{m-1,m-1} x^{m-1} y^{m-1} + m^*_{m-1,m} x^{m-1} y^m \\ &= (n-3) + 2y + m(n-2) y^{m-1} + 2mxy^m + m(n-3) x^{m-1} y^{m-1} + 2m x^{m-1} y^m. \end{aligned}$$

□

Theorem 3.7. *Let $B_{m,n}$ be a stacked book graph. The nbd deficient M-polynomial of the graph $B_{m,n}$ is*

$NDfM(B_{m,n};x,y) =$

$$\begin{cases} my^{m(m-1)}, & m \geq 1, n = 1, \\ mx^{m-1}y^{m-1} + 2mx^{m-1}y^{m(m-1)} & \\ +x^{m(m-1)}y^{m(m-1)}, & m \geq 1, n = 2, \\ 2xy + 4xy^2 + x^2y^2, & m = 1, n = 3, \\ 2mx^my^{2m} + 2mx^my^{m^2} & \\ +mx^{2m}y^{m^2-m+2} + 2x^{m^2-m+2}y^{m^2}, & m \geq 2, n = 3, \\ 2mx^my^{2m-1} + 2mx^my^{m^2} & \\ +mx^{2m-1}y^{2m-1} + 2mx^{2m-1}y^{m^2-m+1} & \\ +2x^{m^2-m+1}y^{m^2} + x^{m^2-m+1}y^{m^2-m+1}, & m \geq 1, n = 4, \\ 2mx^my^{2m-1} + 2mx^my^{m^2} & \\ +m(n-5)x^{2m-2}y^{2m-2} + 2mx^{2m-2}y^{2m-1} & \\ +m(n-4)x^{2m-2}y^{m^2-m} + 2mx^{2m-1}y^{m^2-m+1} & \\ +(n-5)x^{m^2-m}y^{m^2-m} + 2x^{m^2-m}y^{m^2-m+1} & \\ +2x^{m^2-m+1}y^{m^2}, & m \geq 1, n \geq 5. \end{cases}$$

Proof. The stacked book graph $B_{m,n}$ has $(m+1)n$ vertices and $(2mn+n-m-1)$ edges. Let $nd_f(u)$ be the nbd deficient degree of vertex u in the graph $B_{m,n}$. $n^*_{i,j}$ is the number of edges of the graph $B_{m,n}$ which have end vertices of nbd deficient degree i and j. Then, from Equation (1.6) the nbd deficient M-polynomial of the graph $B_{m,n}$ is

$$NDfM(B_{m,n};x,y) = \sum_{\sigma \leq i \leq j \leq \Psi} n^*_{i,j}(B_{m,n})x^iy^j,$$

where $\sigma = min\{nd_f(u) : u \in \mathcal{V}\}$, $\Psi = max\{nd_f(v) : v \in \mathcal{V}\}$.

There are following six cases as follows:

Case(*i*) $m \geq 1$, $n = 1$.

The graph $B_{m,1}$ has m edges with $\sigma = 0$ and $\Psi = m(m-1)$. *i.e.*, $n^*_{0,m(m-1)} = m$. Now, we get nbd deficient M-polynomial of the graph $B_{m,1}$ by

$$\begin{aligned} NDfM(B_{m,1};x,y) &= \sum_{0 \leq i \leq j \leq m(m-1)} n^*_{i,j}(B_{m,1})x^iy^j \\ &= n^*_{0,m(m-1)}x^0y^{m(m-1)} \\ &= my^{m(m-1)}. \end{aligned}$$

Case(*ii*) $m \geq 1$, $n = 2$.

The graph $B_{m,2}$ has $3m+1$ edges with $\sigma = m-1$ and $\Psi = m(m-1)$. There are three types of edges of nbd deficient degree of end vertices $(m-1, m-1); (m-1, m(m-1)); (m(m-1), m(m-1))$. Then,

$$n^*_{m-1,m-1} = m, n^*_{m-1,m(m-1)} = 2m, \text{ and } n^*_{m(m-1),m(m-1)} = 1.$$

Now, we get nbd deficient M-polynomial of the graph $B_{m,2}$ by

$$\begin{aligned} &NDfM(B_{m,2};x,y) \\ &= \sum_{m-1 \leq i \leq j \leq m(m-1)} n^*_{i,j}(B_{m,2})x^iy^j \\ &= n^*_{m-1,m-1}x^{m-1}y^{m-1} + n^*_{m-1,m(m-1)}x^{m-1}y^{m(m-1)} + n^*_{m(m-1),m(m-1)}x^{m(m-1)}y^{m(m-1)} \\ &= mx^{m-1}y^{m-1} + 2mx^{m-1}y^{m(m-1)} + x^{m(m-1)}y^{m(m-1)}. \end{aligned}$$

Case(*iii*) $m = 1$, $n = 3$.

The graph $B_{1,3}$ has 7 edges with $\sigma = 1$ and $\Psi = 2$. There are three types of edges of nbd deficient degree of end vertices $(1,1); (1,2); (2,2)$. Then,

$$n^*_{1,1} = 2, n^*_{1,2} = 4, \text{ and } n^*_{2,2} = 1.$$

Now, we get nbd deficient M-polynomial of the graph $B_{1,3}$ by

$$\begin{aligned}ND_fM(B_{1,3};x,y) &= \sum_{1\le i\le j\le 2} n^*_{i,j}(B_{1,3})x^iy^j\\ &= n^*_{1,1}x^1y^1 + n^*_{1,2}x^1y^2 + n^*_{2,2}x^2y^2\\ &= 2xy + 4xy^2 + x^2y^2.\end{aligned}$$

Case(*iv*) $m \ge 2$, $n = 3$.

In this case, the graph $B_{m,3}$ has $(5m+2)$ edges with $\sigma = m$ and $\Psi = m^2$. There are three types of edges of nbd deficient degree of end vertices $(m,2m); (m,m^2); (2m,m^2-m+2); (m^2-m+2,m^2)$. Then,

$$n^*_{m,2m} = 2m,\ n^*_{m,m^2} = 2m,\ n^*_{2m,m^2-m+2} = m,\ \text{and } n^*_{m^2-m+2,m^2} = 2.$$

Now, we get nbd deficient M-polynomial of the stacked book graph $B_{m,3}$ by

$$\begin{aligned}ND_fM(B_{m,3};x,y) &= \sum_{m\le i\le j\le m^2} n^*_{i,j}(B_{m,3})x^iy^j\\ &= n^*_{m,2m}x^my^{2m} + n^*_{m,m^2}x^my^{m^2} + n^*_{2m,m^2-m+2}x^{2m}y^{m^2-m+2} + n^*_{m^2-m+2,m^2}x^{m^2-m+2}y^{m^2}\\ &= 2mx^my^{2m} + 2mx^my^{m^2} + mx^{2m}y^{m^2-m+2} + 2x^{m^2-m+2}y^{m^2}.\end{aligned}$$

Case(*v*) $m \ge 1$, $n = 4$.

The graph $B_{m,4}$ has $7m+3$ edges with $\sigma = m$ and $\Psi = m^2$. There are six types of edges of nbd deficient degree of end vertices $(m,2m-1); (m,m^2); (2m-1,2m-1); (2m-1,m^2-m+1); (m^2-m+1,m^2-m+1); (m^2-m+1,m^2)$. Then,

$$\begin{aligned}&n^*_{m,2m-1} = 2m, n^*_{m,m^2} = 2m, n^*_{2m-1,2m-1} = m,\\ &n^*_{2m-1,m^2-m+1} = 2m, n^*_{m^2-m+1,m^2-m+1} = 1,\\ &\text{and } n^*_{m^2-m+1,m^2} = 2.\end{aligned}$$

Now, we get nbd deficient M-polynomial of the graph $B_{m,4}$ by

$$\begin{aligned}ND_fM(B_{m,4};x,y) &= \sum_{m\le i\le j\le m^2} n^*_{i,j}(B_{m,4})x^iy^j\\ &= n^*_{m,2m-1}x^my^{2m-1} + n^*_{m,m^2}x^my^{m^2} + n^*_{2m-1,2m-1}x^{2m-1}y^{2m-1}\\ &\quad + n^*_{2m-1,m^2-m+1}x^{2m-1}y^{m^2-m+1} + n^*_{m^2-m+1,m^2-m+1}x^{m^2-m+1}y^{m^2-m+1} + n^*_{m^2-m+1,m^2}x^{m^2-m+1}y^{m^2}\\ &= 2mx^my^{2m-1} + 2mx^my^{m^2} + mx^{2m-1}y^{2m-1} + 2mx^{2m-1}y^{m^2-m+1} + x^{m^2-m+1}y^{m^2-m+1} + 2x^{m^2-m+1}y^{m^2}.\end{aligned}$$

Case(*vi*) $m \ge 1$, $n \ge 5$

In this case, the graph $B_{m,n}$ has $(2mn+n-m-1)$ edges with $\sigma = m$ and $\Psi = m^2$. There are nine types of edges of nbd deficient degree of end vertices as $(m,2m-1); (m,m^2); (2m-2,2m-2); (2m-2,2m-1); (2m-2,m^2-m); (2m-2,m^2-m+1); (m^2-m,m^2-m); (m^2-m,m^2-m+1); (m^2-m+1,m^2)$. Then,

$$\begin{aligned}&n^*_{m,2m-1} = 2m,\ n^*_{m,m^2} = 2m,\ n^*_{2m-2,2m-2} = m(n-5), n^*_{2m-2,2m-1} = 2m,\ n^*_{2m-2,m^2-m}\\ &= m(n-4),\ n^*_{2m-1,m^2-m+1}\\ &= 2m,\ n^*_{m^2-m,m^2-m} = n-5,\ n^*_{m^2-m,m^2-m+1} = 2,\\ \text{and}\quad &n^*_{m^2-m+1,m^2} = 2.\end{aligned}$$

Now, we get nbd deficient M-polynomial of the graph $B_{m,n}$ by

$$\begin{aligned}ND_fM(B_{m,n};x,y) &= \textstyle\sum_{m\le i\le j\le m^2} n^*_{i,j}(B_{m,n})x^iy^j\\ &= n^*_{m,2m-1}x^my^{2m-1} + n^*_{m,m^2}x^my^{m^2} + n^*_{2m-2,2m-2}x^{2m-2}y^{2m-2} + n^*_{2m-2,2m-1}x^{2m-2}y^{2m-1}\\ &\quad + n^*_{2m-2,m^2-m}x^{2m-2}y^{m^2-m} + n^*_{2m-1,m^2-m+1}x^{2m-1}y^{m^2-m+1}\\ &\quad + n^*_{m^2-m,m^2-m}x^{m^2-m}y^{m^2-m} + n^*_{m^2-m,m^2-m+1}x^{m^2-m}y^{m^2-m+1} + n^*_{m^2-m+1,m^2}x^{m^2-m+1}y^{m^2}\\ &= 2mx^my^{2m-1} + 2mx^my^{m^2} + m(n-5)x^{2m-2}y^{2m-2} + 2mx^{2m-2}y^{2m-1} + m(n-4)x^{2m-2}y^{m^2-m}\\ &\quad + 2mx^{2m-1}y^{m^2-m+1} + (n-5)x^{m^2-m}y^{m^2-m} + 2x^{m^2-m}y^{m^2-m+1} + 2x^{m^2-m+1}y^{m^2}.\end{aligned}$$

□

From these above polynomials, we can find some deficient topological indices of the graph $B_{m,n}$. In the next section, we will compute deficient topological indices of the graph $B_{m,n} (m \ge 1, n \ge 3)$ using deficient M-polynomial of said graph.

4 Deficient Topological Indices for Stacked Book Graph $B_{m,n}(m \geq 1, n \geq 3)$

Using the M-polynomial 3.6 case (*iii*) of the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ and Table 1.2, we compute the six deficient degree-based indices for the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ in the following theorem.

Theorem 4.1. *Let $\mathcal{G}$ be a stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$. Then, the deficient degree-based topological indices for the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ are*

(*i*) *deficient first Zagreb index*
$M_1^*(\mathcal{G}) = 2 + 8m - 3mn - 2m^2 + 3m^2n,$

(*ii*) *deficient second Zagreb index*
$M_2^*(\mathcal{G}) = -3m + mn + 6m^2 - 2m^2n - m^3 + m^3n,$

(*iii*) *deficient redefined third Zagreb index*

$$ReZ_3^*(\mathcal{G}) = 6m - 2mn - 14m^2 + 6m^2n + 14m^3 - 6m^3n - 2m^4 + 2m^4n.$$

Proof. From Theorem 3.6, the deficient M-polynomial of the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ is

$$D_fM(\mathcal{G}; x, y) = (n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m = f(x, y).$$

We calculate the following expressions:

$$\begin{aligned}
D_xf(x,y) =& x\frac{\partial}{\partial x}f(x,y)\\
=& x\frac{\partial}{\partial x}\left\{(n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m\right\}\\
=& 2mxy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m(m-1)x^{m-1}y^m,
\end{aligned}$$

$$\begin{aligned}
D_yf(x,y) =& y\frac{\partial}{\partial y}f(x,y)\\
=& y\frac{\partial}{\partial y}\left\{(n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m\right\}\\
=& 2y + m(m-1)(n-2)y^{m-1} + 2m^2xy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m^2x^{m-1}y^m.
\end{aligned}$$

Now,

$$[D_x + D_y]f(x,y) = 2y + m(m-1)(n-2)y^{m-1} + 2m(m+1)xy^m + 2m(n-3)(m-1)x^{m-1}y^{m-1} + 2m(2m-1)x^{m-1}y^m.$$

$$\begin{aligned}
[D_xD_y]f(x,y) =& x\frac{\partial}{\partial x}D_yf(x,y)\\
=& x\frac{\partial}{\partial x}\left\{2y + m(m-1)(n-2)y^{m-1} + 2m^2xy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m^2x^{m-1}y^m\right\}\\
=& 2m^2xy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m^2(m-1)x^{m-1}y^m,
\end{aligned}$$

and

$$[D_xD_y(D_x + D_y)]f(x,y) = 2m^2(m+1)xy^m + 2m(n-3)(m-1)^3x^{m-1}y^{m-1} + 2m^2(m-1)(2m-1)x^{m-1}y^m.$$

Using the above equations and formulae in Table 1.2, we have

(*i*) Deficient first Zagreb index

$$\begin{aligned}
M_1^*(\mathcal{G}) =& [(D_x + D_y)\, f(x,y)]_{x=y=1}\\
=& [2y + m(m-1)(n-2)y^{m-1} + 2m(m+1)xy^m + 2m(n-3)(m-1)x^{m-1}y^{m-1}\\
&+ 2m(2m-1)x^{m-1}y^m]_{x=y=1}\\
=& 2 + 8m - 3mn - 2m^2 + 3m^2n.
\end{aligned}$$

(*ii*) Deficient second Zagreb index

$$\begin{aligned}
M_2^*(\mathcal{G}) =& [(D_xD_y)f(x,y)]_{x=y=1}\\
=& [2m^2xy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m^2(m-1)x^{m-1}y^m]_{x=y=1}\\
=& -3m + mn + 6m^2 - 2m^2n - m^3 + m^3n.
\end{aligned}$$

(iii) Deficient redefined third Zagreb index

$$\begin{aligned}ReZ_3^*(\mathcal{G}) &= [(D_xD_y(D_x+D_y))\, f(x,y)]_{x=y=1}\\ &= \left[2m^2(m+1)xy^m + 2m(n-3)(m-1)^3x^{m-1}y^{m-1} + 2m^2(m-1)(2m-1)x^{m-1}y^m\right]_{x=y=1}\\ &= 6m - 2mn - 14m^2 + 6m^2n + 14m^3 - 6m^3n - 2m^4 + 2m^4n.\end{aligned}$$

□

Theorem 4.2. *Let $\mathcal{G}$ be a stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$. Then, the deficient degree-based indices for the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ are*

(i) *deficient forgotten index*

$$F^*(\mathcal{G}) = 2 - 4m + 3mn + 12m^2 - 6m^2n - 2m^3 + 3m^3n,$$

(ii) *deficient inverse sum indeg index*

$$ISI^*(\mathcal{G}) = \frac{2m^2}{m+1} + \frac{m(m-1)(n-3)}{2} + \frac{2m^2(m-1)}{2m-1},$$

and

(iii) *deficient augmented Zagreb index*

$$A^*(\mathcal{G}) = \frac{2m^4}{(m-1)^3} + \frac{m(n-3)(m-1)^6}{8(m-2)^3} + \frac{2m^4(m-1)^3}{(2m-3)^3}.$$

Proof. From Theorem 3.6, the deficient M-polynomial of the stacked book graph $B_{m,n}(m \geq 1, n \geq 3)$ is

$$D_fM(\mathcal{G};x,y) = (n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m = f(x,y).$$

We calculate the following expressions:

$$\begin{aligned}D_xf(x,y) &= x\frac{\partial}{\partial x}f(x,y)\\ &= x\frac{\partial}{\partial x}\left\{(n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m\right\}\\ &= 2mxy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m(m-1)x^{m-1}y^m,\\ D_x^2f(x,y) &= x\frac{\partial}{\partial x}D_xf(x,y)\\ &= x\frac{\partial}{\partial x}\left\{2mxy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m(m-1)x^{m-1}y^m\right\}\\ &= 2mxy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m(m-1)^2x^{m-1}y^m,\\ D_yf(x,y) &= y\frac{\partial}{\partial y}f(x,y)\\ &= y\frac{\partial}{\partial y}\left\{(n-3) + 2y + m(n-2)y^{m-1} + 2mxy^m + m(n-3)x^{m-1}y^{m-1} + 2mx^{m-1}y^m\right\}\\ &= 2y + m(m-1)(n-2)y^{m-1} + 2m^2xy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m^2x^{m-1}y^m.\\ D_y^2f(x,y) &= y\frac{\partial}{\partial y}D_yf(x,y)\\ &= y\frac{\partial}{\partial y}\left\{2y + m(m-1)(n-2)y^{m-1} + 2m^2xy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m^2x^{m-1}y^m\right\}\\ &= 2y + m(m-1)^2(n-2)y^{m-1} + 2m^3xy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m^3x^{m-1}y^m.\\ [D_xD_y]f(x,y) &= x\frac{\partial}{\partial x}D_yf(x,y)\\ &= x\frac{\partial}{\partial x}\left\{2y + m(m-1)(n-2)y^{m-1} + 2m^2xy^m + m(n-3)(m-1)x^{m-1}y^{m-1} + 2m^2x^{m-1}y^m\right\}\\ &= 2m^2xy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m^2(m-1)x^{m-1}y^m.\\ JD_xD_yf(x,y) &= [D_xD_yf(x,y)]_{x=y}\\ &= \left[2m^2xy^m + m(n-3)(m-1)^2x^{m-1}y^{m-1} + 2m^2(m-1)x^{m-1}y^m\right]_{x=y}\end{aligned}$$

$$=2m^2x^{m+1}+m(n-3)(m-1)^2x^{2(m-1)}+2m^2(m-1)x^{2m-1}.$$

Now,

$$\begin{aligned}
D_x^3D_y^3f(x,y) =&2m^4xy^m+m(n-3)(m-1)^6x^{m-1}y^{m-1}+2m^4(m-1)^3x^{m-1}y^m.\\
JD_x^3D_y^3f(x,y) =&\left[D_x^3D_y^3f(x,y)\right]_{x=y}\\
=&\left[2m^4xy^m+m(n-3)(m-1)^6x^{m-1}y^{m-1}\ +2m^4(m-1)^3x^{m-1}y^m\right]_{x=y}\\
=&2m^4x^{m+1}+m(n-3)(m-1)^6x^{2(m-1)}+2m^4(m-1)^3x^{2m-1}.\\
Q_{-2}JD_x^3D_y^3f(x,y) =&x^{-2}\left[JD_x^3D_y^3f(x,y)\right]\\
=&x^{-2}\left[2m^4x^{m+1}+m(n-3)(m-1)^6x^{2(m-1)}\ +2m^4(m-1)^3x^{2m-1}\right]\\
=&2m^4x^{m-1}+m(n-3)(m-1)^6x^{2(m-2)}+2m^4(m-1)^3x^{2m-3}.\\
S_xQ_{-2}JD_x^3D_y^3f(x,y) =&\int_0^x\frac{1}{t}\left\{Q_{-2}JD_x^3D_y^3f(x,y)\right\}_{x=t}dt\\
=&\int_0^x\frac{1}{t}\left\{2m^4t^{m-1}+m(n-3)(m-1)^6t^{2(m-2)}\ +2m^4(m-1)^3t^{2m-3}\right\}dt\\
=&\frac{2m^4}{m-1}x^{m-1}+\frac{m(n-3)(m-1)}{2(m-2)}^6x^{2(m-2)}+\frac{2m^4(m-1)^3}{2m-3}x^{2m-3}.
\end{aligned}$$

Now,

$$\begin{aligned}
[D_x^2+D_y^2]f(x,y) =&2y+m(m-1)^2(n-2)y^{m-1}+2m(m^2+1)xy^m+2m(n-3)(m-1)^2x^{m-1}y^{m-1}\\
&+2m(2m^2-2m+1)x^{m-1}y^m.\\
[S_xJD_xD_y]f(x,y) =&\int_0^x\frac{1}{t}\left\{JD_xD_yf(x,y)\right\}_{x=t}dt\\
=&\int_0^x\frac{1}{t}\left\{2m^2t^{m+1}+m(n-3)(m-1)^2\ t^{2(m-1)}+2m^2(m-1)t^{2m-1}\right\}dt\\
=&\frac{2m^2}{m+1}x^{m+1}+\frac{m(n-3)(m-1)}{2}x^{2(m-1)}+\frac{2m^2(m-1)}{2m-1}x^{2m-1},
\end{aligned}$$

and

$$[S_x^3Q_{-2}JD_x^3D_y^3]f(x,y)=\frac{2m^4}{(m-1)^3}x^{m-1}+\frac{m(n-3)(m-1)}{8(m-2)^3}^6x^{2(m-2)}+\frac{2m^4(m-1)^3}{(2m-3)^3}x^{2m-3}.$$

Using the above equations and formulae in Table 1.2, we get

(*i*) Deficient forgotten index

$$\begin{aligned}
F^*(\mathcal{G}) =&\left[\left(D_x^2+D_y^2\right)f(x,y)\right]_{x=y=1}\\
=&[2y+m(m-1)^2(n-2)y^{m-1}+2m(m^2+1)xy^m+2m(n-3)(m-1)^2x^{m-1}y^{m-1}\\
&+2m(2m^2-2m+1)x^{m-1}y^m]_{x=y=1}\\
=&2-4m+3mn+12m^2-6m^2n-2m^3+3m^3n.
\end{aligned}$$

(*ii*) Deficient inverse sum indeg index

$$\begin{aligned}
ISI^*(\mathcal{G}) =&\left[(S_xJD_x.D_y)\,f(x,y)\right]_{x=1}\\
=&\left[\frac{2m^2}{m+1}x^{m+1}+\frac{m(n-3)(m-1)}{2}x^{2(m-1)}\ +\frac{2m^2(m-1)}{2m-1}x^{2m-1}\right]_{x=1}\\
=&\frac{2m^2}{m+1}+\frac{m(n-3)(m-1)}{2}+\frac{2m^2(m-1)}{2m-1}.
\end{aligned}$$

(*iii*) Deficient augmented Zagreb index

$$\begin{aligned}
A^*(\mathcal{G}) =&\left[\left(S_x^3Q_{-2}JD_x^3D_y^3)\right)f(x,y)\right]_{x=1}\\
=&\left[2\frac{m^4}{(m-1)^3}x^{m-1}+\frac{m(n-3)(m-1)}{8(m-2)^3}^6x^{2(m-2)}\ +\frac{2m^4(m-1)^3}{(2m-3)^3}x^{2m-3}\right]_{x=1}\\
=&\frac{2m^4}{(m-1)^3}+\frac{m(n-3)(m-1)}{8(m-2)^3}^6+\frac{2m^4(m-1)^3}{(2m-3)^3}.
\end{aligned}$$

□

5 Conclusion

In this article, we proved some graph theoretical results on the deficient degree of vertices in a simple graph. For the stacked book graph, we calculated the deficient degree vertex polynomial, nbd deficient degree vertex polynomial, deficient M-polynomial, and nbd deficient M-polynomial. These results can be used to determine almost all the deficient topological indices of organic compounds in chemical graph theory.

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