ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, Vol. 55(2) (2025), 205-210

STRUCTURAL PROPERTIES OF V-ALGEBRAS

Pulak Sabhapandit[1], Manoj Kumar Sarma[2], Kulajit Pathak[3] and R. L. Prasad[4]

[1]Department of Mathematics, Biswanath College, Biswanath Chariali, Assam, India-784176.
[2]Department of Mathematics, Guwahati College, Guwahati, Assam, India-781021.
[3]Department of Mathematics, B.H. College, Howly, Assam, India-781316.
[4]Department of Mathematics, Retd. Dean and Professor, M. U., Bodh Gaya, Bihar, India-824231.
Email: pulaksabhapandit@gmail.com, mksghy3009@gmail.com, kulajitpathak79@gmail.com, drlakhaprasad@gmail.com

(Received: September 04, 2025; In format: September 25, 2025; Revised: November 07, 2025; Accepted: November 20, 2025)

DOI: https://doi.org/10.58250/jnanabha.2025.55224

Abstract

After the introduction of the concept of BCK, BCH, dual BCK and MV-algebras [2, 6], many authors have studied their properties in the last two decades. Prasad and Rani [7] introduced the concept of a U-algebra and studied some of its properties and applications. The existence of certain examples motivated us to introduce the concept of a V-algebra, which has properties different from those of U-algebras. The structures studied in [1, 2, 3, 4] are distinct from the structures of U-algebras and V-algebras[5] in several respects. Some structural properties of V-algebras are discussed in this paper.
2020 Mathematical Sciences Classification: 06F35, 03G25, 03B52
Keywords and Phrases: U-algebra, V-algebra, BCK-algebra, BCH-algebra, MV-algebra.

1 Introduction

Firstly, we mention the definition and some properties of a $U-$algebra:

Definition 1.1. *An U-algebra $(X;*,u)$ is a triplet consisting of a non-empty set X, a binary operation $*$ and an initial element u subject to the following conditions:*

(U1) $u*s=s$.
(U2) $s*s=s$.
(U3) $(s*t)*v=(s*v)*t$.
(U4) $s*(t*v)=(s*t)*(s*v)$.

for all $s,t,v\in X$.

Lemma 1.1. *The following conditions are also satisfied in a $U-$algebra:*

(U5) $s*u=s$.
(U6) $s*t=t*s$.
(U7) $(s*(s*t))*t=s*t$.
(U8) $s*t=u\Rightarrow s=u,t=u$.
(U9) $(s*t)*s=s*(s*t)=s*t$.

for all $s,t,v\in X$.

Note 1.1. The base of the above system is the following operation defined on $\{0,1\}$:

***Table* 1.1**

$*$	0	1
0	0	1
1	1	1

However, if we change the binary operation on $\{0,1\}$ as follows, then it gives rise to another system, which we call a V-algebra.

***Table* 1.2**

$*$	0	1
0	0	0
1	1	1

Definition 1.2. *A V-algebra is a system* $(X;*,u)$ *consisting of a non-empty set* X*, a binary operation* $*$ *and a fixed element* u *satisfying the following conditions:*

(V1) $u*s=u$.
(V2) $\equiv (U5)$ $s*u=s$.
(V3) $\equiv (U3)$ $(s*t)*v=(s*v)*t$.
(V4) $\equiv (U4)$ $s*(t*v)=(s*t)*(s*v)$.

for all $s,t,v\in X$.

Lemma 1.2. *Let* $(X;*,u)$ *be a V-algebra. Then for all* $s,t\in X$ *we have:*

(V5) $s*s=s$.
(V6) $s=s*(s*t)$.
(V7) $(s*(s*t))*t=s*t$.
(V8) $s*(t*s)=(s*t)*s$.

Proof. **(V5):** Putting $t=v=u$ in (V4), we get
$s*(u*u)=(s*u)*(s*u)$
$\implies s*u=s*s$
$\implies s=s*s$.
(V6): Putting $t=u$ and $v=t$ in (V4), we get
$s*(u*t)=(s*u)*(s*t)$
$\implies s=s*(s*t)$.
(V7): $(s*(s*t))*t=(s*t)*(s*t)=s*t$ by (V5).
(V8): Putting $v=s$ in (V4), we get $s*(t*s)=(s*t)*(s*s)=(s*t)*s$. □

Definition 1.3. *Let* $(X;*,u)$ *be a Valgebra and let* $E\subseteq X$*. Then* E *is called:*

(a) a pre subalgebra of X *if* $x,y\in E\implies x*y\in E$;
(b) a subalgebra of X *if* $u\in E$ *and* $x,y\in E\implies x*y\in E$;
(c) an extended subalgebra of X *if there exists* $v\in E$ $(v\neq u)$ *such that* $v*x=v$ *and* $x*v=x$ *for all* $x\in E$ *and* $x,y\in E\implies x*y\in E$.

2 Some structural properties

Theorem 2.1. *Let* X *be an integral domain with* $|X|\geq 3$*, zero element* 0 *and unit element* 1*. Then* $(X;*,0)$ *is a V-algebra if and only if*
$$x*y=1\cdot x+0\cdot y.$$

Proof. Assume that $(X,*,0)$ is a V-algebra and $x*y$ is expressible in quadratic form as
$$x*y=Ax^2+Bxy+Cy^2+Dx+Ey+F, \tag{2.1}$$
where A,B,C,D,E,F are constants in X. Then condition (V5) implies
$$x=x*x=Ax^2+Bx^2+Cx^2+Dx+Ex+F$$
$$\Rightarrow\ A+B+C=0,\quad D+E=1,\quad F=0. \tag{2.2}$$

Again, condition $(V1)$ implies
$$0=0*x=Cx^2+Ex+F$$
$$\Rightarrow\ C=0,\quad E=0,\quad F=0. \tag{2.3}$$

From condition $(V2)$, we get

$$x = x * 0 = Ax^2 + Dx + F$$
$$\Rightarrow\ A = 0, \quad D = 1, \quad F = 0. \tag{2.4}$$

From (2.2), (2.3), and (2.4), we get

$$B = 0, \quad D = 1, \quad E = 0, \quad F = 0.$$

Putting these values in (2.1), we have

$$x * y = 1 \cdot x + 0 \cdot y = x.$$

On the other hand, if $x * y = x$ in X and 0 is an initial element, then $(X, *, 0)$ is a V-algebra. □

Corollary 2.1. *Every finite set $E = \{a_0, a_1, \ldots, a_n\}$ is a V-algebra with initial element a_0 and binary operation $*$ defined as $a_i * a_j = a_i$, for all $a_i, a_j \in E$.*

Theorem 2.2. *The Cartesian product of two V-algebras is a V-algebra under a suitable binary operation.*

Proof. Let $(X; *, u)$ and $(Y; \circ, v)$ be two V-algebras and let

$$Z = X \times Y.$$

For $(s_1, t_1), (s_2, t_2) \in Z$, we define

$$(s_1, t_1) \otimes (s_2, t_2) = \big(s_1 * s_2,\ t_1 \circ t_2\big). \tag{2.5}$$

Taking (u, v) as the initial element of Z, we see that

$$(Z; \otimes, (u, v))$$

is a V-algebra. □

Now we discuss structures of V-algebras on some finite sets.

Example 2.1. Let X contain only one element u. Then the binary operation $*$ is defined only as

$$u * u = u,$$

and $(X; *, u)$ is a V-algebra.

Let $X = \{u, s\}$. In view of conditions (V1), (V2), and (V5), a binary operation table is of the form

***Table* 2.1**

$*$	u	s
u	u	u
s	s	s

Then $(X; *, u)$ is a V-algebra.

Let $X = \{u, s, t\}$ be a set and we take u as an initial element. In order to make X a V-algebra a binary operation $*$ may be defined on X in the form as shown in the *Table 2.2 where p and q are to be determined.*

***Table* 2.2**

$*$	u	s	t
u	u	u	u
s	s	s	p
t	t	q	t

Also,

$$s = s * (s * t)$$

implies that either $s * t = u$ or s or t, where $s * t = s$.

Now, $s * t = t \implies s * t = s$. *Hence* $s * t = u$ *or* s, *i.e.* $p = u$ *or* s.

Similarly, $q = u$ *or* t.

So we have four possibilities:

$$(p, q) = (u, u),\ (u, t),\ (s, u),\ (s, t).$$

Simple computations show that in all the four cases, $(X; *, u)$ *becomes a* V*-algebra.*

Remark 2.1. *Every set* X *with* $|X| \leq 3$ *is a* V*-algebra under suitable binary operations.*

Example 2.2. Now we consider the case in which $|X| = 4$. Let $X = \{u, s, t, v\}$. In view of conditions (V1), (V2), and (V5), we consider two binary operations $*$ and $\circ$ given by the following tables.

***Table* 2.3**

$*$	u	s	t	v
u	u	u	u	u
s	s	s	u	t
t	t	u	t	v
v	v	t	v	u

***Table* 2.4**

$\circ$	u	s	t	v
u	u	u	u	u
s	s	s	u	t
t	t	u	t	v
v	v	t	v	v

From Table 2.3,

$$v * (s * t) = v * u = v \quad \text{and} \quad (v * s) * (v * t) = t * v = u,$$

imply that $(V4)$ is not satisfied for the binary operation $*$. This means that $(X; *, u)$ is not a V-algebra.

However, $(X; \circ, u)$ is a V-algebra.

Example 2.3. We consider a system where $X = \{u, a, b\}$, u being the initial element, and the binary operation $*$ is given by

***Table* 2.5**

$*$	u	a	b
u	u	u	u
a	a	a	u
b	b	u	b

Then $(X; *, u)$ is a V-algebra (Example 2.1).

Let $Y = X \times X = \{e, l, m, n, p, q, r, s, t\}$ where

$$e = (u, u), \quad l = (u, a), \quad m = (u, b), \quad n = (a, u), \quad p = (a, a), \quad q = (a, b),$$
$$r = (b, u), \quad s = (b, a), \quad t = (b, b).$$

Let a binary operation $\circ$ be extended in Y from the binary operation $*$ given by Table 2.5. The binary operation table for $\circ$ is as follows:

***Table* 2.6**

$\circ$	e	l	m	n	p	q	r	s	t
e	e	e	e	e	e	e	e	e	e
l	l	l	l	l	l	l	l	l	l
m	m	e	m	m	e	m	m	e	m
n	n	n	n	n	n	n	n	n	n
p	p	p	p	p	p	p	p	p	p
q	q	n	q	q	n	q	q	n	q
r	r	r	r	e	e	e	r	r	r
s	s	s	s	l	l	l	s	s	s
t	t	r	t	m	e	m	t	r	t

Then $(Y; \circ, e)$ is a V-algebra (by Theorem 2.2).

3 Homomorphisms and Multipliers

Let $(X; *, u)$ be a Valgebra. Let ϕ and ψ be functions of X into itself.

Definition 3.1. *(a) ϕ is said to be a* homomorphism *if* $\phi(s * t) = \phi(s) * \phi(t), \ s, t \in X.$ (3.1)
(b) ψ is called a multiplier *if* $\psi(s * t) = \psi(s) * t, \ s, t \in X.$ (3.2)

Example 3.1. Let a be a fixed element of X. The function ϕ_a, defined as $\phi_a(s) = a * s, \ s \in X.$ (3.3)
is a homomorphism. This follows from condition (V4).
Again, if ψ_a is defined as $\psi_a(s) = s * a, \ s \in X,$
then condition $(V3)$ implies that ψ_a is a multiplier on X.

Remark 3.1. *Since $\phi_a(u) = a * u = a$, a homomorphism may not map the initial element to itself.*

Definition 3.2. *The* kernel *of a function ϕ, denoted as $K(\phi)$, is defined as*

$$K(\phi) = \{x \in X \ ; \ \phi(x) = u\}. \tag{3.4}$$

Lemma 3.1. *The kernels of a homomorphism and a multiplier are pre-subalgebras of X.*

Proof. Let $K(\phi)$ and $K(\psi)$ be their respective kernels.
For $s, t \in K(\phi)$, we have $\phi(s * t) = \phi(s) * \phi(t) = u * u = u$,
and $\phi(t * s) = \phi(t) * \phi(s) = u * u = u$.
So $K(\phi)$ is a pre-subalgebra of X.
Again, for $s, t \in K(\psi)$, we have $\psi(s * t) = \psi(s) * t = u * t = u$,
and $\psi(t * s) = \psi(t) * s = u * s = u$.
So $K(\psi)$ is also a pre-subalgebra of X. □

Note 3.1. In general $K(\phi)$ (or $K(\psi)$) may be a sub-algebra (or an extended sub-algebra).

Example 3.2. We recall Table 2.6 and observe the following:

$$K(\phi_m) = \{l, p, s\}, \ K(\phi_r) = \{n, p, q\}, \ K(\phi_t) = \{p\}, \ K(\psi_p) = \{m, r, t\}.$$

All these sets are extended sub-algebras of X.

Lemma 3.2. *Every constant function is a homomorphism but not a multiplier.*

Proof. Let c be a fixed element of X and let $f(s) = c$ for all $s \in X$. Then $f(s * t) = c = c * c = f(s) * f(t)$, which implies that a constant function is a homomorphism.

For the other part, we recall Table 2.5 and let $f(s) = b$ for all $s \in X$. Then $f(b * a) = f(u) = b$ and $f(b) * a = b * a = u \ \Rightarrow \ f$ is not a multiplier. □

4 Conclusion

In this paper, the notion of V-algebra is introduced and illustrated in Examples 2.1, 2.2, and 2.3. The concepts of pre-subalgebra, subalgebra, and extended subalgebra are introduced in Definition 1.3. Some properties of V-algebra are discussed in Lemma 1.2, Theorem 2.1, and Theorem 2.2.

Two interesting mappings are developed through this work, namely the *homomorphism* and the *multiplier map*. Examples of both these maps are illustrated in Example 3.1. Furthermore, the kernel of a homomorphism is defined in Definition 3.2, and a property of the kernel is discussed in Lemma 3.1. Finally, a property of constant functions regarding homomorphisms and multiplier maps is discussed in Lemma 3.2.

References

[1] R. Ahmed and R. Rani, Some algebraic structure on poset ad loset, *Turkish Journal of Computer and Mathematics Educations,* **12**(11) (2021), 3877-3880.

[2] Q. P. Hu and X. Li, On BCH-algebras, *Math. Seminar Notes,* **11**(2) (1983), 313-320.

[3] Y. Imai and K. Iseki, On axiom systems of propositional calculi, XIV, *Proc. Japan Acad.,* **42**(1) (1966), 19-22.

[4] K. Iseki, An algebra related with a propositional calculus, *Proc. Japan Acad.,* **42**(1) (1966), 26-29.

[5] K. Pathak, M. Sarma, P. Sadhapandit and R. L. Prasad, Some specific weak V-algebras, *Nonlinear Studies (NS),* **30**(2) (2023), 445-449.

[6] K. Pathak, P. Sadhapandit and H. S. Mondal, On multipliers and reversal maps on BCK/BCI- algebras, *Jñānābha,* **54**(1) (2024), 224-229.

[7] R. L. Prasad and R. Rani, U-algebra: A study and applications, *Design Engineering,* **8** (2021), 276-285.