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ANALYTICAL SOLUTION OF LINEAR OPTIMIZATION PROBLEMS USING GAUSS ELIMINATION AND LU DECOMPOSITION METHODS

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Abstract

This paper presents a novel approach to address Linear Optimization Problems *(LOPs)* utilizing numerical analysis techniques, specifically the Gauss Elimination *(GE)* Method and *LU* Decomposition *(LUD)* Method for matrices. These methods centred on square matrices offer direct solutions without the need for iterative processes. The study demonstrates the superiority of these techniques over the Simplex Method.
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1 Introduction

Optimization is the essential part of Operation Research *(OR)* which is the art of decision making. Generally all *LOP* are related with maximization (or minimization) of linear function subject to set of constraints. The Simplex method of the linear optimization was developed by Dantzig [2] (Also see Stevenson [11]). He demonstrated how to use an objective function to find the optimal solution from amongst the several feasible solutions to the *LOP*. Further, development of computers last three decades has made it easy for the Simplex Method to solve large scale *LOPs* very quickly [3, 13]. However in 1984, Karmakar [6] developed a new algorithm for solving very large scale *LOP*. Further in 2010, this method was modified by Pandian and Natarajan [9].

Numerical analysis is the investigation of algorithms that require a numerical approximation for the problems of mathematical analysis. Generally it has application in all fields of engineering and physical sciences, but in the 21^{th} century it has found application in many other different fields as life sciences, social sciences, medicine, and business. The activities of scientific computations have been introduced in different regions [3]. The rapid development of high speed digital computers and the increasing desire for numerical answer to applied problems have led to enhanced demands in the dealing with the methods and techniques of numerical analysis. Now a day the numerical methods have uses in the scientific research field so its application has come to the fundamental importance of distinct areas. One very important reason is that numerical methods can give solution when ordinary analytical method fails. For example when finding the roots of transcendental equations or in solving the differential equations, we use numerical methods. So without understanding of numerical methods we cannot apply its application in applied science or engineering field.

As an aspect of mathematics and computer science that generates, analyses, implements algorithms, the growth in power and the revolution in computing have raised the use of realistic mathematical models in science and engineering. Complex numerical analysis is required to provide solutions to the more elaborate models of the world. Numerical linear algebra is relevant for data analysis and stochastic differential equations. Numerical methods continue this long tradition of practical mathematical calculations. The field of numerical analysis predates the invention of modern computers by many centuries. Linear interpolation was already used more than 2000 years ago. Many great mathematicians of the past were preoccupied by numerical analysis and they provided a lot of important algorithms like Newton's method, Lagrange interpolation polynomial, Gaussian elimination, and Euler's method etc. [4].

In this paper, we use numerical analysis techniques which are GE method and LUD method of matrix for the system of LOP and solve it. These ideas have been adopted from Householder [5]. By applying numerical analysis techniques we get the solution after initial iteration. GE method gets the solution after the initial iteration. In LUD method the objective function is considered as a constraint together with the linear inequalities which forms a system of linear inequalities. Here we get the solution just after the initial iteration.

2 Some Basic Terminology [7, 10, 12]

Optimal (optimum) solution A feasible solution to LOP is said to be optimal solution if it is also optimize the objective function Z of the problem.

Coefficient Matrix All the coefficient of the set of linear system of equations in matrix form is Coefficient Matrix.

Constant Matrix All the constant of the set of linear system of equations in matrix form is Constant Matrix.

Augmented Matrix Combination matrix of coefficient matrix and constant matrix is augmented matrix. If A is coefficient matrix and B is constant matrix then $[A \mid B]$ is augmented matrix.

Elementary row transformation If a matrix converts into another matrix by only row operation (addition and subtraction of any two rows, multiplication and division by any scalar of any rows) is Elementary row transformation.

Upper Triangular Matrix A square matrix is upper triangular matrix if all elements below principal diagonal are zero.

Lower Triangular Matrix A square matrix is lower triangular matrix if all elements above principal diagonal are zero.
The matrix $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ is called lower triangular matrix.

Unit Lower Triangular Matrix A lower triangular matrix is said to unit lower triangular matrix if all elements of principal diagonal are one. Lower triangular matrix, $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ is called unit lower triangular matrix.

3 Methods to Solve *LOPs*

Simplex Method All Optimization problems can be solved by Simplex Method but here we focus linear optimization problems because it's comparing Numerical analysis techniques. Given problem always optimize to maximization if it is not optimize to maximization then it's convert to maximization, objective function. All inequalities convert into equalities by introducing non negative slack and surplus variables. If we cannot get the initial basic matrix $(B) = I_n$ (Identity matrix) then we introduce one more variable which is Artificial variables and last we solve the optimization problem by general Simplex method, Two-phase method or Big-M method according to situation.

- This is the iteration method. We construct the Simplex tables step by step.
- When all $\Delta_j = z_j - c_j \geq 0$ in the Simplex table then stop the iteration and we get the optimum values and optimal solution.

***GE* Method** The Gauss Elimination Method is a direct approach that involves transforming the initial system of simultaneous equations into an equivalent upper triangular system. Once this transformation is achieved, the desired solution can be determined through a process known as back substitution.

Steps to Solve a System of Linear Inequalities Using *GE* Method

- Transform the Linear Optimization Problem in to canonical form i.e., if the objective function is maximization then all inequalities sing must be ' $\leq$ ' types and if objective function is minimization then all inequalities sing must be ' $\geq$ ' types.
- Convert all inequalities in to equalities then system of n linear equations in n variables $x_1, x_2, x_3, \ldots, x_n$ as a matrix equation $AX = B$, where $A = [a_{ij}]$ is $n \times n$ matrix of real coefficients, $X = [x_i]$ is $n \times 1$ matrix of variables and $B = [b_i]$ is $n \times 1$ matrix of constants.
- Construct augmented matrix $[A \mid B]$ and convert A matrix into upper triangular matrix. i.e., $[U \mid B']$ where U is the upper triangular matrix and B' is the transform form of B.
- Now solve $UX = B'$, we will get the solution by back substitution.

Numerical Illustration Maximize $Z = 3x_1 + 5x_2 + 4x_3$.
Subject to the constraints:

$$
\begin{aligned}
&2x_1 + 3x_2 \leq 8,\\
&2x_2 + 5x_3 \leq 10,\\
&3x_1 + 2x_2 + 4x_3 \leq 15, x_1 \geq 0, x_2 \geq 0 \text{ and } x_3 \geq 0.
\end{aligned}
$$

Solution

(a) By *GE* method The above system of inequalities can be written as,

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 5 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 15 \end{bmatrix}. \tag{3.1}$$

The augmented matrix is

$$\begin{bmatrix} 2 & 3 & 0 & \vdots & 8 \\ 0 & 2 & 5 & \vdots & 10 \\ 3 & 2 & 4 & \vdots & 15 \end{bmatrix}.$$

By elementary row transformation, we convert the coefficient matrix into upper triangular matrix

$$\begin{bmatrix} 2 & 3 & 0 & \vdots & 8 \\ 0 & 2 & 5 & \vdots & 10 \\ 0 & 0 & -\frac{41}{5} & \vdots & -\frac{62}{5} \end{bmatrix}.$$

Also (3.1) can be written as

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -\frac{41}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ -\frac{62}{5} \end{bmatrix}.$$

Now by back substitution method, we get the solution
$x_1 = 89/41, x_2 = 50/41$ and $x_3 = 62/41$ and Max. $Z = 765/41$.

(b) By Simplex Method We solve this *LOP* by Simplex method then we get the optimal solution in **4**$^{\text{th}}$ iterations, $x_1 = 89/41, x_2 = 50/41$ and $x_3 = 62/41$, Maximum of $Z = 765/41$ and obtain the same result from both methods.

***LU* Decomposition** In this method, the objective function is considered as a constraint which together with the given inequalities forms a system of linear inequalities [1].
Let us consider the LOP
Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_{n-1}x_{n-1}$.
Subject to constraints:
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1,n-1}x_{n-1} \leq b_1$,
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2,n-1}x_{n-1} \leq b_2$,
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3,n-1}x_{n-1} \leq b_3$,
....,
$a_{n-1,1}x_1 + a_{n-1,2}x_2 + a_{n-1,3}x_3 + \ldots + a_{n-1,n-1}x_{n-1} \leq b_{n-1}$,
and $x_1, x_2, x_3, \ldots, x_{n-1} \geq 0$.
To find: Z
Subject to:
$-c_1x_1 - c_2x_2 - c_3x_3 - \ldots - c_{n-1}x_{n-1} + Z \leq 0$,
$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1,n-1}x_{n-1} \leq b_1$,
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2,n-1}x_{n-1} \leq b_2$,
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3,n-1}x_{n-1} \leq b_3$,
$a_{n-1,1}x_1 + a_{n-1,2}x_2 + a_{n-1,3}x_3 + \ldots + a_{n-1,n-1}x_{n-1} \leq b_{n-1}$,
$-x_1, -x_2, -x_3, \ldots, -x_{n-1}, -Z \leq 0$.
The system of linear equations is $AX = \boldsymbol{b}$, where

$$A = \begin{bmatrix} -c_1 & -c_2 & -c_3 & & -c_{1,n-1} & 1 \\ a_{11} & a_{12} & a_{13} & \cdots & a_{2,n-1} & 0 \\ & \vdots & & \ddots & \vdots & \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} & 0 \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ Z \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Here objective function is considered as a constraint and Z is considered as a variable.

WhenLUdecomposition method is applied to solve any LOP, four types of cases may arise.

Case I. When the given LOP has the number of inequalities equal to the number of variables then LU decomposition method is applied to the linear system of equations $AX = \boldsymbol{b}$ as follows:

First we consider the objective function as a constraint which together with inequalities forms a system of linear inequalities. The above inequalities are written in matrix form. Let A be the coefficient matrix, $\boldsymbol{b}$ be the constant matrix and X be the variable matrix.
Now $AX = \boldsymbol{b} \Rightarrow LUX = \boldsymbol{b}$.
Putting $UX = Y$ in (i) we get $LY = \boldsymbol{b}$. From this we can obtain the value of Y and put it in $UX = Y$ we get the value of X.

Example Maximize $Z = 5x_1 + 3x_2$.
Subject to constraint:
$3x_1 + 5x_2 \leq 15$,
$5x_1 + 2x_2 \leq 10$,
and $x_1, x_2 \geq 0$.

Solution (a) By *LU* decomposition method

We write the above LOP as follows.

$$\begin{aligned} &-5x_1 - 3x_2 + Z \leq 0, \\ &3x_1 + 5x_2 \leq 15, \\ &5x_1 + 2x_2 \leq 10 \text{ and } -x_1, -x_2, -Z \leq 0. \end{aligned}$$

Now the above inequalities can be written in matrix form, $AX = \boldsymbol{b}$.

i.e., $\begin{bmatrix} -5 & -3 & 1 \\ 3 & 5 & 0 \\ 5 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 10 \end{bmatrix}$.

Here $A = \begin{bmatrix} -5 & -3 & 1 \\ 3 & 5 & 0 \\ 5 & 2 & 0 \end{bmatrix}$.

Let $A = LU$

Then we can also write

$LU = A$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -5 & -3 & 1 \\ 3 & 5 & 0 \\ 5 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{31}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} -5 & -3 & 1 \\ 3 & 5 & 0 \\ 5 & 2 & 0 \end{bmatrix}.$$

Now comparing both matrices, we get

- $u_{11} = -5, u_{12} = -3, u_{13} = 1;$
- $l_{21}u_{11} = 3 \Rightarrow l_{21} = -3/5;$
  $l_{21}u_{12} + u_{22} = 5 \quad \Rightarrow u_{22} = 16/5;$
  $l_{21}u_{13} + u_{23} = 0 \quad \Rightarrow u_{23} = 3/5$
- $l_{31}u_{11} = 5 \quad \Rightarrow l_{31} = -1;$
  $l_{31}u_{12} + l_{32}u_{22} = 2 \quad \Rightarrow l_{32} = -5/16;$
  $l_{31}u_{13} + l_{31}u_{23} + u_{33} = 0 \quad \Rightarrow u_{33} = 19/16.$
  Hence $LUX = \boldsymbol{b}$.
  Let $UX = Y$, then from $LY = \boldsymbol{b}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -1 & -5/16 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 10 \end{bmatrix}.$$

By simplifying, we get

$$y_1 = 0, y_2 = 15, y_3 = 235/16.$$

Finally, we write $UX = Y$ i.e.,

$$\begin{bmatrix} -5 & -3 & 1 \\ 0 & 16/5 & 3/5 \\ 0 & 0 & 19/16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 235/16 \end{bmatrix}.$$

By back substitution method we get $Z = 235/19, x_2 = 45/19$ and $x_1 = 20/19$.

(b) By Simplex Method We solve this *LOP* by Simplex method then we get the optimal solution in $\mathbf{3}^{\text{rd}}$ iteration, $x_1 = 20/19$ and $x_2 = 45/19$, Maximum of $Z = 235/19$. We obtain the same result from both methods.

Case II. If *LOP* has the number of inequalities less than the number of variables then we add inequalities in the system, till the number of inequalities equals the number of variables. We can add the inequalities in the system as below:

Consider the first constraint in given *LOP*

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1,n-1}x_{n-1} \leq b_1.$$

Then, choose any non-zero coefficient in this inequality $a_{1j} \neq 0$ and add the inequality $a_{1j}x_j \leq b_1$ in the system. Continuing in this way till the number of inequalities reaches the number of variables. Now the reduced *LOP* can be solved as like Case I.

Example Maximize $Z = 2x_1 + 3x_2$.

Subject to constraint:

$x_1 + x_2 \leq 1$ and $x_1, x_2 \geq 0$.

Solution

(a) By LU decomposition method We write the above LOP as follows:

$$\begin{aligned} -2x_1 - 3x_2 + Z &\leq 0, \\ x_1 + x_2 &\leq 1, \end{aligned}.$$

Since one constraint is present, so we add one more constraint

$$x_2 \leq 1 \text{ and } -x_1, -x_2, -Z \leq 0.$$

Now the above inequalities can be written in matrix form $AX = \boldsymbol{b}$, i.e.,

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Here

$$A = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let $A = LU$.

Then we can write $LU = A$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{31}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now comparing both matrices, we get

- $u_{11} = -2, u_{12} = -3, u_{13} = 1, u_{22} = -1/2, u_{23} = 1/2, u_{33} = 1$
- $l_{21} = -1/2, l_{32} = -2, l_{31} = 0.$

Hence $LUX = \boldsymbol{b}$

Let $UX = Y$, then from $LY = \boldsymbol{b}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

By simplifying, we get

$$y_1 = 0, y_2 = 1, y_3 = 3.$$

Finally we write $UX = Y$

$$\begin{bmatrix} -2 & -3 & 1 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

By back substitution method we get $Z = 3, x_2 = 1$ and $x_1 = 0$.

(b) By Simplex method When we solve it by simplex method then we get the optimal solution in 2^{nd} iteration $x_1 = 0$ and $x_2 = 1$, Maximum of $Z = 3$. We obtain the same result from both methods.

Case III. When the given LOP has the number of variables less than the number of inequalities then we introduce the slack variables in the suitable inequalities and add $+1$ on RHS of each of these inequalities. This can be easily understood by the following example.

Example Maximize Z $= 2x_1 + 3x_2$.

Subject to constraints:

$x_1 + x_2 \leq 1,$

$6x_1 + 2x_2 \leq 3,$

$2x_1 + 6x_2 \leq 3$ and $x_1, x_2 \geq 0.$

Solution (a) By LU decomposition method

We write the above LOP as follows

$$\begin{aligned}&-2x_1 - 3x_2 + Z \leq 0,\\ &x_1 + x_2 \leq 1,\\ &6x_1 + 2x_2 \leq 3,\\ &2x_1 + 6x_2 \leq 3 \text{ and } -x_1, -x_2, -Z \leq 0.\end{aligned}$$

Here number of inequalities is greater than the number of variables so, we introduce one more slack variable in the first constraint and add +1 on RHS of that inequality.

$$\begin{aligned}&\text{i.e., } x_1 + x_2 + s_1 \leq 1 + 1\\ &\Rightarrow x_1 + x_2 + s_1 \leq 2, \qquad .\\ &\text{and } -x_1, -x_2, -s_1 \leq 0.\end{aligned}$$

Now the above inequalities are written in matrix form $AX = \boldsymbol{b}$

i.e.,

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

Here

$$A = \begin{bmatrix} -2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \end{bmatrix}.$$

Let $A = LU$.

Therefore

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} & l_{21}u_{14} + u_{24} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} & l_{31}u_{14} + l_{32}u_{24} + u_{34} \\ l_{41}u_{11} & l_{41}u_{12} + l_{42}u_{22} & l_{41}u_{13} + l_{42}u_{23} + l_{43}u_{33} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 6 & 2 & 0 & 0 \\ 2 & 6 & 0 & 0 \end{bmatrix}.$$

Comparing both matrices, we get

- $u_{11} = -2, u_{12} = -3, u_{13} = 1, u_{14} = 0, u_{22} = -1/2, u_{23} = 1/2, u_{24} = 1,\ u_{33} = -4 u_{34} = -14, u_{44} = -8$
- $l_{21} = -1/2, l_{31} = -3, l_{32} = 1, l_{41} = -1, l_{42} = -6, l_{43} = -1.$

Hence $LUX = \boldsymbol{b}$,

Now let $UX = Y$, then from $LY = \boldsymbol{b}$, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ -1 & -6 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

By simplifying, we get

$$y_1 = 0, y_2 = 2, y_3 = -25, y_4 = -10.$$

Finally we write $UX = Y$ as

$$\begin{bmatrix} -2 & -3 & 1 & 0 \\ 0 & -1/2 & 1/2 & 1 \\ 0 & 0 & -4 & -14 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -25 \\ -10 \end{bmatrix}.$$

By back substitution method, we get

$$s_1 = 5/4, Z = 15/8, x_2 = 3/8 \text{ and } x_1 = 3/8.$$

(b) By Simplex method When we solve this *LOP* by simplex method, then we get the optimal solution in 3rd iteration, $x_1 = 3/8$ and $x_2 = 3/8$ and Maximum of $Z = 15/8$. Hence we obtain the same answer from both methods.

Case IV. If the upper triangular matrix has zero row, then the given *LOP* has no solution or the solution is infeasible.

Example Maximize $Z = 3x_1 + 4x_2$.
Subject to constraints:
$x_1 - x_2 \leq -1$,
$-x_1 + x_2 \leq 0$; and $x_1, x_2 \geq 0$.

Solution (a) By *LU* decomposition method

We write the above *LOP*, as follows

$$-3x_1 - 4x_2 + Z \leq 0,$$
$$x_1 - x_2 \leq -1,$$
$$-x_1 + x_2 \leq 0,$$
$$\text{and } -x_1, -x_2, -Z \leq 0.$$

Now the above inequalities are written in matrix form, $AX = \boldsymbol{b}$
i.e.,

$$\begin{bmatrix} -3 & -4 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Here

$$A = \begin{bmatrix} -3 & -4 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Let $A = LU$ where L is unit lower triangular matrix and U upper triangular matrix. Then we can also write

$$LU = A$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ u_{12} & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{31}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} -3 & -4 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}.$$

Comparing both matrices, we get

- $u_{11} = -3, u_{12} = -4, u_{13} = 1, u_{22} = -7/3, u_{23} = 1/3, u_{33} = 0,$
- $l_{21} = -1/3, l_{31} = 1/3, l_{32} = -1.$

Hence $LUX = \boldsymbol{b}$.

Here $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ 1/3 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -3 & -4 & 1 \\ 0 & -7/3 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$.

Since U has one zero row, the given *LOP* has no solution (or infeasible solution).

(b) By Simplex method When we solve this *LOP* by simplex method then we get the optimal solution in 2^{nd} iteration. Therefore, *LOP* has no solution or an infeasible solution. Also this solution is verified with *LU* decomposition method.

4 Comparisons of Numerical Analysis Techniques and Simplex Method Similarity

- Both methods are iteration method.
- Both methods give an actual solution.

Difference

- In Simplex Method takes at least two iterations but in Numerical Analysis Techniques (Gauss elimination and *LU* decomposition) take exactly two iterations.

5 Conclusions and Remarks

Numerical Analysis Techniques involve fewer calculations compared to the Simplex Method. While the Simplex Method introduces slack variables, the *GE* Method does not require the use of any additional variables. These numerical techniques not only demonstrate a higher speed of computation than the Simplex Method but also prove to be increasingly valuable, adopting a more systematic and mechanical approach.

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