ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, **Vol. 55(2) (2025), 196-204**

GENERALIZED-TWO-VARIABLE-MITTAG-LEFFLER FUNCTION AND ATANGANA-BALEANU RIEMANN-LIOUVILLE DERIVATIVES OF GENERALIZED MITTAG-LEFFLER FUNCTIONS

Madan Lal[1], Amit Punia[2,*] and Rajeev Kumar Gupta[3]
[1,2,3]Department of Mathematics and Statistics, Jai Narain Vyas University,
Jodhpur, Rajasthan, India-342001
Email: madan.lakhani288@gmail.com, [*]rs_maths_amitpunia@jnvu.edu.in,drrkbgupta@yahoo.co.in
[*]corresponding author

(Received: July 23*, 2025; In format: September* 08*, 2025; Revised: November* 17*, 2025; Accepted: November* 20*, 2025)*

DOI: https://doi.org/10.58250/jnanabha.2025.55223

Abstract

This paper studies the fractional differentiation of generalized Mittag-Leffler functions using the Atangana–Baleanu Riemann–Liouville (*AB–RL*) operator. Exact series representations are derived for the *AB–RL* derivative of these functions, including some special cases. The analysis leads to a compact reformulation involving a two-variable function, termed the Generalized–Two–Variable–Mittag–Leffler (*GML*) function, which includes several classical Mittag-Leffler families. We establish its structural properties and limiting cases, and provide graphical illustrations to demonstrate the impact of key parameters such as the memory index μ. These results enrich the analytical structure of fractional calculus involving generalized special functions and highlight the relevance of the *GML* function in memory-influenced systems.

2020 Mathematical Sciences Classification: 26A33, 33E12, 34A08.
Keywords and Phrases: Fractional calculus, *AB–RL* derivative, Mittag-Leffler function, Generalized special functions, Two–Variable-Generalized–Mittag–Leffler function

1 Introduction and preliminaries

Mittag-Leffler functions appear naturally in the solution of fractional differential equations and have applications across viscoelasticity, anomalous diffusion, epidemiology, control theory and related fractional derivative studies [1, 3, 5, 6, 8, 14, 15]. Recent developments in fractional calculus have led to new derivative definitions involving non-singular and singular kernels. One such approach is the *AB–RL* type derivative with a non-singular Mittag-Leffler kernel, introduced in [2]. This operator captures non-local effects with fading memory in a smooth and physically consistent manner, making it suitable for modeling complex fractional dynamics.

In this paper, we explore the action of this operator on the generalized Mittag-Leffler function introduced by Shukla and Prajapati [13]:

$$E^{\rho,q}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} z^k}{\Gamma(\alpha k+\beta)\, k!}, \quad (\alpha,\beta,\rho \in \mathbb{C},\ q \in (0,1)\cup\mathbb{N}), \tag{1.1}$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\rho) > 0$, and $(\rho)_{qk} = \dfrac{\Gamma(\rho+qk)}{\Gamma(\rho)}$ is the generalized Pochhammer symbol. As shown in [13], this function includes several classical functions as special cases.

Exponential Function. When $\alpha = 1$, $\beta = 1$, $\rho = 1$, and $q = 1$, we recover

$$E^{1,1}_{1,1}(z) = e^z.$$

Classical Mittag-Leffler Function. For $\rho = 1$ and $q = 1$, we obtain the Mittag-Leffler Function [9],

$$E^{1,1}_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

Two-parameter Wiman Function. For $\rho = 1$, $q = 1$, and general β, we obtain the Wiman function [16]:

$$E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}.$$

Prabhakar Function. For $q = 1$ and general ρ, we obtain the Prabhakar function [12]:

$$E^{\rho,1}_{\alpha,\beta}(z) = E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{\Gamma(k\alpha + \beta)\, k!}.$$

Hence, the function $E^{\rho,q}_{\alpha,\beta}(z)$ serves as a unified framework encompassing these classical Mittag-Leffler-type functions as special cases. Consequently, the derivatives of these classical functions can be directly obtained from the derivative of the generalized function.

Definition 1.1 ([2]). *Let $\mu \in (0,1)$ and $u \in C^1(0,T]$. The Atangana–Baleanu Riemann–Liouville (AB–RL) fractional derivative of order μ of a function $u(t)$ is defined by*

$$({}^*D^{\mu}_0 u)(t) := \frac{B(\mu)}{1-\mu} \cdot \frac{d}{dt} \int_0^t u(s)\, E_{\mu}\left(-\mu_{\mu}(t-s)^{\mu}\right) ds, \tag{1.2}$$

where $E_{\mu}(\cdot)$ is the one-parameter Mittag-Leffler function, $\mu_{\mu} > 0$ is a constant, and $B(\mu)$ is a normalization function satisfying $B(0) = B(1) = 1$.

Definition 1.2. *We define the generalized Mittag-Leffler-type function of two variables associated with the AB–RL fractional derivative as*

$$\mathcal{E}^{\rho,q}_{\mu,\alpha,\beta}(x,y) := \sum_{k=0}^{\infty} \frac{(\rho)_{qk}\, x^k}{k!} \cdot E_{\mu,\alpha k+\beta}(y), \tag{1.3}$$

where $(\rho)_{qk}$ is the generalized Pochhammer symbol, and $E_{\mu,\nu}(y)$ is the two-parameter Mittag-Leffler function. The parameters $\mu, \alpha, \beta, \rho, x, y \in \mathbb{C}$, with $\Re(\mu), \Re(\alpha), \Re(\beta), \Re(\rho) > 0$, and $q \in \mathbb{N}$.

Definition 1.3. *Let $\mu, \rho, \alpha, \beta, x, y \in \mathbb{C}$ with $\Re(\mu), \Re(\rho), \Re(\alpha), \Re(\beta) > 0$, and $q \in \mathbb{N}$. We define the Generalized–Two–Variable–Mittag–Leffler (GML) function as*

$$\mathcal{GML}^{\rho,q}_{\mu,\alpha,\beta}(x,y) := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\rho)_{qk}}{k!} \cdot \frac{x^k y^m}{\Gamma(\mu m + \alpha k + \beta)}. \tag{1.4}$$

This function arises naturally in the study of *AB–RL* fractional derivatives applied to the generalized Mittag-Leffler functions.

2 Main Results

In this section, we derive the *AB–RL* fractional derivative of the Shukla function (1.1), which is an entire function [4], along with its special cases. While classical fractional derivatives of generalized Mittag-Leffler functions have been studied in [10], to the best of our knowledge, the explicit action of the *AB–RL* derivative on the Shukla function has not yet been systematically studied.

Theorem 2.1. *Let $\alpha, \beta, \rho, \lambda \in \mathbb{C}$ with $\Re(\alpha), \Re(\beta), \Re(\rho) > 0$, $\mu \in (0,1)$ and $q \in \mathbb{N}$. Then the AB-RL fractional derivative of the function*

$$u(t) = t^{\beta-1} E^{\rho,q}_{\alpha,\beta}(\lambda t^{\alpha})$$

is given by the following double series representation:

$$({}^*D^{\mu}_0 u)(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\rho)_{qk} \lambda^k (-\mu_{\mu})^m}{k!\, \Gamma(\mu m + \alpha k + \beta)}\, t^{\mu m + \alpha k + \beta - 1}. \tag{2.1}$$

Proof. We start by expanding the generalized Mittag-Leffler function as

$$E^{\rho,q}_{\alpha,\beta}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} \lambda^k t^{\alpha k}}{\Gamma(\alpha k + \beta)\, k!}.$$

Then the function $u(t)$ becomes

$$u(t) = t^{\beta-1}E^{\rho,q}_{\alpha,\beta}(\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(\rho)_{qk}\lambda^k}{\Gamma(\alpha k+\beta)\,k!} \cdot t^{\alpha k+\beta-1}.$$

We now apply the *AB-RL* fractional derivative term-by-term:

$$({}^*D^\mu_0 u)(t) = \sum_{k=0}^{\infty} \frac{(\rho)_{qk}\lambda^k}{\Gamma(\alpha k+\beta)\,k!} \cdot ({}^*D^\mu_0 t^{\alpha k+\beta-1}).$$

It follows from direct calculation that:

$$({}^*D^\mu_0 t^\nu) = \frac{B(\mu)}{1-\mu}\sum_{m=0}^{\infty} \frac{(-\mu_\mu)^m\,\Gamma(\nu+1)}{\Gamma(\mu m+\nu+1)}\,t^{\mu m+\nu},$$

where $\nu = \alpha k+\beta-1$. Hence:

$$({}^*D^\mu_0 t^{\alpha k+\beta-1}) = \frac{B(\mu)}{1-\mu}\sum_{m=0}^{\infty} \frac{(-\mu_\mu)^m\Gamma(\alpha k+\beta)}{\Gamma(\mu m+\alpha k+\beta)}t^{\mu m+\alpha k+\beta-1}.$$

Substituting back:

$$({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\rho)_{qk}\lambda^k(-\mu_\mu)^m}{k!\,\Gamma(\mu m+\alpha k+\beta)}t^{\mu m+\alpha k+\beta-1}.$$

□

Corollary 2.1. *Let $\alpha,\beta,\rho,\lambda \in \mathbb{C}$ with $\Re(\alpha),\Re(\beta),\Re(\rho) > 0$, $\mu \in (0,1)$, and $q \in \mathbb{N}$. Then the Atangana–Baleanu Riemann–Liouville (AB-RL) fractional derivative of the generalized Mittag-Leffler function*

$$u(t) = t^{\beta-1}E^{\rho,q}_{\alpha,\beta}(\lambda t^\alpha)$$

can be expressed as:

$$({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot \mathcal{E}^{\rho,q}_{\mu,\alpha,\beta}(\lambda t^\alpha, -\mu_\mu t^\mu), \tag{2.2}$$

Proof. We begin from Theorem 2.1, which yields the double-series representation:

$$({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\rho)_{qk}\lambda^k(-\mu_\mu)^m}{k!\,\Gamma(\mu m+\alpha k+\beta)} \cdot t^{\mu m+\alpha k+\beta-1}.$$

Rewriting the inner summation as a Mittag-Leffler function:

$$E_{\mu,\alpha k+\beta}(-\mu_\mu t^\mu) = \sum_{m=0}^{\infty} \frac{(-\mu_\mu t^\mu)^m}{\Gamma(\mu m+\alpha k+\beta)},$$

we regroup terms:

$$({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu}\sum_{k=0}^{\infty} \frac{(\rho)_{qk}\lambda^k}{k!} \cdot t^{\alpha k+\beta-1} \cdot E_{\mu,\alpha k+\beta}(-\mu_\mu t^\mu).$$

Finally, factoring out $t^{\beta-1}$, we obtain:

$$({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot \sum_{k=0}^{\infty} \frac{(\rho)_{qk}(\lambda t^\alpha)^k}{k!} \cdot E_{\mu,\alpha k+\beta}(-\mu_\mu t^\mu),$$

which matches the definition of $\mathcal{E}^{\rho,q}_{\mu,\alpha,\beta}(x,y)$. □

Remark 2.1. *The result of Theorem* 2.1 *also generalizes the case of a polynomial multiplier. For any $\gamma \in \mathbb{C}$, one can write:*

$$u(t) = t^\gamma \cdot E^{\rho,q}_{\alpha,\beta}(\lambda t^\alpha) \quad \Rightarrow \quad ({}^*D^\mu_0 u)(t) = \frac{B(\mu)}{1-\mu}\sum_{k=0}^{\infty} \frac{(\rho)_{qk}\lambda^k}{k!}t^{\alpha k+\gamma} \cdot E_{\mu,\alpha k+\gamma+1}(-\mu_\mu t^\mu).$$

In particular, setting $\gamma = \beta - 1$ recovers the main result.

Corollary 2.2. *Let $\alpha, \beta, \rho, \lambda \in \mathbb{C}$ with $\Re(\alpha), \Re(\beta), \Re(\rho) > 0$, $\mu \in (0,1)$, and $q \in \mathbb{N}$. Then the Atangana–Baleanu Riemann–Liouville (AB-RL) fractional derivative of the generalized Mittag-Leffler function*

$$u(t) = t^{\beta-1} E^{\rho,q}_{\alpha,\beta}(\lambda t^{\alpha})$$

can also be represented using the Generalized–Two–Variable–Mittag–Leffler (GML) function as:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot \mathcal{GML}^{\rho,q}_{\mu,\alpha,\beta}(\lambda t^{\alpha}, -\mu_{\mu} t^{\mu}), \tag{2.3}$$

Proof. From Theorem 2.1, we have the double-series expression:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\rho)_{qk} \lambda^{k} (-\mu_{\mu})^{m}}{k!\,\Gamma(\mu m + \alpha k + \beta)} \cdot t^{\mu m + \alpha k + \beta - 1}.$$

This can be factored as:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\rho)_{qk}}{k!} \cdot \frac{(\lambda t^{\alpha})^{k} (-\mu_{\mu} t^{\mu})^{m}}{\Gamma(\mu m + \alpha k + \beta)},$$

which is exactly the definition of the *GML* function. □

Corollary 2.3. *Let $\mu, \alpha, \rho, \lambda \in \mathbb{C}$ with $\Re(\mu), \Re(\alpha), \Re(\rho) > 0$, and $q \in \mathbb{N}$. Consider the generalized Mittag-Leffler function:*

$$u(t) = E^{\rho,q}_{\alpha,1}(\lambda t^{\alpha}).$$

Then its Atangana–Baleanu RL-type fractional derivative is given by:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \cdot \mathcal{GML}^{\rho,q}_{\mu,\alpha,1}(\lambda t^{\alpha}, -\mu_{\mu} t^{\mu}), \tag{2.4}$$

i.e., a direct expression in terms of the GML function with $\beta = 1$.

Remark 2.2. *Corollary 2.3 shows that setting $\beta = 1$ in the general AB-RL derivative of $u(t) = t^{\beta-1} E^{\rho,q}_{\alpha,\beta}(\lambda t^{\alpha})$ directly yields the AB-RL derivative of the generalized Mittag-Leffler function $E^{\rho,q}_{\alpha,1}$, which appears naturally in the GML framework with the parameter $\beta = 1$.*

Remark 2.3. *The GML function provides an alternative representation to the compact form involving the two-variable Mittag-Leffler-type function $\mathcal{E}^{\rho,q}_{\mu,\alpha,\beta}$ given in Corollary 2.1. Both representations are mathematically equivalent:*

$$\mathcal{GML}^{\rho,q}_{\mu,\alpha,\beta}(x,y) = \sum_{k=0}^{\infty} \frac{(\rho)_{qk} x^{k}}{k!} \cdot \sum_{m=0}^{\infty} \frac{y^{m}}{\Gamma(\mu m + \alpha k + \beta)} = \mathcal{E}^{\rho,q}_{\mu,\alpha,\beta}(x,y),$$

Remark 2.4. *An important advantage of expressing the AB–RL derivative in terms of the GML function is that it enables compact analytical manipulation. For instance, the difference between two AB–RL derivatives of distinct orders μ_1 and μ_2 applied to the same function can be written as:*

$$({}^{*}D^{\mu_1}_{0} u)(t) - ({}^{*}D^{\mu_2}_{0} u)(t) = t^{\beta-1} \left[\frac{B(\mu_1)}{1-\mu_1} \cdot \mathcal{GML}^{\rho,q}_{\mu_1,\alpha,\beta}(x, y_1) - \frac{B(\mu_2)}{1-\mu_2} \cdot \mathcal{GML}^{\rho,q}_{\mu_2,\alpha,\beta}(x, y_2) \right],$$

where $x = \lambda t^{\alpha}$, $y_1 = -\mu_{\mu_1} t^{\mu_1}$, and $y_2 = -\mu_{\mu_2} t^{\mu_2}$. This expression shows the structural advantage of the GML formulation in representing operator differences in a unified analytic framework.

Theorem 2.2. *If $\rho = 1$ and $q = 1$, then the generalized Mittag-Leffler function reduces to the two-parameter Mittag-Leffler function $E_{\alpha,\beta}$, and:*

$$({}^{*}D^{\mu}_{0}[t^{\beta-1} E_{\alpha,\beta}(\lambda t^{\alpha})])(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \lambda^{k} t^{\alpha k + \beta - 1} E_{\mu,\alpha k + \beta}(-\mu_{\mu} t^{\mu}). \tag{2.5}$$

Proof. We begin from Corollary 2.1, which states:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \frac{(\rho)_{qk} \lambda^{k}}{k!} \cdot t^{\alpha k + \beta - 1} \cdot E_{\mu,\alpha k + \beta}(-\mu_{\mu} t^{\mu}).$$

Now substitute $\rho = 1, q = 1$, so that:

$$(\rho)_{qk} = (1)_{k} = k!,$$

which cancels the denominator. Hence:

$$({}^{*}D^{\mu}_{0} u)(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \lambda^{k} t^{\alpha k + \beta - 1} E_{\mu,\alpha k + \beta}(-\mu_{\mu} t^{\mu}),$$

□

Theorem 2.3. *If $\lambda = 0$, then $u(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}$, and:*

$$({}^*D_0^\mu u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot E_{\mu,\beta}(-\mu_\mu t^\mu). \tag{2.6}$$

Proof. Set $\lambda = 0$ in the generalized Mittag-Leffler function:

$$E_{\alpha,\beta}^{\rho,q}(0) = \frac{1}{\Gamma(\beta)} \quad \Rightarrow \quad u(t) = t^{\beta-1} \cdot \frac{1}{\Gamma(\beta)}.$$

Now, from Corollary 2.1 the *AB-RL* derivative is:

$$({}^*D_0^\mu u)(t) = \frac{B(\mu)}{1-\mu} \sum_{k=0}^{\infty} \frac{(\rho)_{qk}\lambda^k}{k!} \cdot t^{\alpha k+\beta-1} \cdot E_{\mu,\alpha k+\beta}(-\mu_\mu t^\mu).$$

Since $\lambda = 0$, all terms with $k \geq 1$ vanish, and only the $k = 0$ term remains:

$$({}^*D_0^\mu u)(t) = \frac{B(\mu)}{1-\mu} \cdot \frac{(\rho)_0}{0!} \cdot t^{\beta-1} \cdot E_{\mu,\beta}(-\mu_\mu t^\mu).$$

Since $(\rho)_0 = 1$, we get:

$$({}^*D_0^\mu u)(t) = \frac{B(\mu)}{1-\mu} \cdot t^{\beta-1} \cdot E_{\mu,\beta}(-\mu_\mu t^\mu).$$

□

3 Special Cases and Properties of the Generalized-Two-Variable-Mittag-Leffler Function

The Generalized–Two–Variable–Mittag–Leffler function

$$\mathcal{GML}_{\mu,\alpha,\beta}^{\rho,q}(x,y) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\rho)_{qk}}{k!} \cdot \frac{x^k y^m}{\Gamma(\mu m + \alpha k + \beta)}$$

encompasses several well-known special functions under specific parameter settings. Table 3.1 summarizes key special cases and their corresponding classical counterparts.

Table 3.1: *Special cases of the Generalized–Two–Variable–Mittag–Leffler function $\mathcal{GML}_{\mu,\alpha,\beta}^{\rho,q}(x,y)$*

Conditions	**Reduced Form**	**Function Name**
$\mu = 0,\ \lvert y\rvert < 1$	$\frac{1}{1-y}\sum_{k=0}^{\infty}\frac{(\rho)_{qk}x^k}{\Gamma(\alpha k+\beta)k!}$	Scaled Generalized ML $\frac{1}{1-y}E_{\alpha,\beta}^{\rho,q}(x)$
$\mu = 0,\ q = 1,\ \lvert y\rvert < 1$	$\frac{1}{1-y}\sum_{k=0}^{\infty}\frac{(\rho)_{k}x^k}{\Gamma(\alpha k+\beta)k!}$	Scaled Prabhakar function $E_{\alpha,\beta}^{\rho}(x)$
$\mu = 0,\ \rho = 1,\ q = 1,\ \lvert y\rvert < 1$	$\frac{1}{1-y}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}$	Scaled two-parameter ML function $E_{\alpha,\beta}(x)$
$\mu = 0,\ \rho = 1,\ q = 1,\ \beta = 1,\ \lvert y\rvert < 1$	$\frac{1}{1-y}\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}$	Scaled standard ML function $E_\alpha(x)$
$\rho = 1,\ q = 0$	$\sum_{k=0}^{\infty}\frac{x^k}{k!}E_{\mu,\alpha k+\beta}(y)$	Mittag-Leffler generating form
$x = 0$	$E_{\mu,\beta}(y)$	Two-parameter ML function

Remark 3.1. *The presence of the doubly-indexed gamma term in the denominator, $\Gamma(\mu m + \alpha k + \beta)$, makes the GML function structurally different from standard hypergeometric, Fox–Wright, Srivastava–Daoust, or Lauricella-type families in their usual forms. In the limiting case $\mu = 0$, the GML function reduces to a scaled generalized Mittag-Leffler function as:*

$$\mathcal{GML}_{0,\alpha,\beta}^{\rho,q}(x,y) = \frac{1}{1-y} \cdot E_{\alpha,\beta}^{\rho,q}(x), \quad \textit{for } |y| < 1,$$

demonstrating the internal consistency and usefulness of the GML framework in unifying these cases.

Property 3.1. *If $\alpha = \mu$, then the GML function*

$$\mathcal{GML}^{\rho,q}_{\mu,\mu,\beta}(x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\rho)_{qk}}{k!} \cdot \frac{x^k y^m}{\Gamma(\mu(k+m)+\beta)}$$

becomes symmetric in x and y under the common fractional order μ.

Property 3.2. *When $\mu = 1$, $\alpha = 1$, and $\beta = 2$, the GML function reduces to*

$$\mathcal{GML}^{1,1}_{1,1,2}(x,y) = \int_0^1 e^{x(1-s)} e^{ys}\, ds. \tag{3.1}$$

Proof. We begin by considering the *GML* function at the given values of parameters:

$$\mathcal{GML}^{1,1}_{1,1,2}(x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^k y^m}{\Gamma(k+m+2)}.$$

We aim to show that:

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^k y^m}{\Gamma(k+m+2)} = \int_0^1 e^{x(1-s)} e^{ys}\, ds.$$

Now expand both exponentials as power series:

$$e^{x(1-s)} = \sum_{k=0}^{\infty} \frac{x^k(1-s)^k}{k!}, \quad e^{ys} = \sum_{m=0}^{\infty} \frac{y^m s^m}{m!}.$$

Then multiply and integrate term-by-term:

$$\int_0^1 e^{x(1-s)} e^{ys}\, ds = \int_0^1 \left(\sum_{k=0}^{\infty} \frac{x^k(1-s)^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{y^m s^m}{m!}\right) ds.$$

Interchanging summation and integration:

$$\begin{aligned}\int_0^1 e^{x(1-s)} e^{ys}\, ds &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^k y^m}{k!m!} \cdot \int_0^1 (1-s)^k s^m\, ds \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^k y^m}{k!m!} \cdot \frac{\Gamma(m+1)\Gamma(k+1)}{\Gamma(k+m+2)} \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{x^k y^m}{\Gamma(k+m+2)}. \end{aligned} \tag{3.2}$$

which is exactly the form of $\mathcal{GML}^{1,1}_{1,1,2}(x,y)$. □

Before proceeding to the graphical representation, we present a few explicit examples illustrating how suitable specialisations of the *GML* function relate to the two-variable Mittag–Leffler forms studied by Pathan and Kumar [11].

3.1 Relations with Two-Variable Mittag–Leffler Structures

Pathan and Kumar [11] studied the two-variable generalized Mittag–Leffler function

$$E^{(\delta)}_{\alpha,\beta;\gamma}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\delta)_{m+n}}{m!\,n!\,\Gamma(\alpha m + \beta n + \gamma)}\, x^m y^n. \tag{3.3}$$

They analysed several important special forms arising from (3.3), including Prabhakar-type, Wiman-type, and hypergeometric-type reductions [7]. Since the *GML* function presented in this work also possesses a two-variable Mittag–Leffler structure governed by a doubly-indexed gamma term, it is natural to examine how its specialisations relate to the forms discussed by Pathan and Kumar. Representative examples of these correspondences are given below.

Example 1. Prabhakar-type correspondence

Among the notable reductions examined by Pathan and Kumar is the Prabhakar-type form

$$E^{(\delta)}_{\alpha,\alpha,\gamma}(x,y) = E^{\delta}_{\alpha,\gamma}(x+y) = \sum_{n=0}^{\infty} \frac{(\delta)_n (x+y)^n}{n!\,\Gamma(\alpha n + \gamma)}. \tag{3.4}$$

For the *GML* function, the parameter choice $\mu = 0,\ q = 1,\ |y| < 1$ (see Table 3.1) yields

$$\mathcal{GML}^{\rho,1}_{0,\alpha,\beta}(x,y) = \frac{1}{1-y}\, E^{\rho}_{\alpha,\beta}(x) = \frac{1}{1-y}\, E^{(\rho)}_{\alpha,\alpha,\beta}(x,0), \qquad |y| < 1. \tag{3.5}$$

Apart from the natural scaling factor $(1-y)^{-1}$ that arises in the two-variable *GML* construction, both expressions contain the same Prabhakar-type function. Thus this specialisation of the *GML* function aligns structurally with the Prabhakar-type form described by Pathan and Kumar.

Example 2. Wiman-type correspondence
Pathan and Kumar also obtained a Wiman-type reduction by suppressing the second variable.

$$E^{(1)}_{\mu,\alpha,\beta}(x,0) = E_{\mu,\beta}(x). \tag{3.6}$$

An analogous reduction is obtained by setting $x = 0$ in the *GML* function (see Table 3.1), giving

$$\mathcal{GML}^{\rho,q}_{\mu,\alpha,\beta}(0,y) = E_{\mu,\beta}(y) = E^{(1)}_{\mu,\alpha,\beta}(y,0). \tag{3.7}$$

Thus suppressing one variable in the *GML* function produces precisely the one-variable Mittag–Leffler form that appears in the Wiman-type case studied by Pathan and Kumar.

Example 3. Hypergeometric-type correspondence
Pathan and Kumar further showed that, for $(\alpha, \beta) = (1, 1)$, their two-variable function can be written in confluent hypergeometric form

$$E^{(\delta)}_{1,1;\gamma}(x,y) = \frac{1}{\Gamma(\gamma)}\, {}_1F_1(\delta;\gamma;\, x+y)\,, \tag{3.8}$$

where ${}_1F_1$ denotes Kummer's confluent hypergeometric function [7].

A closely parallel representation arises for the *GML* function. Starting from the expression obtained earlier in equation (3.1) (corresponding to the specialisation $(\mu, \alpha, \beta, \rho, q) = (1, 1, 2, 1, 1)$), and using the standard identity

$$\int_0^1 e^{\lambda s}\, ds = {}_1F_1(1;2;\lambda),$$

one obtains

$$\mathcal{GML}^{1,1}_{1,1,2}(x,y) = e^{x}\, {}_1F_1(1;2;\, y-x)\,. \tag{3.9}$$

Hence the *GML* function also admits a confluent hypergeometric representation for appropriate parameter choices, mirroring the hypergeometric-type structure discussed by Pathan and Kumar.

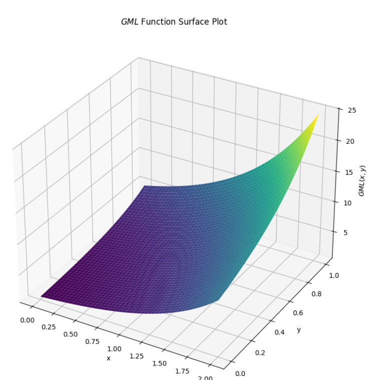
These examples show that the principal special forms discussed by Pathan and Kumar, namely the Prabhakar, Wiman and hypergeometric types, arise naturally from suitable specialisations of the *GML* function developed in this work. This indicates that the *GML* framework fits within the same general class of two variable Mittag-Leffler structures considered by Pathan and Kumar, and that the present *GML* form itself arises naturally through the action of the *AB-RL* fractional derivative.

4 Graphical Representation and Parametric Behavior

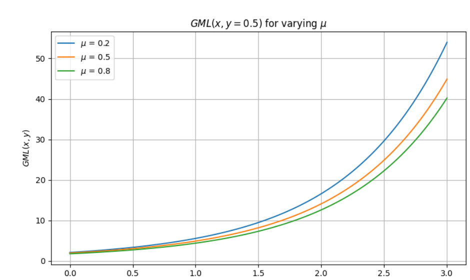
To further illustrate the analytic behavior of the *GML* function

$$\mathcal{GML}^{\rho,q}_{\mu,\alpha,\beta}(x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\rho)_{qk}}{k!} \cdot \frac{x^k y^m}{\Gamma(\mu m + \alpha k + \beta)},$$

we provide numerical visualizations that highlight its dependence on the memory parameter μ, as well as the local variables x and y.



(a) *3D surface plot of the GML function over $x \in [0,2]$, $y \in [0,1]$; parameters: $\mu = 0.7$, $\alpha = 1.2$, $\beta = 1$, $\rho = 1$, $q = 1$.*



(b) *Effect of varying $\mu \in \{0.2, 0.5, 0.8\}$ on the GML function for fixed $y = 0.5$; other parameters: $\alpha = 1.2$, $\beta = 1$, $\rho = 1$, $q = 1$.*

Figure 4.1: *GML function visualizations under varying parameters.*

The plot in Figure 4.1a illustrates a monotonic increase of the *GML* function as both x and y increase. This reflects the joint influence of the spatial input and memory variable, suggesting the function's relevance in modeling long-range coupled dynamics in fractional systems.

As shown in Figure 4.1b, smaller values of μ (indicating stronger memory) produce higher *GML* values. This aligns with theoretical expectations: a smaller μ corresponds to a heavier memory effect, amplifying the contribution of past states. The curve becomes flatter as $\mu \to 1$, consistent with convergence toward classical (non-fractional) behavior.

These graphical results support the analytical structure of the *GML* function. They demonstrate its flexibility and sensitivity to fractional parameters, justifying its role as a two-variable analytic output arising from generalized fractional operators, and affirming its potential for modeling memory-influenced dynamics.

5 Conclusion

In this study, we analyzed the Atangana–Baleanu Riemann–Liouville (*AB–RL*) fractional derivative of a broad class of functions involving the generalized Mittag-Leffler function. Explicit analytical expressions for the *AB–RL* fractional derivative were derived and subsequently reformulated into more compact representations, including a single-series form and a two-variable formulation.

A central aspect of this work is the study of a two-variable special function, referred to as the Generalized–Two–Variable–Mittag–Leffler (*GML*) function. This function emerges naturally from the *AB–RL* derivative structure of the generalized Mittag-Leffler function. The *GML* function recovers several known functions such as the Prabhakar function, the classical Mittag-Leffler functions, and their scaled variants as limiting cases, and thus serves as a convenient interpolating structure between fractional and classical regimes under suitable parameter limits.

In addition, we demonstrated that important special forms appearing in the two-variable Mittag-Leffler framework of Pathan and Kumar admit clear structural analogues within the *GML* setting. This correspondence shows that the *GML* function fits naturally into the broader family of two-variable Mittag-Leffler constructions. The theoretical results were supplemented by graphical illustrations highlighting the function's parametric sensitivity with respect to the memory index μ and local parameters α, β, ρ, and q, thus providing deeper insight into its analytic behavior.

The *GML* function's and the associated *AB–RL* derivative framework provide a useful basis for modeling systems exhibiting nested memory and nonlocal interactions. Potential directions for future research include investigating the *GML* functions integral transforms, asymptotic behavior, convergence properties, and developing efficient numerical schemes. Moreover, its application to the solution of fractional differential equations arising in viscoelasticity, control theory, and anomalous diffusion presents a promising avenue for further exploration.

Acknowledgement. The authors gratefully acknowledge all the references and sources cited throughout this paper, particularly the foundational works on generalized Mittag-Leffler functions and fractional calculus. The results obtained in this work were inspired in part by earlier studies, especially those in [10], which provided key insights into the fractional behavior of special functions. The authors also express their sincere appreciation to the anonymous reviewers for their constructive comments and suggestions, which helped to improve the clarity and presentation of the manuscript. This work was supported by the University Grants Commission (UGC), Government of India, under the Junior Research Fellowship scheme (Ref. No. 211610214096 dated 19.04.2022) awarded to Amit Punia.

References

[1] M.A. Almalahi, K. Aldwoah, F. Alqarni, M. Hleili, K. Shah and F.M.O. Birkea, On modified Mittag-Leffler coupled hybrid fractional system constrained by Dhage hybrid fixed point in Banach algebra, *Sci. Rep.* , **14** (2024), 30264. https://doi.org/10.1038/s41598-024-81568-8. [*Erratum: Sci. Rep.* , **15** (2025), 5534. https://doi.org/10.1038/s41598-025-89846-9.]

[2] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.* , **20** (2016), 763–769.

[3] L. D'Costa and N. Menaria, New integral transforms of the extended k-generalized Mittag-Leffler function with graphical representations, *Eur. J. Pure Appl. Math.* , **17** (2024), 4164–4179. https://ejpam.com/index.php/ejpam/article/view/5215.

[4] R. Gorenflo, A.A. Kilbas, F. Mainardi and S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications, Springer Monographs in Mathematics*, Springer, Berlin, 2014.

[5] M.K. Gurjar, Integral averages for multi-index Mittag-Leffler functions via double Dirichlet measures and related fractional derivatives, *Jñānābha*, **51**(1) (2021), 8–18.
[6] H.J. Haubold, A.M. Mathai and R.K. Saxena, Mittag-Leffler functions and their applications, *J. Appl. Math.* , **2011** (2011), 298628, 51 pages. https://doi.org/10.1155/2011/298628.
[7] H. Hochstadt, *The Functions of Mathematical Physics*, Dover Publications, New York, 1986.
[8] F. Mainardi, Why the Mittag-Leffler function can be considered the queen function of the fractional calculus?, *Entropy*, **22** (2020), 1359. https://doi.org/10.3390/e22121359.
[9] G. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *C. R. Acad. Sci. Paris*, **137** (1903), 554–558.
[10] D. Pang, W. Jiang and A.U.K. Niazi, Fractional derivatives of the generalized Mittag-Leffler functions, *Adv. Differ. Equ.* , **2018** (2018), 415. https://doi.org/10.1186/s13662-018-1855-9.
[11] M. A. Pathan and H. Kumar, Generalized multivariable Cauchy residue theorem and non-zero zeros of multivariable and multiparameters generalized Mittag–Leffler functions, *Southeast Asian Bulletin of Mathematics*, **43** (2019), 733–749.
[12] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.* , **19** (1971), 7–15.
[13] A.K. Shukla and J.C. Prajapati, On a generalization of Mittag-Leffler function and its properties, *J. Math. Anal. Appl.* , **336** (2007), 797–811.
[14] H.G. Sun, Y. Zhang, D. Baleanu, W. Chen and Y.Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci. Numer. Simulat.* , **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019.
[15] T.G. Thange and S.M. Chhatraband, A new α-Laplace Transform on time scales, *Jñānābha*, **53**(2) (2023), 151–160.
[16] A. Wiman, Über den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$, *Acta Math.* , **29** (1905), 191–201.