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$\mathcal{Z}$-TENSOR ON THREE DIMENSIONAL TRANS-PARA-SASAKIAN MANIFOLD

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Abstract

This article attempts to give classification of trans-para-Sasakian manifold M as a para-Sasakian or para-cosymplectic or a manifold of constant curvature or an Einstein manifold. The methods used are based on cyclic parallel, codazzi type and semisymmetric restrictions on $\mathcal{Z}$-tensor of trans-para-Sasakian manifold. We conclude with an example which verifies some of the proved results.
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1 Introduction

The trans-Sasakian structure on a manifold M as detailed by Oubin [9], is an almost contact metric structure that encompasses both Sasakian and cosymplectic structures, and it has a close relationship with locally conformal Khler manifolds. Marrero [6, 7] demonstrated the trans-Sasakian structure of type (α, β), where α and β are smooth functions defined on M. The specific types of trans-Sasakian manifolds, namely $(0, 0)$, $(\alpha, 0)$, and $(0, \beta)$, correspond to cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively. Zamkovy [12] defined the trans-para-Sasakian manifold and showed that it is analogous to the trans-Sasakian manifold. The $(0, 0)$, $(1, 0)$, $(-1, 0)$ and $(0, 1)$ trans-para-Sasakian manifolds correspond to the para-cosymplectic, para-Sasakian, quasi-para-Sasakian and para-Kenmotsu manifolds respectively.
Mantica and Molinari [8] introduced $\mathcal{Z}$-tensor and defined it as a $(0, 2)-$type tensor on a Riemannian manifold M as follows:

$$\mathcal{Z}(X, Y) = S(X, Y) + fg(X, Y), \tag{1.1}$$

where X and Y are arbitrary vector fields on M, S is Ricci tensor, g is Riemannian metric and f is smooth function on semi-Riemannian manifold M. If $f = 0$ and $\mathcal{Z} = 0$ in (1.1), M becomes $\mathcal{Z}$-Einstein and Einstein manifold respectively. Unal [10] conducted a study on the $\mathcal{Z}$-tensor within $N(\kappa)$-contact metric manifolds, while Prakash focused on $\mathcal{Z}$-symmetries in para-Sasakian three-manifolds.
This research examines trans-para-Sasakian manifolds in three dimensions. In Section 2, we outline several fundamental formulas, and Section 3 discusses $\mathcal{Z}$-covariantly constant, $\mathcal{Z}$-cyclic parallel, $\mathcal{Z}$-recurrent and $\mathcal{Z}$-codazzi type tensors. Furthermore, Section 4 addresses the conditions $\mathcal{Z} \circ R = 0$, $R \circ \mathcal{Z} = 0$, $Q \circ \mathcal{Z} = 0$, and $\mathcal{Z} \circ Q = 0$.

2 Preliminaries

A smooth $(2n + 1)$-dimensional manifold M is said to be an almost paracontact manifold if it admits a $(1, 1)-$tensor field ϕ, a vector field ξ and a 1-form η satisfying the following conditions:
$(i)\phi^2 X = X - \eta(X)\xi$ and $\eta(\xi) = 1$,
$(ii)$the tensor field ϕ induces an almost paracomplex structure on each fiber of $D = ker(\eta)$, that is the eigen distribution D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigen values 1 and -1 respectively having equal dimension n.
From the definition it follows that $\phi \circ \xi = 0$, $\eta \circ \phi = 0$.
An almost paracontact manifold M is said to be an almost paracontact metric manifold if there is a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all vector fields X and Y on M.
An almost paracontact structure is said to be a paracontact metric structure if $d\eta(X,Y) = g(X, \phi Y)$. Almost paracontact metric structure is said to be normal, if the $(1,2)-$type torsion tensor $N_\phi = [\phi,\phi] - 2d\eta \times \xi$, where $[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ vanishes. The Manifold M with the paracontact metric structure (ϕ, ξ, η, g) is called paracontact metric manifold.
A paracontact metric manifold is said to be para-Sasakian [12] if and only if

$$(\nabla_X \phi)(Y) = -g(X,Y)\xi + \eta(Y)X,$$

where X, Y are vector fields on M and ∇ is Levi-Civita connection of metric g.
Simillarly, a paracontact metric manifold is said to be para-Kenmotsu manifold [14] if and only if

$$(\nabla_X \phi)(Y) = \eta(Y)\phi X + g(X, \phi Y)\xi.$$

A paracontact metric manifold is said to be para-cosymplectic manifold [2] if and only if

$$(\nabla_X \phi)(Y) = 0.$$

A trans-para-contact metric manifold is trans-para-Sasakian manifold [13] if and only if

$$(\nabla_X \phi)(Y) = \alpha(-g(X,Y)\xi + \eta(Y)X) + \beta(g(X,\phi Y)\xi + \eta(Y)\phi X), \tag{2.2}$$

where X and Y are vector fields, α and β are smooth functions on the manifold M.
Making use of equation (2.2), we obtain

$$\nabla_X \xi = -\alpha \phi X - \beta(X - \eta(X)\xi). \tag{2.3}$$

In a trans-para-Sasakian manifold, the curvature tensor R is in the form

$$\begin{aligned} R(X,Y)\xi = &-(\alpha^2+\beta^2)(\eta(Y)X - \eta(X)Y) - 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) - X(\alpha)\phi Y + Y(\alpha)\phi X \\ &+ Y(\beta)\phi^2 X - X(\beta)\phi^2 Y, \end{aligned} \tag{2.4}$$

where X, Y and Z are vector fields on M.
Considering α and β as constants in last equation, we get

$$R(X,Y)\xi = (\alpha^2+\beta^2)(\eta(X)Y - \eta(Y)X) + 2\alpha\beta(\eta(X)\phi Y - \eta(Y)\phi X). \tag{2.5}$$

Three-dimensional semi-Riemannian manifold has curvature tensor R in the form

$$R(X,Y)Z = g(Y,Z)R^{\#}X - g(X,Z)R^{\#}Y + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}(g(Y,Z)X - g(X,Z)Y), \tag{2.6}$$

where R, S and $R^{\#}$ are semi-Riemannian curvature tensor, Ricci tensor and Ricci operator respectively. Putting $Y = Z = \xi$ in (2.6) and using (2.5), we get

$$R^{\#}X = \left(\frac{r}{2} + 2(\alpha^2+\beta^2)\right)X - \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(X)\xi. \tag{2.7}$$

Utilizing (2.7) in (2.6), we have

$$\begin{aligned} R(X,Y)Z = &\left(\frac{r}{2} + 2(\alpha^2+\beta^2)\right)(g(Y,Z)X - g(X,Z)Y) - \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)(g(Y,Z)\eta(X)\xi \\ &- g(X,Z)\eta(Y)\xi) + \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)(\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X). \end{aligned} \tag{2.8}$$

Using $X = \xi$ in (2.8), we obtain

$$R(\xi, X)Y = -(\alpha^2+\beta^2)(g(X,Y)\xi - \eta(Y)X). \tag{2.9}$$

3 $\mathcal{Z}$-tensor satisfying certain conditions on three-dimensional trans-para-Sasakian manifold.

Let M be a 3-dimensional trans-para-Sasakian manifold.

Definition 3.1. *The $\mathcal{Z}$-tensor in M is cyclic parallel or M admits $\mathcal{Z}$-cyclic parallel tensor if*

$$(\nabla_X \mathcal{Z})(Y,W) + (\nabla_Y \mathcal{Z})(W,X) + (\nabla_W \mathcal{Z})(X,Y) = 0, \tag{3.1}$$

where X, Y and W are arbitary vector field on M.

Theorem 3.1. *In a three-dimensional trans-para-Sasakian manifold M admitting $\mathcal{Z}$-cyclic parallel tensor (i) the associated function f is constant and (ii) either scalar curvature $r = -6(\alpha^2+\beta^2)$ for $\beta \neq 0$ or M is para-Sasakian or M is para-cosymplectic.*

Proof. Making use of (2.7) in (1.1), we get

$$\mathcal{Z}(X,Y) = \left(\frac{r}{2} + (\alpha^2+\beta^2) + f\right) g(X,Y) - \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right) \eta(X)\eta(Y). \tag{3.2}$$

Taking covarinat derivative along W in (3.2), we obtain

$$\begin{aligned}(\nabla_W \mathcal{Z})(X,Y) = {} & \left(\frac{(Wr)}{2} + (Wf)\right) g(X,Y) - \frac{(Wr)}{2}\eta(X)\eta(Y) - \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right) \\ & (-\alpha g(\phi W, X)\eta(Y) - \alpha g(\phi W, Y)\eta(X) - \beta g(X,W)\eta(Y) \\ & - \beta g(Y,W)\eta(X) + 2\beta\eta(X)\eta(Y)\eta(W)).\end{aligned} \tag{3.3}$$

Utilizing (3.3) in (3.1), we have

$$\begin{aligned}&\left(\frac{(Xr)}{2} + (Xf)\right) g(Y,W) + \left(\frac{(Yr)}{2} + (Yf)\right) g(X,W) + \left(\frac{(Wr)}{2} + (Wf)\right) g(X,Y) - \frac{(Xr)}{2}\eta(Y)\eta(W) \\ &- \frac{(Yr)}{2}\eta(X)\eta(W) - \frac{(Wr)}{2}\eta(X)\eta(Y) - \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)(-2\beta g(X,W)\eta(Y) - 2\beta g(X,Y)\eta(W) \\ &- 2\beta g(Y,W)\eta(X) - 2\beta\eta(X)\eta(Y)\eta(W)) = 0.\end{aligned} \tag{3.4}$$

Setting $X = W = \xi$ in (3.4), we get

$$(Yf) = -2(\xi f)\eta(Y), \tag{3.5}$$

for all vector fields Y. Taking $Y = \xi$ in (3.5), we have $(\xi f) = 0$. Which when substituted in (3.5) gives $(Yf) = 0$. Therefore f is a constant.

Contracting (3.4) with respect to $X = W = e_i$, where $\{e_i\}_{i=1}^3$ is an orthonormal frame on M, and summing over $i = 1$ to 3, we obtain

$$2(Yr) - (\xi r)\eta(Y) + 4\beta\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(Y) = 0. \tag{3.6}$$

Taking $Y = \xi$ in (3.4), we get

$$(\xi r) = -4\beta\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right). \tag{3.7}$$

Using (3.7) in (3.6), we obtain

$$(Yr) + 4\beta\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(Y) = 0, \tag{3.8}$$

for allvector field Y. Last equation becomes

$$Dr = -4\beta\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right). \tag{3.9}$$

When we compute the covariant derivative along X in equation (3.9) and incorporate (2.3), then take the inner product with ξ, we find that

$$g(\nabla_X Dr, Y) = -2\beta(Xr)\eta(Y) + 4\beta\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)(\alpha g(\phi X, Y) + \beta g(X,Y) - \beta\eta(X)\eta(Y)). \tag{3.10}$$

Interchanging X and Y and subtracting obtained equation with (3.10), making use of $g(\nabla_X Dr, Y) = g(\nabla_Y Dr, X)$, we obtain

$$\beta(Xr)\eta(Y) - \beta(Yr)\eta(X) - 2\alpha\beta g(\phi X, Y) = 0. \tag{3.11}$$

Setting $Y = \xi$ in (3.11), we get

$$(Xr) = (\xi r)\eta(Y), \tag{3.12}$$

for all vector field X, (3.11) becomes $dr = (\xi r)\eta$. By applying d to both sides of the equation and taking the wedge product, we find that $(\xi r) = 0$. Using this result in the earlier equation and performing direct calculations, we conclude that r is a constant. Substituting r =constant into the equation (3.11), we find that $\alpha\beta(\frac{r}{2} + 3(\alpha^2+\beta^2)) = 0$.

For the case where $\alpha\beta \neq 0$, the scalar curvature can be expressed as $r = -6(\alpha^2+\beta^2)$. If $\alpha\beta = 0$, then the manifold is para-cosymplectic when both α and β are equal zero.

Definition 3.2. *A trans-para-Sasakian manifold M is $\mathcal{Z}$-recurrent, if*

$$(\nabla_X \mathcal{Z})(Y,W) = A(X)\mathcal{Z}(Y,W), \tag{3.13}$$

where X, Y and W are arbitary vector fields and A is a 1-form on M.

Theorem 3.2. *If a three-dimensional trans-para-Sasakian manifold M is $\mathcal{Z}$-recurrent, then either the manifold M is para-cosymplectic or scalar curvature $r = -6(\alpha^2+\beta^2)$ and $f = 2(\alpha^2+\beta^2)$.*

Proof. *Here we consider 1-form $A(X) = \eta(X)$ and utilizing (3.3) in (3.13), we get*

$$\begin{aligned}(S(Y,W)+fg(Y,W))\eta(X) = \left(\frac{(Xr)}{2}+(Xf)\right)g(Y,W) - \frac{(Xr)}{2}\eta(Y)\eta(W) - \left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)\\ (-\alpha g(\phi X,Y)\eta(W) - \alpha g(\phi X,W)\eta(Y) - \beta g(X,Y)\eta(Z)\\ - \beta g(X,Z)\eta(Y) + 2\beta\eta(X)\eta(Y)\eta(W)).\end{aligned} \tag{3.14}$$

Taking $Y = W = \xi$ in (3.14), we get

$$(Xf) = (f - 2(\alpha^2+\beta^2))\eta(X). \tag{3.15}$$

Using (3.15) in (3.14), we have

$$\begin{aligned}&\frac{(Xr)}{2}g(Y,W) - 2(\alpha^2+\beta^2)g(Y,W)\eta(X) - \frac{(Xr)}{2}\eta(Y)\eta(W) - \left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)(-\alpha g(\phi X,Y)\eta(W)\\ &- \alpha g(\phi X,W)\eta(Y) - \beta g(X,Y)\eta(W) - \beta g(X,W)\eta(Y) + 2\beta\eta(X)\eta(Y)\eta(W))\\ &= \left(\frac{r}{2}+\alpha^2+\beta^2\right)g(Y,W)\eta(X) - \left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)\eta(X)\eta(Y)\eta(W).\end{aligned} \tag{3.16}$$

On contracting (3.16) with respect to Y and W, we have

$$(Xr) = (r + 6(\alpha^2+\beta^2))\eta(X). \tag{3.17}$$

Utilizing (3.17) in (3.15), we get

$$\begin{aligned}&(\frac{r}{2}+3(\alpha^2+\beta^2))(\alpha g(\phi X,Y) + \alpha g(\phi X,W)\eta(Y) + \beta g(X,Y)\eta(W) + \beta g(X,W)\eta(Y)\\ &- 2\beta\eta(X)\eta(Y)\eta(W)) = 0.\end{aligned} \tag{3.18}$$

On taking $X = W = e_i$, where $\{e_i\}_{i=1}^3$ is an orthonal frame on M, we obtain $\beta(\frac{r}{2}+3(\alpha^2+\beta^2)) = 0$. i.e., either $\beta = 0$ or $r = -6(\alpha^2+\beta^2)$.
In the first case M is para-cosymplectic and for the second case substitution of $r = -6(\alpha^2+\beta^2)$ in (3.14) yields $(Xf) = (f - 2(\alpha^2+\beta^2))\eta(X)$ for all vector fields X. Consequently, we have

$$Df = (f - 2(\alpha^2+\beta^2))\xi. \tag{3.19}$$

From equation (2.3), we perform the covariant derivative of (3.19) along X and then compute the scalar product with Y, yielding

$$\begin{aligned}g(\nabla_X Df, Y) =& (Xf)\eta(Y) + f\alpha g(\phi X,Y) - f\beta(g(X,Y) - \eta(X)\eta(Y))\\ &+ 2(\alpha^2+\beta^2)(\alpha g(\phi X,Y) - \beta\eta(X)\eta(Y)).\end{aligned} \tag{3.20}$$

Interchanging X, Y, and subtracting the obtained equation from (3.20), and them making use of $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$, we obtain

$$(Xf)\eta(Y) - (Yf)\eta(X) + (2f\alpha + 2(\alpha^2+\beta^2))g(\phi X,Y) = 0. \tag{3.21}$$

Setting $Y = \xi$ in (3.21), we get $(Xf) = (\xi f)\eta(X)$. Using same calculation as in Theorem 3.1, we obtain f =constant. Utilizing this in (3.20), we get $f = 2(\alpha^2+\beta^2)$. This completes our proof.

Definition 3.3. *A $\mathcal{Z}$- tensor in M is said to be Codazzi type if it satisfies the following:*

$$(\nabla_X \mathcal{Z})(Y,W) = (\nabla_Y \mathcal{Z})(X,W), \tag{3.22}$$

where X, Y and W are arbitary vector fields on M.

Theorem 3.3. *In a three-dimensional trans-para-Sasakian manifold M if the $\mathcal{Z}$-tensor is a Codazzi type with scalar curvature r and associated function f as constants along Reeb vector field ξ then manifold M is either para cosymplectic or of constant curvature $r = -6(\alpha^2+\beta^2)$.*

Proof. Making use of (3.3) in (3.22), we get

$$\left(\frac{(Xr)}{2}+(Xf)\right)g(Y,W)-\left(\frac{(Yr)}{2}+(Yf)\right)g(X,W)-\frac{(Xr)}{2}\eta(Y)\eta(W)+\frac{(Yr)}{2}\eta(X)\eta(W)$$
$$+\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)(2\alpha g(\phi X,Y)\eta(W)+\alpha g(\phi X,W)\eta(Y)-\alpha g(\phi Y,W)\eta(X)+\beta g(X,W)\eta(Y)$$
$$-\beta g(Y,W)\eta(X))=0. \quad (3.23)$$

Tracing (3.23) *with respect to* $X=W=\{e_i\}_{i=1}^3$ *where* e_i *is an orthonormal frame on* M, *we have*

$$\frac{(Yr)}{2}+2(Yf)+\frac{(\xi r)}{2}\eta(Y)-2\beta(\frac{r}{2}+3(\alpha^2+\beta^2))\eta(Y)=0. \quad (3.24)$$

Choosing $Y=\xi$ *in* (3.24), *we get* $(\xi r)+(\xi f)-2\beta(\frac{r}{2}+3(\alpha^2+\beta^2))$.
If r *and* f *both are constants along the Reeb vector field* ξ *then from* (3.24) *it follows that either* M *is cosymplectic or* M *is of constant scalar curvature* $r=-6(\alpha^2+\beta^2)$.

Definition 3.4. *A* $\mathcal{Z}$*-tensor in* M *is covariantly constant if* $(\nabla_W\mathcal{Z})(X,Y)=0$.

Theorem 3.4. *In a three-dimensional trans-para-Sasakian manifold,* $\mathcal{Z}$*-tensor is covariantly constant if the smooth function* f *and scalar curvature* r *are constants.*

Proof. Let us assume $\mathcal{Z}$- tensor is covariantly constant i.e., $(\nabla_W\mathcal{Z})(X,Y)=0$. Utilizing this in (3.3), we get

$$\left(\frac{(Wr)}{2}+(Wf)\right)g(X,Y)-\frac{(Wr)}{2}\eta(X)\eta(Y)-\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)(-\alpha g(\phi W,X)\eta(Y)-\alpha g(\phi W,Y)\eta(X)$$
$$-\beta g(X,W)\eta(Y)-\beta g(Y,W)\eta(X)+2\beta\eta(X)\eta(Y)\eta(W))=0. \quad (3.25)$$

Setting $X=Y=\xi$ *in* (3.25), *we arrive at* $(Wf)=0$, *for all vector fields* W, *and hence we have* $Df=0$. *On integrating it we get* $f=$*constant. Using* $f=$*constant in* (3.25), *we obtain*

$$\frac{(Wr)}{2}(g(X,Y)-\eta(X)\eta(Y))-\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)(-\alpha g(\phi W,X)\eta(Y)-\alpha g(\phi W,Y)\eta(X)$$
$$-\beta g(X,W)\eta(Y)-\beta g(Y,W)\eta(X)+2\beta\eta(X)\eta(Y)\eta(W))=0. \quad (3.26)$$

Tracing (3.26) *with respect to* $X=W=e_i$ *where* $\{e_i\}_{i=1}^3$ *is an orthonormal frame on* M, *we have* $\beta(\frac{r}{2}+3(\alpha^2+\beta^2))=0$. *Then we have either* $\beta=0$ *or* $r=-6(\alpha^2+\beta^2)$. *We follow as in proof of Theorem* (2.3) *and get that manifold is either para-cosymplectic or of constant curvature* $r=-6(\alpha^2+\beta^2)$.
On the other hand, if $\beta\neq 0$, $r=-6(\alpha^2+\beta^2)$, f *and* r *are constants in (3.3), then we obtain* $(\nabla_W\mathcal{Z})(X,Y)=0$. *i.e.,* $\mathcal{Z}$*- tensor is covariantly constant.*

Definition 3.5. *The* $\mathcal{Z}$*- tensor in* M *is* $\eta-$*parallel if*

$$(\nabla_W\mathcal{Z})(\phi X,\phi Y)=0, \quad (3.27)$$

where X, Y *and* W *are arbitary vector fields.*

Theorem 3.5. *In a three-dimensional trans-para-Sasakian manifold admitting* $\eta-$ *parallel-*$\mathcal{Z}$*-tensor,* $r+2f$ *a constant.*

Proof. Replacing X by ϕX in (3.2), we get

$$\mathcal{Z}(\phi X,Y)=(\frac{r}{2}+(\alpha^2+\beta^2)+f)g(\phi X,Y). \quad (3.28)$$

Setting $Y=\xi$ *in* (3.28), *we obtain*

$$\mathcal{Z}(\phi X,\xi)=0. \quad (3.29)$$

Replacing Y *by* ϕY *in* (3.28), *we have*

$$\mathcal{Z}(\phi X,\phi Y)=(\frac{r}{2}+(\alpha^2+\beta^2)+f)(-g(X,Y)+\eta(X)\eta(Y)). \quad (3.30)$$

Differntiating (3.30) *using* (2.2), (2.3) *and* (3.30), *we get*

$$(\nabla_W\mathcal{Z})(\phi X,\phi Y)=((Wf)+\frac{(Wr)}{2})(-g(X,Y)+\eta(X)\eta(Y)). \quad (3.31)$$

Utlizing (3.27) *in* (3.31), *we obtain*

$$\left((Wf)+\frac{(Wr)}{2}\right)(-g(X,Y)+\eta(X)\eta(Y))=0. \quad (3.32)$$

i.e., $\left((Wf)+\frac{(Wr)}{2}\right)=0$, *for all vector field* W. *Terefore we have* $D(2f+r)=0$. *On integrating it we arrive at* $2f+r=C$, *where* $C=$ *constant.*

4 Three-dimensional trans-para-Sasakian manifold satisfying semi-symmetric conditions with respect to $\mathcal{Z}$-tensor

Theorem 4.1. *A trans-para-Sasakian manifold satisfying $R(\xi, X) \circ \mathcal{Z}(Y, W) = 0$ is an Einstein manifold.*
Proof. *Let*

$$\mathcal{Z}(R(\xi, X)Y, W) + \mathcal{Z}(Y, R(\xi, X)W) = 0. \tag{4.1}$$

Utilizing (2.8) *in* (4.1), *we get*

$$-(\alpha^2 + \beta^2)(g(X, Y)\mathcal{Z}(\xi, W) - \eta(Y)\mathcal{Z}(X, W) - g(X, W)\mathcal{Z}(Y, W) - \mathcal{Z}(X, Y)\eta(W)) = 0. \tag{4.2}$$

Putting $X = \xi$ *in* (1.1) *and using obtained equtaion in* (4.2), *we obtain*

$$(\alpha^2 + \beta^2)(S(X, W) + 2(\alpha^2 + \beta^2)g(X, W)) = 0. \tag{4.3}$$

Dividing the last equation by $(\alpha^2 + \beta^2)$, *we get*

$$S(X, W) = -2(\alpha^2 + \beta^2)g(X, W). \tag{4.4}$$

i.e., manifold is Einstein. On tracing (4.4), *we obtain scalar curvature* $r = -6(\alpha^2 + \beta^2)$.

Theorem 4.2. *A trans-para-Sasakian manifold satisfying $\mathcal{Z}(X, Y) \circ R(U, V)W = 0$ is an Einstein manifold.*
Proof. *Let* $\mathcal{Z}(X, Y) \circ R(U, V)W = 0$. *Then we have*

$$(X \wedge_{\mathcal{Z}} Y)R(U, V)W + R((X \wedge_{\mathcal{Z}} Y)U, V) + R(U, (X \wedge_{\mathcal{Z}} Y)V)W + R(U, V)(X \wedge_{\mathcal{Z}} Y)W = 0. \tag{4.5}$$

From (3.23), *we get*

$$\begin{aligned}&\mathcal{Z}(Y, R(U, V)W)X - \mathcal{Z}(X, R(U, V)W)Y + \mathcal{Z}(Y, U)R(X, V)W \\ &- \mathcal{Z}(X, U)R(Y, V)W + \mathcal{Z}(Y, V)R(U, X)W - \mathcal{Z}(X, V)R(U, Y)W \\ &+ \mathcal{Z}(Y, W)R(U, V)X - \mathcal{Z}(X, W)R(U, V)Y = 0.\end{aligned} \tag{4.6}$$

Taking inner product with ξ *in* (4.6), *we obtain*

$$\begin{aligned}&\mathcal{Z}(Y, R(U, V)W)\eta(X) - \mathcal{Z}(X, R(U, V)W)\eta(Y) + \mathcal{Z}(Y, U)\eta(R(X, V)W) \\ &- \mathcal{Z}(X, V)\eta(R(Y, V)W) + \mathcal{Z}(Y, V)\eta(R(U, X)W) - \mathcal{Z}(X, V)\eta(R(U, Y)W) \\ &+ \mathcal{Z}(Y, U)\eta(R(U, V)X) - \mathcal{Z}(X, W)\eta(R(U, V)Y) = 0.\end{aligned} \tag{4.7}$$

Taking $X = U = W = \xi$ *in* (4.7), *we arrive at*

$$\begin{aligned}&\mathcal{Z}(Y, R(\xi, V)\xi) - \mathcal{Z}(\xi, R(\xi, V)\xi)\eta(Y) + \mathcal{Z}(Y, \xi)\eta(R(\xi, V)\xi) - \mathcal{Z}(\xi, V)\eta(R(Y, V)\xi) + \mathcal{Z}(Y, V)\eta(R(\xi, \xi)\xi) \\ &- \mathcal{Z}(\xi, V)\eta(R(\xi, Y)\xi) + \mathcal{Z}(Y, \xi)\eta(R(\xi, V)\xi) - \mathcal{Z}(\xi, \xi)\eta(R(\xi, V)Y) = 0.\end{aligned} \tag{4.8}$$

Using (1.1), (2.7) *and* (2.8) *in* (4.8), *we get*

$$S(Y, V) = -2(f - (\alpha^2 + \beta^2))g(Y, V) + 2(f - 2(\alpha^2 + \beta^2))\eta(Y)\eta(V). \tag{4.9}$$

On contracting last equation, we obtain scalar curvature $r = -(4f - 2(\alpha^2 + \beta^2))$.

A (1,3) curvature tensor known as Q- tensor is defined [11] by

$$Q(X, Y)Z = R(X, Y)Z - \frac{f}{2}(g(Y, Z)X - g(X, Z)Y), \tag{4.10}$$

where X, Y, Z are arbitary vector fields and f is a smooth function on the manifold. It follows from (4.10) that $\mathcal{Z}$-tensor is a trace of Q-tensor.

Theorem 4.3. *A trans-para-Sasakian manifold satisfying $Q(X, Y) \circ \mathcal{Z}(U, V) = 0$ is an Einstein manifold.*
Proof. *Let* $\mathcal{Z}(Q(X, Y)U, V) + \mathcal{Z}(U, Q(X, Y)V) = 0$.
Setting $X = U = \xi$ *in last equation, we obtain*

$$\mathcal{Z}(Q(\xi, Y)\xi, V) + \mathcal{Z}(\xi, Q(\xi, Y)V) = 0. \tag{4.11}$$

Making use of (4.10) *in* (4.11), *we obtain*

$$\mathcal{Z}(\xi, V)\eta(Y) - \mathcal{Z}(Y, V) + \mathcal{Z}(\xi, \xi)g(Y, V) - \mathcal{Z}(\xi, Y)\eta(V) = 0. \tag{4.12}$$

Using (1.1) *in* (4.12), *we arrive at*

$$S(Y, V) = -2(\alpha^2 + \beta^2)g(Y, V). \tag{4.13}$$

On contracting last equation with respect to $Y = V = e_i$ $i = 1, 2, 3$, *we obtain scalar crvature* $r = -6(\alpha^2+\beta^2)$.

Theorem 4.4. *A trnas-para-Sasakian manifold satisfying $\mathcal{Z}(X,Y)\circ Q(U,V)T=0$ is an Einstein manifold.*
Proof. Let $\mathcal{Z}(X,Y)\circ Q(U,V)T=0$.
Taking $X=U=T=\xi$ in last equation, we get

$$\mathcal{Z}(Y,Q(\xi,V)\xi)\xi-\mathcal{Z}(\xi,Q(\xi,V)\xi)Y+\mathcal{Z}(Y,\xi)Q(\xi,V)T-\mathcal{Z}(\xi,\xi)Q(Y,V)\xi-\mathcal{Z}(\xi,V)Q(\xi,Y)\xi$$
$$+\mathcal{Z}(Y,\xi)Q(\xi,V)\xi-\mathcal{Z}(\xi,\xi)Q(\xi,V)Y=0. \tag{4.14}$$

Taking scalar product with ξ in (4.14) and making use of (1.1), (2.9) and (4.10), we get

$$((\alpha^2+\beta^2)+\frac{f}{2})(S(Y,V)+2(\alpha^2+\beta^2)g(Y,V))=0. \tag{4.15}$$

If $((\alpha^2+\beta^2)+\frac{f}{2})\neq 0$, then the manifold is Einstein and scalar curvature $r=-6(\alpha^2+\beta^2)$.

Example 4.1: We consider the 3-dimensional manifold $M=\{(x,y,z)\in R^3\ z\neq 0\}$, with (x,y,z) as the standard co-ordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1=-z\frac{\partial}{\partial x},\ e_2=-z\frac{\partial}{\partial y},\ e_3=-z\frac{\partial}{\partial z}.$$

The vectors e_1, e_2 and e_3 are linearly independent at each point of M.
Let g be a semi-Riemannian metric defined by

$$g(e_1,e_1)=g(e_3,e_3)=1,\ g(e_2,e_2)=-1,\ g(e_i,e_j)=0,\ i,j\in\{1,2,3\}\text{ and } i\neq j.$$

Let η be a 1-form on M defined by $\eta(X)=g(X,e_3)$, for all X on M and $e_3=\xi$. Let ϕ be a $(1,1)$ tensor field on M defined by

$$\phi e_1=e_2,\ \phi e_2=e_1,\ \phi e_3=0.$$

Using linearity property of ϕ and g, we have

$$\phi^2X=X-\eta(X)\xi,\ g(\phi X,\phi Y)=-g(X,Y)+\eta(X)\eta(Y).$$

Thus the structure (ϕ,ξ,η,g) defines an almost paracontact structure on M. Let ∇ be the Levi-civita-connection with respect to the metric g. Then we have

$$[e_1,e_2]=0,\ [e_1,e_3]=e_1,\ [e_2,e_3]=e_2.$$

From Koszul formula

$$2g(\nabla_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z)-g([Y,Z],X)+g([Z,X],Y).$$

We can easily calculate

$$\nabla_{e_1}e_1=-e_3,\ \nabla_{e_1}e_2=0,\ \nabla_{e_1}e_3=e_1,$$
$$\nabla_{e_2}e_1=0,\ \nabla_{e_2}e_2=e_3,\ \nabla_{e_2}e_3=e_2,$$
$$\nabla_{e_3}e_1=0,\ \nabla_{e_3}e_2=0,\ \nabla_{e_3}e_3=0.$$

These equations show that $\nabla_X\xi$ and $(\nabla_X\phi)(Y)$ satisfies equations (2.2) and (2.3) and we obtain $\alpha=0$ and $\beta=-1$. Thus M is a trans-para-Sasakian manifold of type $(0,-1)$. With the help of above equations it is easy to calculate

$$R(e_1,e_2)e_1=e_2,\ R(e_1,e_2)e_2=e_1,\ R(e_1,e_2)e_3=0,$$
$$R(e_1,e_3)e_1=e_3,\ R(e_1,e_3)e_2=0,\ R(e_1,e_3)e_3=-e_1,$$
$$R(e_2,e_3)e_1=0,\ R(e_2,e_3)e_2=-e_3,\ R(e_2,e_3)e_3=-e_2.$$

From these curvature tensor components we can calculate Ricci tensor as follows

$$S(e_1,e_1)=S(e_3,e_3)=-2\text{ and }S(e_2,e_2)=2.$$
$$S(e_1,e_2)=S(e_1,e_3)=S(e_2,e_3)=0.$$

We get $r=-6$. Using $\alpha=0$ and $\beta=-1$ in Theorem 3.2 and Theorem 4.1, we get $r=-6$.
Using (3.2), we obtain the following

$$\mathcal{Z}(e_1,e_1) = (\tfrac{r}{2} + (\alpha^2+\beta^2) + f),\ \mathcal{Z}(e_2,e_2) = -(\tfrac{r}{2} + (\alpha^2+\beta^2) + f),\ \mathcal{Z}(e_3,e_3) = f - 2(\alpha^2+\beta^2).$$
$$\mathcal{Z}(e_1,e_2) = \mathcal{Z}(e_1,e_3) = \mathcal{Z}(e_2,e_1) = \mathcal{Z}(e_2,e_3) = \mathcal{Z}(e_3,e_1) = \mathcal{Z}(e_3,e_2) = 0.$$

Using previous equations, we obtain

$$(\nabla_{e_1}\mathcal{Z})(e_1,e_3) = -(\tfrac{r}{2} + 3(\alpha^2+\beta^2)) \text{ and } (\nabla_{e_1}\mathcal{Z})(e_1,e_3) = \eta(e_1)\mathcal{Z}(e_1,e_3) = 0.$$

$$(\nabla_{e_1}\mathcal{Z})(e_1,e_1) = \tfrac{(e_1 r)}{2} + (e_1 f) \text{ and } (\nabla_{e_1}\mathcal{Z})(e_1,e_1) = \eta(e_1)\mathcal{Z}(e_1,e_1) = 0.$$

$$(\nabla_{e_2}\mathcal{Z})(e_2,e_3) = (\tfrac{r}{2} + 3(\alpha^2+\beta^2)) \text{ and } (\nabla_{e_2}\mathcal{Z})(e_2,e_3) = 0.$$

From the above equations, we arrive at $r = -6(\alpha^2+\beta^2)$.
Let $\mathcal{Z}(e_1,e_1) - \mathcal{Z}(e_2,e_2) + \mathcal{Z}(e_3,e_3) = 0$, we get $f = 2(\alpha^2+\beta^2)$. For $\alpha = 0$, $\beta = -1$, we have $r = -6$ and $f = 2$. This verifies Theorem 3.2 and Theorem 4.1.

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