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THE APPROXIMATION ON ANALYTIC FUNCTION OF LAPLACE-STIELTJES TRANSFORMS WITH PROXIMATE ORDER

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Abstract

In this paper, we investigate the growth of $M_u(\sigma, F)$ defined by Laplace-Stieltjes transforms of proximate order which is convergent in the half plane. We obtain the error in approximating Laplace-Stieltjes transform of finite order in the half plane by Dirichlet polynomials.
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1 Introduction

Let Laplace-Stieltjes transform

$$F(s) = \int_0^\infty e^{sx} d\alpha(x), \qquad s = \sigma + it, \tag{1.1}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, x](0 < x < \infty)$, σ and t are real variables. If $\alpha(x)$ is a step function and satisfies

$$\alpha(x) = \begin{cases} a_1 + a_2 + \cdots + a_n & , \quad \lambda_n < x < \lambda_{n+1} \\ 0 & , \quad 0 \le x < \lambda_1 \\ \frac{\alpha(x+)+\alpha(x-)}{2} & , \quad x > 0 \end{cases},$$

where the sequence $\{\lambda_n\}_{n=1}^\infty$ is

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots, \lambda_n \to \infty \quad as \quad n \to \infty, \tag{1.2}$$

which satisfies the following conditions

$$\limsup_{n\to\infty}(\lambda_{n+1} - \lambda_n) = h < +\infty, \quad \limsup_{n\to\infty} \frac{n}{\lambda_n} = E < +\infty. \tag{1.3}$$

Set

$$A_n^* = \sup_{\lambda_n \le x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^x e^{ity} d\alpha(y) \right|,$$

$$\limsup_{n\to\infty} \frac{\log A_n^*}{\lambda_n} = 0. \tag{1.4}$$

In 1963, Yu [20] obtained Valiron-Knopp-Bohr formula.

Theorem 1.1 ([20])**.** *If Laplace-Stieljes transforms* (1.1) *satisfies* (1.3) *and* $\limsup_{n\to\infty} \frac{\log n}{\lambda_n} < +\infty$*,*

$$\limsup_{n\to\infty} \frac{\log A_n^*}{\lambda_n} < \sigma_u^F \le \limsup_{n\to\infty} \frac{\log A_n^*}{\lambda_n} + \limsup_{n\to\infty} \frac{\log n}{\lambda_n},$$

where σ_u^F is called the abscissa of uniformly convergent.

It follows that from (1.3), (1.4) *and Theorem* 1.1 *such that $\sigma_u^F = 0$. i.e. $F(s)$ is an analytic in the half plane $H = \{s = \sigma + \iota t : \sigma < 0, t \in \mathbb{R}\}$. Set*

$$M_u(\sigma, F) = \sup_{0<x<+\infty, -\infty<t<+\infty} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right|,$$

and

$$\mu(\sigma, F) = \max_{n \in N} \{A_n^* e^{\lambda_n \sigma}\} (\sigma < +\infty).$$

2 Definitions

Definition 2.1 ([18])**.** *If the Laplace-Stieltjes transform* (1.1) *satisfies $\sigma_u^F = 0$, then*

$$\rho = \limsup_{\sigma \to 0^-} \frac{\log^+ \log M_u(\sigma, F)}{\log\left(-\frac{1}{\sigma}\right)}, and$$

is called order of F(s) in $Res = \sigma < 0$, where $\log^+ x = \max(\log x, 0)$.

If $\rho \in (0, \infty)$, we say that $F(s)$ is an analytic function of finite order in the left half plane. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Laplace-Stieltjes ([1]-[16], [19], [21]) *for some results.*

Let $\rho(r)$ be an non-negative, continuous, monotonic function and let it have a left-hand derivative and a right-hand derivative in every $r (> r_0)$, such that

$$\lim_{r \to +\infty} \rho(r) = \rho, \qquad \lim_{r \to +\infty} \rho'(r) r \log r = 0, \tag{2.1}$$

and set $U(r) = r^{\rho(r)}$, which is a strictly increasing function of r in $r \geq r_0' > r_0$. Let

$$t = rU(r), \qquad r = W(t), \qquad r > 0, t > 0, \tag{2.2}$$

be two reciprocally inverse functions. From [6], *for any positive real number k, we have*

$$\lim_{r \to +\infty} \frac{U(kr)}{U(r)} = k^\rho, \qquad \lim_{t \to +\infty} \frac{W(kt)}{W(t)} = k^{\frac{1}{\rho+1}}. \tag{2.3}$$

Definition 2.2 ([18])**.** *If the Laplace-Stieltjes transform* (1.1) *of (F(s),*

$$\limsup_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} = 1, \tag{2.4}$$

$\rho(-\frac{1}{\sigma})$ and $U(-\frac{1}{\sigma})$ are called the proximate order and the type of (1.1) in $Res = \sigma < 0$.

We denote $\overline{L}_\beta$ be the class of all the functions $F(s)$ of the form (1.1) which are analytic in the half plane $Res < \beta (-\infty < \alpha < +\infty)$ and the sequence $\{\lambda_n\}$ satisfies (1.2) and (1.3), and L denote be the class of all the function $F(s)$ of the form (1.1) which are analytic in the half plane $Res < 0$ and (1.1) which are analytic in the half plane $Res < \beta (-\infty < \alpha < +\infty)$ and the sequence $\{\lambda_n\}$ satisfies (1.2), (1.3) and (1.4). Therefore, if $-\infty < \beta < 0$ and $F(s) \in L$, then $F(s) \in \overline{L}_\beta$; if $0 < \beta < +\infty$ and $F(s) \in \overline{L}_\beta$, then $F(s) \in L$. If $A_n^* = 0$ for $n \geq k+1$, and $A_n^* \neq 0$ then $F(s)$ will be called an exponential polynomial of degree k usually denoted by p_k, i.e. $p_k(s) = \int_0^{\lambda_k} \exp(sy) d\alpha(y)$. We denote $\prod_n$ to the class of all exponential polynomial of degree n. i.e.

$$\prod_n = \{\sum_{i=1}^n b_i \exp(s\lambda_i); (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n\}$$

For $F(s) \in \overline{L}_\beta$, $-\infty < \beta < +\infty$, we denote $E_n(F, \beta)$ the error in approximating the function $F(s)$ by exponential polynomial of degree n in uniform norm as

$$E_n(F, \beta) = \inf_{p \in \prod_n} \|F - P\|_\beta, \qquad n = 1, 2, \ldots,$$

where

$$\|F - P\|_\beta = \max_{-\infty < t < +\infty} |F(\beta + it) - P(\beta + it)|.$$

3 Lemmas

To prove our results we use the Lemmas [17];

Lemma 3.1 ([17]). *Let α and λ be any positive real numbers,then*

$$\phi(\sigma) = \alpha U\left(-\frac{1}{\sigma}\right) - \lambda\sigma, \qquad \sigma < 0$$

obtained the minimum,

$$\alpha^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda}{W(\lambda)}(1+o(1)), \lambda\to\infty \quad in \quad \sigma = \frac{(\alpha\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1)), \lambda\to\infty.$$

Lemma 3.2 ([17]). *Let b and λ be any positive real numbers, then*

$$\phi(\sigma) = \frac{x}{W(bx)} + \sigma x,$$

obtained the maximum,

$$\frac{\rho^\rho}{b(\rho+1)^{(\rho+1)}}U\left(-\frac{1}{\sigma}\right)(1+o(1)), \sigma\to 0^- \quad in \quad x = \frac{1}{b}\frac{\rho}{\rho+1}^{\rho+1}\frac{1}{\sigma}U\left(-\frac{1}{\sigma}\right)(1+o(1)), \sigma\to 0^-$$

Lemma 3.3 ([17]). *Let $A > 0$ and $\{\lambda_{n_\nu}\}$ be a strictly increasing sequence tending to $\infty(\nu\to\infty)$ and satisfy $\{\lambda_{n_1} > Ar_o^{'}U(r_o^{'})\}$. If $\lim_{\nu\to\infty}\frac{\lambda_{n_\nu+1}}{\lambda_{n_\nu}} = 1$, then there exists a monotone decreasing positive sequence $\{\sigma_\nu\}$ convergent to zero satisfying*

$$\lambda_{n_\nu} = -\frac{A}{\sigma_\nu}U\left(-\frac{1}{\sigma_\nu}\right), \qquad \lim_{\nu\to\infty}\frac{\left(-\frac{1}{\sigma_\nu+1}\right)U\left(-\frac{1}{\sigma_\nu+1}\right)}{\left(-\frac{1}{\sigma_\nu}\right)U\left(-\frac{1}{\sigma_\nu}\right)} = 1.$$

We investigate the approximation of analytic function defined by Laplace-Stieljes transform and obtain the relations between the error $E_n(F,\beta)$ and the growth order of $F(s)$.

4 Mains Results

Theorem 4.1. *Let the Laplace-Stieljes transforms* (1.1) *of finite order $\rho(0<\rho<\infty)$ satisfies* (1.2), (1.3), (1.4) *and*

$$\lim_{n\to\infty}\frac{\log^+\log n}{\log^+\lambda_n} \le \frac{\rho}{\rho+1}. \tag{4.1}$$

Then

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U\left(-\frac{1}{\sigma}\right)} = 1 \iff \limsup_{n\to\infty}\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} = 1, \tag{4.2}$$

where $B = (1+\rho)^{(1+\rho)}\rho^{-\rho}$ and $U(r)$ are defined by (2.3).

Proof. We prove the sufficient part of the Theorem, for any $\epsilon > 0$, there exist $N_0 \in \mathbb{N}$, as $n > N_0$, we have

$$\log^+ A_n^* < (1+\epsilon)BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right),$$

$$i.e. \qquad \lambda_n < (1+\epsilon)B\frac{\lambda_n}{\log^+ A_n^*}U\left(\frac{\lambda_n}{\log^+ A_n^*}\right).$$

Since, $r = W(t)$ and $t = rU(r)$ are two reciprocal inverse functions and monotonically increasing functions, we get

$$W\left(\frac{\lambda_n}{B(1+\epsilon)}\right) \le \frac{\lambda_n}{\log^+ A_n^*}.$$

$$i.e. \qquad \log^+ A_n^* \le \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\epsilon)}\right)}.$$

Thus, there exist a positive constant D such that

$$i.e. \qquad A_n^* \le D \exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\epsilon)}\right)}\right], n = 0, 1, 2, 3, \ldots \quad . \tag{4.3}$$

Let

$$I_k(x; it) = \int_{\lambda_k}^{x} e^{ity} d\alpha(y), \lambda_k \le x \le \lambda_{k+1},$$

for any $t \in \mathbb{R}$, we get

$$|I_k(x; it)| \le A_k^*. \tag{4.4}$$

Thus for any x, $\lambda_k \le x \le \lambda_{k+1}$, $\sigma > 0$, we get

$$\begin{aligned} \int_0^x e^{(\sigma+it)y} d\alpha(y) &= \sum_{k=1}^{n-1} \int_{\lambda_k}^{\lambda_{k+1}} e^{\sigma y} d_y I_k(y; it) + \int_{\lambda_k}^{x} e^{\sigma y} d_y I_k(y; it) \\ &= \sum_{k=1}^{n-1} \left[e^{\sigma\lambda_{k+1}} I_k(\lambda_{k+1}; it) + \sigma \int_{\lambda_k}^{\lambda_{k+1}} e^{\sigma y} I_k(y; it) dy \right] \\ &\quad + e^{\sigma x} I_n(x; it) + \sigma \int_{\lambda_k}^{x} e^{\sigma y} I_n(y; it) dy. \end{aligned} \tag{4.5}$$

Then for any $\sigma < 0$ and any $t \in \mathbb{R}$, we get

$$\begin{aligned} \left| \int_0^x e^{(\sigma+it)y} d\alpha(y) \right| &\le \sum_{k=1}^{n-1} A_k^* (e^{\sigma\lambda_{k+1}} + |e^{\sigma\lambda_{k+1}} - e^{\sigma\lambda_k}|) + A_n^* (e^{\sigma x} + |e^{\sigma x} - e^{\sigma\lambda_n}|) \\ &\le \sum_{k=1}^{n} A_k^* e^{\sigma\lambda_k}. \end{aligned} \tag{4.6}$$

From (4.3) and (4.6), we get

$$\begin{aligned} M_u(\sigma, F) &\le \sum_{k=1}^{n} A_k^* e^{\sigma\lambda_k} \le D \sum_{n=0}^{\infty} \exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\epsilon)}\right)} + \lambda_n \sigma\right] \\ &\le D \sup_{n \ge 0} \left\{ \exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\epsilon)}\right)} + \lambda_n (1+\epsilon)\sigma\right] \right\} \sum_{n=0}^{\infty} e^{\sigma\epsilon\lambda_n}. \end{aligned} \tag{4.7}$$

From (4.1) there exist $\rho_1 \in (0, \rho)$ such that

$$\limsup_{n\to\infty} \frac{\log\log^+ n}{\log \lambda_n} < \frac{\rho_1}{1+\rho_1}. \tag{4.8}$$

Thus there exist $N_1 \in \mathbb{N}$ such that

$$\lambda_n > (\log n)^{\frac{\rho_1+1}{\rho_1}} > 1, n > N_1. \tag{4.9}$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} e^{\sigma\epsilon\lambda_n} &\le N_1 + 1 + \sum_{n=N_1+1}^{\infty} e^{\sigma\epsilon(\log n)^{\frac{\rho_1+1}{\rho_1}}} \\ &\le D_1 + \int_{N_1}^{T} \frac{dx}{x^{-\epsilon\sigma}} \\ &\le D_2 + \frac{1}{(1+\epsilon\sigma)} T^{1+\epsilon\sigma}, \end{aligned}$$

where $T = e^{\frac{\rho_1}{\epsilon\sigma}}$ and D_1 ,D_2 are two real constant by Lemma 3.2, we get

$$M_u(\sigma, F) \le D \exp\left[(1+\epsilon) U\left(\frac{-1}{(1+\epsilon)\sigma}\right)(1+o(1))\right]\left(D_2 + \frac{1}{(1+\epsilon\sigma)} T^{1+\epsilon\sigma}\right).$$

Thus

$$\log^+ M_u(\sigma, F) \le (1+2\epsilon) U\left(-\frac{1}{\sigma}\right)(1+o(1)). \tag{4.10}$$

Hence

$$\limsup_{\sigma\to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} \le 1.$$

Suppose that

$$\limsup_{\sigma\to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} = \beta < 1. \tag{4.11}$$

Set $\eta > 0$ and $\beta + 2\eta < 1$ then there exists $\sigma_0 > 0$,

$$\log^+ M_u(\sigma, F) < (\beta+\eta) U\left(-\frac{1}{\sigma}\right), (0 < \sigma < \sigma_0).$$

Let

$$I(x; \sigma + it) = \int_0^x e^{\sigma + ity} d\alpha(y).$$

From (1.3), there exists $K > 0$ satisfying $0 < \lambda_{n+1} - \lambda_n < K(n = 1, 2, \dots)$.
For $\sigma(< 0)$ sufficiently tending to zero, it follows that $e^{-K\sigma} < \frac{3}{2}$,

$$\begin{aligned}\int_{\lambda_n}^x e^{ity} d\alpha(y) &= \int_{\lambda_n}^x e^{-\sigma y} d_y I(x; \sigma+it) \\ &= \left\{I(x;\sigma+it)e^{-\sigma y}\right\}_{\lambda_n}^x + \sigma \int_{\lambda_n}^x e^{-\sigma y} I(x;\sigma+it) dy.\end{aligned}$$

For any $\sigma < 0$ and any $x \in (\lambda_n, \lambda_{n+1}]$, we get

$$\begin{aligned}\left|\int_{\lambda_n}^x e^{ity} d\alpha(y)\right| &\le M_u(\sigma, F)\left[|e^{-\sigma x} + e^{-\sigma\lambda_n}| + |e^{-\sigma x} - e^{-\sigma \lambda_n}|\right] \\ &\le 3M_u(\sigma, F) e^{-\lambda_n \sigma}. \end{aligned} \tag{4.12}$$

From (4.10), we have

$$\log^+ A_n^* < (\beta + 2\eta) U\left(-\frac{1}{\sigma}\right) + \lambda_n \sigma. \tag{4.13}$$

When n is sufficiently large from Lemma 3.1, we get

$$\begin{aligned}\log^+ A_n^* &\le (\beta+2\eta)^{\frac{1}{\rho+1}} \frac{\rho+1}{\rho^{\frac{1}{\rho+1}}} \frac{\lambda_n}{W(\lambda_n)}(1+\eta) \\ &= (B(\beta+2\eta))^{\rho+1} \frac{\lambda_n}{W(\lambda_n)}(1+\eta), \\ i.e., W(\lambda_n) &\le \frac{\lambda_n}{\log^+ A_n^*}(B(\beta+2\eta))^{\rho+1}(1+\eta).\end{aligned}$$

For $x > x_0 = r_0' U(r_0')$, the function $W(x)$ is monotonic increasing, we get

$$\lambda_n \le \frac{\lambda_n}{\log^+ A_n^*}(B(\beta+2\eta))(1+\eta)^{\rho+1}(1+o(1)) U\left(\frac{\lambda_n}{\log^+ A_n^*}\right).$$

Therefore,

$$\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} \le (\beta+\eta)^{\rho+2}(1+o(1)).$$

Thus,

$$\limsup_{n\to+\infty} \frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} \le \beta < 1.$$

□

Theorem 4.2. *Let $F(s) \in L$ be of the finite order $\rho(0 < \rho < \infty)$ and $-\infty < \beta < 0$, then*

$$\lim_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} = 1 \iff \limsup_{n \to \infty} \frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)} = 1; \tag{4.14}$$

there exists a increasing, positive integer $\{n_\nu\}$ satisfying

$$\limsup_{n \to \infty} \frac{\log^+ A_{n_\nu}^*}{BU\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right)} = 1 \qquad \lim_{\nu \to \infty} \frac{\lambda_{n_\nu + 1}}{\lambda_{n_\nu}} = 1. \tag{4.15}$$

Proof. We prove the sufficient part of the Theorem 4.2, and sufficient large value of ν, we have

$$\log^+ A_n^* > (1-\epsilon) BU\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right),$$

$$\frac{\lambda_{n_\nu}}{(1-\epsilon)B} > \frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*} U\left(\frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*}\right).$$

Since $r = W(t)$ and $t = rU(r)$, are two reciprocally inverse functions and monotonically increasing function, we get

$$W\left(\frac{\lambda_{n_\nu}}{(1-\epsilon)B}\right) > \frac{\lambda_{n_\nu}}{\log^+ A_{n_\nu}^*},$$

i.e.,

$$\log^+ A_{n_\nu}^* > \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{B(1-\epsilon)}\right)}.$$

We take a positive real sequence $\{\sigma_\nu\}$ satisfying

$$\begin{aligned} \lambda_{n_\nu} &= \left(\frac{\rho}{\rho+1}\right)^{\rho+1} (1-\epsilon) BU\left(-\frac{1}{\sigma_\nu}\right) \frac{1}{\sigma_\nu}(1+\epsilon) \\ &= \rho(1-\epsilon^2)\frac{1}{\sigma_\nu} U\left(-\frac{1}{\sigma_\nu}\right). \end{aligned}$$

From Lemma 3.3, we have the sequence $\{\sigma_\nu\}$ monotonically decreasing to zero, then for any sufficiently small $\sigma < 0$, there exist $\nu \in \mathbb{N}$ such that $\sigma_{\nu+1} \le \sigma \le \sigma_\nu$. By Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned} \log^+ \mu(\sigma, F) &\ge \log^+ A_{n_\nu}^* + \lambda_{n_\nu}\sigma - \log A \\ &\ge \log^+ A_{n_\nu}^* + \lambda_{n_\nu}\sigma_\nu - \log A \\ &\ge \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{B(1-\epsilon)}\right)} + \lambda_{n_\nu}\sigma_\nu - \log A \\ &\ge (1+o(1))U\left(-\frac{1}{\sigma_{\nu+1}}\right) \\ &\ge (1+o(1))U\left(-\frac{1}{\sigma}\right). \end{aligned}$$

From (4.12), we have $\log^+ M_u(\sigma, F) \ge \log^+ \mu(\sigma, F) + \log \frac{1}{3}$. Then from this and the above inequality, we can get

$$\liminf_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} \ge \liminf_{\sigma \to 0^-} \frac{\log^+ \mu(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} \ge 1.$$

Using Theorem 4.1, we get

$$\lim_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{U\left(-\frac{1}{\sigma}\right)} = 1.$$

We prove the necessary part of the Theorem 4.2 in the following way.

If $\lim_{\sigma\to 0^-}\frac{log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})}=1$ by Theorem 4.1, we can easily get (4.14) of Theorem 4.2. Then we will prove (4.15) of Theorem 4.2 in the following. We take a positive decreasing sequence $\{\eta_i\}(0<\eta_i<1)$, $\eta_i\to 0(i\to\infty)$. Let

$$S_i=\left\{n:\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)}>1-\eta_i\right\}. \tag{4.16}$$

For all i, $S_i\neq\emptyset$, $S_i\subset S_{i-1}$. For each i, we arrange $n\in S_i$ in an increasing sequence $\{n_\nu^i\}_{\nu=1}^\infty$, then we consider two cases as follows.

Case I Suppose that $\lim_{\nu\to\infty}\frac{\lambda_{n_\nu^{(i)}+1}}{\lambda_{n_\nu^{(i)}}}=1$, for any i. Then there exist $N_i\in S_i(i\in\mathbb{N})$, when $n_\nu^{(i)}\geq N_i$, we have

$$\frac{\lambda_{n_\nu^{(i)}+1}}{\lambda_{n_\nu^{(i)}}}\leq 1+\eta_k. \tag{4.17}$$

Note $S_{i+1}\subset S_i$, take $N_{i+1}>N_i$, denoted by $S_i^{'}$, the subset of S_i,

$$S_i^{'}=\{n\in S_i: N_i\leq n\leq N_{i+1}\},$$

thus the element of $S_i^{'}$ satisfies (2.16) and (2.17).
Let $S=\bigcup_{i=1}^\infty S_i^{'}$ and arrange $n\in S_i^{'}$ in an increasing sequence $\{n_\nu\}$, (2.15) is proved.

Case II If there exist $i\in\mathbb{N}$ satisfying $\lim_{\nu\to\infty}\frac{\lambda_{n_\nu^{(i)}+1}}{\lambda_{n_\nu^{(i)}}}\neq 1$, then since $\lambda_{n_\nu^{(i)}+1}>\lambda_{n_\nu^{(i)}}$, we get $\lim_{\nu\to\infty}\frac{\lambda_{n_\nu^{(i)}+1}}{\lambda_{n_\nu^{(i)}}}>1$. Hence, there exist $\{n_{\nu_k}^i\}\subseteq\{n_\nu^i\}$ and $\delta\in(0,\frac{1}{2}(1+\frac{1}{\rho})^{-\rho})$ and it follows that

$$\frac{\lambda_{n_\nu^{(i)}+1}}{\lambda_{n_\nu^{(i)}}}>1+\delta,\qquad \nu=1,2,3,\ldots.$$

Let

$$n_1'=n_1^{(i)},n_2'=n_3^{(i)},\ldots,n_\nu'=n_{2\nu-1}^{(i)},\ldots$$
$$n_1'',=n_2^{(i)},n_2''=n_4^{(i)},\ldots,n_\nu''=n_{2\nu}^{(i)},\ldots,$$

where $\{n_\nu'\}$, $\{n_\nu''\}$ are two increasing sequences of positive integer, and $n_\nu''<n_{\nu+1}'$, $\lambda_{n_\nu''}>(1+\delta)\lambda_{n_\nu'},n=1,2,\ldots$. Take $\gamma=\frac{1}{2}\eta_i>0$ and from (4.17), for any sufficiently large ν, when $n\ni S_i$ satisfies $n_\nu'<n<n_\nu''$, we get

$$\frac{\log^+ A_n^*}{BU\left(\frac{\lambda_n}{\log^+ A_n^*}\right)}\leq 1-\eta_i<1-\gamma,$$

$$thus\qquad \log^+ A_n^*<\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1-\gamma)}\right)},$$

$$i.e.\qquad \log^+(A_n^*e^{\sigma\lambda_n})<\frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1-\gamma)}\right)}+\sigma\lambda_n.$$

For σ sufficiently reaching to zero and from Lemma 4.2, we get

$$\log^+(A_n^*e^{\sigma\lambda_n})\leq(1-\gamma)(1+o(1))U(-\frac{1}{\sigma}),\qquad n_\nu'<n<n_\nu''. \tag{4.18}$$

Take $\mu>0$

$$\frac{1+\mu}{1+\delta}<1-\eta,\qquad 0<\eta<1.$$

Let $\sigma_\nu=\left[W\left(\frac{\lambda_{n_\nu''}}{B(1+\mu)}\right)\right]^{-1}$, then we have the sequence $\{\sigma_\nu\}$ monotonic decreasing to zero,

$$\lambda_{n_\nu''}=B(1+\mu)\left(-\frac{1}{\sigma_\nu}\right)U\left(-\frac{1}{\sigma_\nu}\right). \tag{4.19}$$

From (2.19) $\mu > 0$ and from (2.14) there exist a positive integer $n_0 \in \mathbb{N}$,

$$\log^+(A_n^* e^{\sigma\lambda_n}) < \frac{\lambda_n}{W\left(\frac{\lambda_n}{B(1+\mu)}\right)} + \lambda_n \sigma, \qquad n \geq n_0. \tag{4.20}$$

When $n \geq n''_\nu > n_0$, then $\lambda_n \geq \lambda_{n''_\nu}$. Since $W(t)$ is an increasing function, from (4.19) and (4.20), we have

$$\log^+(A_n^* e^{\sigma\lambda_n}) < \lambda_n \left(\frac{1}{W\left(\frac{\lambda_{n''_\nu}}{B(1+\mu)}\right)} + \sigma_\nu \right) = 0. \tag{4.21}$$

From Lemma 3.2 and for sufficiently large ν when $n_0 \leq n \leq n'_\nu$ there exist $\lambda_n \leq \lambda_{n'_\nu} < \frac{1}{1+\delta}\lambda_{n''_\nu}$ then we have

$$\begin{aligned} \log^+(A_n^* e^{\sigma\lambda_n}) &\leq \frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{W\left(\frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{B(1+\mu)}\right)} + \frac{1}{1+\delta}\lambda_{n''_\nu}\sigma_\nu \\ &\leq (1-\epsilon)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right), \end{aligned}$$

where $n \geq n_0$. From (4.17), (4.20) and (4.21), we get

$$\log^+(A_n^* e^{\sigma\lambda_n}) < (1-\alpha)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right), \qquad 0 < \alpha = min(\epsilon, \gamma) < 1.$$

Hence

$$\mu(\sigma_\nu, F) \leq C \exp\left[(1-\alpha)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right)\right], \tag{4.22}$$

where C is a positive real number.

From (4.10) and for any $\epsilon > 0$, we have

$$\begin{aligned} M_u(\sigma_\nu, F) &\leq \sum_{n=0}^{\infty} A_n^* e^{\sigma\lambda_n} \\ &\leq \mu((1-\epsilon)\sigma_\nu, F) \sum_{n=0}^{\infty} e^{\epsilon\sigma_\nu\lambda_n}. \end{aligned} \tag{4.23}$$

Making an appeal to process of proving Theorem 4.2, (4.23) and (4.22), gives

$$M_u(\sigma_\nu, F) \leq C_1 \exp\left[(1-\alpha)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right)\right]\left[C_2 + \frac{1}{1+\epsilon\sigma_\nu}T^{1+\epsilon\sigma_\nu}\right],$$

where $T = \left[e^{\left(\frac{2}{\epsilon\sigma_\nu}\right)^{\rho_1}}\right]$ and C_1, C_2 are two constants. Thus, when ν is sufficiently large, we get

$$\begin{aligned} \log^+ M_u(\sigma_\nu, F) &\leq (1-\alpha)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right) + (1-\epsilon)\left(\frac{2}{\epsilon\sigma_\nu}\right)^{\rho_1} + C_3 \\ &\leq \left(1 - \frac{\alpha}{2}\right)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right), \end{aligned}$$

where C_3 is a constant.

Thus

$$\limsup_{\nu\to\infty} \frac{\log^+ M_u(\sigma_\nu, F)}{U\left(-\frac{1}{\sigma_\nu}\right)} \leq 1 - \frac{\alpha}{2},$$

which gives the contradiction of the Theorem.

Thus, the necessary part of the Theorem is proved. □

We establish some relation between error $E_n(F, \beta)$ and growth of $F(s)$.

Theorem 4.3. *Let $F(s) \in L$ be of the finite order $\rho (0 < \rho < \infty)$ and $-\infty < \beta < 0$, then*

$$\limsup_{n\to\infty} \frac{\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}{BU\left(\frac{\lambda_{n+1}}{\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}\right)} = 1.$$

Theorem 4.4. *Let $F(s) \in L$ be of the finite order $\rho (0<\rho<\infty)$ and $-\infty<\beta<0$, then*

$$\lim_{\sigma\to 0^-} \frac{\log^+ M_u(\sigma,F)}{U\left(-\frac{1}{\sigma}\right)} = 1 \iff \limsup_{n\to\infty} \frac{\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}{BU\left(\frac{\lambda_{n+1}}{\log^+[E_n(F,\beta)\exp(-\beta\lambda_{n+1})]}\right)} = 1;$$

there exists a increasing, positive integer $\{n_\nu\}$ satisfying

$$\lim_{\nu\to\infty} \frac{\log^+[E_{n_\nu}(F,\beta)\exp(-\beta\lambda_{n_\nu+1})]}{BU\left(\frac{\lambda_{n_\nu+1}}{\log^+[E_{n_\nu}(F,\beta)\exp(-\beta\lambda_{n_\nu+1})]}\right)} = 1, \qquad \lim_{\nu\to\infty}\frac{\lambda_{n_\nu+1}}{\lambda_{n_\nu}} = 1.$$

Theorem 4.5. *If $F(s) \in L$ is of finite order then for any fixed real numberβ, $-\infty<\beta<0$ we have*

$$\limsup_{\sigma\to 0^+} \frac{\log^+ M(\sigma,F)}{U(-\frac{1}{\sigma})} = 1 \iff \limsup_{n\to+\infty} \psi_n(F,\beta,\lambda_n) = 1;$$

where

$$\psi_n(F,\beta,\lambda_n) = \frac{\log^+(E_{n-1}(F,\beta)e^{-\beta\lambda_n})}{BU\left(\frac{\lambda_n}{\log^+(E_{n-1}(F,\beta)e^{-\beta\lambda_n})}\right)}.$$

Proof. We prove the sufficient part of the Theorem. For convenience, let $E_{n-1} = E_{n-1}(F,\beta)$. Suppose that

$$\limsup_{n\to+\infty} \psi_n(F,\beta,\lambda_n) = \limsup_{n\to+\infty} \frac{\log^+(E_{n-1}e^{-\beta\lambda_n})}{BU\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{-\beta\lambda_n})}\right)} = 1. \tag{4.24}$$

Then for sufficiently large positive integer n, we have

$$\log^+(E_{n-1}e^{-\beta\lambda_n}) < (1+\epsilon)BU\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{-\beta\lambda_n})}\right),$$

i.e.

$$\lambda_n < (1+\epsilon)B\frac{\lambda_n}{\log^+(E_{n-1}e^{-\beta\lambda_n})}U\left(\frac{\lambda_n}{\log^+(E_{n-1}e^{-\beta\lambda_n})}\right).$$

Since $r = W(t)$ and $t = rU(r)$ are two reciprocal inverse functions and monotonically increasing functions, therefore

$$W\left(\frac{\lambda_n}{(1+\epsilon)B}\right) \le \frac{\lambda_n}{\log^+(E_{n-1}e^{-\beta\lambda_n})}.$$

Hence

$$\log^+(E_{n-1}e^{-\beta\lambda_n}) \le \frac{\lambda_n}{W\left(\frac{\lambda_n}{(1+\epsilon)B}\right)}.$$

Then there exists a positive constant G, such that

$$E_{n-1}e^{-\beta\lambda_n} \le G\exp\left(\frac{\lambda_n}{W\left(\frac{\lambda_n}{(1+\epsilon)B}\right)}\right). \tag{4.25}$$

For any $\beta<0$, then from the definition of $E_k(\beta,F)$, then exist $P_1 \in \prod_{n-1}$, satisfying

$$\|F-P_1\| \le 2E_{n-1}. \tag{4.26}$$

Since

$$\begin{aligned} A_n^*\exp(\beta\lambda_n) &= \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\{ity\}d\alpha(y)\right|\exp(\beta\lambda_n) \\ &\le \sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^x \exp\{(\beta+it)y\}d\alpha(y)\right| \\ &\le \sup_{-\infty<t<+\infty}\left|\int_{\lambda_n}^\infty \exp\{(\beta+it)y\}d\alpha(y)\right|, \end{aligned}$$

then for any $P\in\prod_{n-1}$, we get

$$\begin{aligned} A_n^*\exp(\beta\lambda_n) &\le |F(\beta+it)-P(\beta+it)| \\ &\le \|F-P\|_\beta. \end{aligned} \tag{4.27}$$

Hence for any $\beta < 0$, and $F(s) \in \overline{L_\beta}$, it follows from that (4.26) and (4.27),

$$A_n^* \exp(\beta\lambda_n) \le 2E_{n-1}, \qquad i.e\ A_n^* \le 2E_{n-1}\exp(-\beta\lambda_n)$$
$$i.e,\ A_n^* e^{-\sigma\lambda_n} \le 2E_{n-1}e^{(\sigma-\beta)\lambda_n}. \tag{4.28}$$

Now from (4.25) and (4.28), we get

$$A_n^* \exp(\beta\lambda_n) \le G\exp\left(\frac{\lambda_n}{W\left(\frac{\lambda_n}{(1+\epsilon)B}\right)}\right).$$

So by Lemma 3.2 and follows that from (1.4), we get

$$M_u(\sigma, F) \le G\exp\left[(1+\epsilon)U\left(\frac{1}{(1-\epsilon)\sigma}\right)(1+0(1))\right],$$
$$\log^+ M_u(\sigma, F) \le (1+2\epsilon)U\left(-\frac{1}{\sigma}\right)(1+0(1)). \tag{4.29}$$

As $\epsilon > 0$, we get

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U\left(-\frac{1}{\sigma}\right)} = 1.$$

Suppose that

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U\left(-\frac{1}{\sigma}\right)} = T < 1. \tag{4.30}$$

Then, there exists any real number $\eta > 0$ and $T + 4\eta < 1$ and for any sufficiently small $\sigma < 0$, we get

$$\log^+ M_u(\sigma, F) < (T+\eta)U\left(-\frac{1}{\sigma}\right). \tag{4.31}$$

Since,

$$\begin{aligned} E_{n-1}(\beta, F) &\le \|F - P_{n-1}\|_\beta \\ &\le |F(\beta+it) - P_{n-1}(\beta+it)| \\ &\le \left|\int_{\lambda_n}^{+\infty}\exp\{(\beta+it)y\}d\alpha(y)\right|. \end{aligned} \tag{4.32}$$

For $\beta < \sigma < 0$ and

$$\left|\int_{\lambda_k}^{+\infty}\exp\{(\beta+it)y\}d\alpha(y)\right| = \lim_{b\to+\infty}\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right|.$$

set,

$$I_{j+k}(b\ ;it) = \int_{\lambda_{j+k}}^{b}\exp(-ity)d\alpha(y),\ \ (\lambda_{j+k} \le b \le \lambda_{j+k+1}),$$

then, we have $|I_{j+k}(b\ ;it)| \le A_{j+k}^*$. Thus,

$$\begin{aligned} &\left|\int_{\lambda_k}^{b}\exp\{(\beta+it)y\}d\alpha(y)\right| \\ &\quad = \left|\sum_{j=k}^{n+k-1}\int_{\lambda_j}^{\lambda_{j+1}}\exp(\beta y)dyI_j(y\ ;it) + \int_{\lambda_{n+k}}^{b}\exp(\beta y)dyI_{n+k}(y\ ;it)\right| \\ &\le \sum_{j=k}^{n+k-1}[A_j^*e^{\lambda_{j+1}\beta} + A_j^*(e^{\lambda_{j+1}\beta} - e^{\lambda_j\beta})] + 2e^{\beta\lambda_{n+k+1}}A_{n+k}^* - e^{\beta\lambda_{n+k}}A_{n+k}^* \\ &\le 2\sum_{j=k}^{n+k}A_j^*e^{\beta\lambda_{j+1}}. \end{aligned}$$

Because $b \to +\infty$ as $n \to +\infty$, it follows that

$$\left|\int_{\lambda_k}^{+\infty} \exp\{(\beta + it)y\} d\alpha(y)\right| \leq 2 \sum_{n=k}^{+\infty} A_n^* \exp\{\beta\lambda_{n+1}\}. \tag{4.33}$$

From (1.3), there exists $\epsilon > 0$ satisfying $0 < \lambda_{n+1} - \lambda_n = \epsilon$ $(n = 1, 2, 3, \ldots,)$. When σ is sufficiently close to 0^-, it follows that$e^{-\epsilon\sigma} < 2$. When $x > \lambda_n$, it follows that

$$\int_{\lambda_n}^{x} \exp\{ity\} d\alpha(y) = \int_{\lambda_n}^{x} e^{-\sigma y} d_y I(y; \sigma + it)$$
$$= (e^{-\sigma y} I(y; \sigma + it))_{\lambda_n}^{x} + \sigma \int_{\lambda_n}^{x} e^{-\sigma y} I(y; \sigma + it) dy.$$

Then for $\sigma < 0$, we get

$$\left|\int_{\lambda_n}^{x} \exp\{ity\} d\alpha(y)\right| \leq M_u(\sigma, F) \left(|e^{-\sigma x} + e^{-\sigma\lambda_n}| + |e^{-\sigma x} - e^{-\sigma\lambda_n}|\right)$$
$$\leq 2M_u(\sigma, F) e^{-\sigma x}.$$

Then for any $\sigma < 0$ and any $x \in (\lambda_n, \lambda_{n+1})$, we have

$$\left|\int_{\lambda_n}^{x} \exp\{ity\} d\alpha(y)\right| \leq 2M_u(\sigma, F) e^{-\sigma x} e^{-\epsilon\sigma}$$
$$\leq 4M_u(\sigma, F) e^{-\sigma x}.$$
$$i.e. \qquad \mu(\sigma, F) \leq 4M_u(\sigma, F). \tag{4.34}$$

Now from (4.32), (4.33) and (4.34), we get

$$E_{n-1} \leq 2 \sum_{k=n}^{+\infty} A_{k-1}^* \exp\{\beta\lambda_k\} \leq 8M_u(\sigma, F) \sum_{k=n}^{+\infty} \exp\{(\beta - \sigma)\lambda_k\}. \tag{4.35}$$

From (1.3) we take $h'(0 < h' < h)$ such that $(\lambda_{n-1} - \lambda_n) \geq h'$ for $n \geq 0$. Then from (4.35) for $\sigma \geq \frac{\beta}{2}$, we get

$$E_{n-1} \leq 8M_u(\sigma, F) \exp\{(\beta - \sigma)\lambda_n\} \sum_{k=n}^{+\infty} \exp\{(\beta - \sigma)(\lambda_k - \lambda_n)\}$$
$$\leq 8M_u(\sigma, F) \exp\{(\beta - \sigma)\lambda_n\} \left(1 - \exp \frac{\beta}{2} h'\right)^{-1}$$
$$i.e. \;\; E_{n-1} \leq KM_u(\sigma, F) \exp\{(\beta - \sigma)\lambda_n\}, \tag{4.36}$$

where K is constant. Thus for sufficiently small $\sigma < 0$ and $-\infty < \beta < \sigma < 0$, we get

$$M_u(\sigma, F) \geq K_1 E_{n-1}(F, \beta) \exp\{-(\beta - \sigma)\lambda_n\} = K_1 E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\} e^{\sigma\lambda_n}. \tag{4.37}$$

Where $K_2 = \left(1 - \exp \frac{\beta}{2} h'\right)$. Hence it follows that from (4.31) and (4.37),

$$\log^+[E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\} e^{\sigma\lambda_n}] \leq \log^+ M_u(\sigma, F) \leq (T + \eta) U\left(-\frac{1}{\sigma}\right).$$
$$\log^+[E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}] \leq (T + \eta) U\left(-\frac{1}{\sigma}\right) - \sigma\lambda_n. \tag{4.38}$$

When n is sufficiently large from Lemma 3.1, we get

$$\log^+[E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}] \leq (T + \eta)^{\frac{1}{\rho+1}} \frac{\rho + 1}{\rho^{\frac{\rho}{\rho+1}}} \frac{\lambda_n}{W(\lambda_n)} (1 + \eta)$$
$$= (B(1 + \eta))^{\rho+1} \frac{\lambda_n}{W(\lambda_n)} (1 + \eta)$$
$$W(\lambda_n) \leq \frac{\lambda_n}{\log^+[E_{n-1}(F, \beta) \exp\{-\beta\lambda_n\}]} (B(1 + \eta))^{\rho+1} (1 + \eta).$$

For $x > x_0 = r_0' U(r_0')$, the function $W(x)$ is monotonically increasing, we get

$$\lambda_n \leq \frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\left(B(1+\eta)\right)^{\rho+1}(1+\eta) \\ U\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}(B(1+\eta_1))^{\rho+1}(1+\eta_1)\right) \\ \leq \frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\left(B(1+\eta)\right)(1+\eta)^{\rho+1}(1+0(1)) \\ U\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\right).$$

Therefore

$$\frac{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}{BU\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\right)} \leq (T+5\eta)^{\rho+2}(1+0(1)).$$

Thus

$$\limsup_{n\to\infty}\frac{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}{BU\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\beta)\exp\{-\beta\lambda_n\}]}\right)} \leq T < 1,$$

which gives the contradiction.

Thus the sufficient part is proved. □

5 Conclusion

In this paper, we investigate the growth of $M_u(\sigma, F)$ defined by Laplace-Stieltjes transforms of proximate order which is convergent in the half plane. We obtain the error in approximating Laplace-Stieltjes transform of finite order in the half plane by Dirichlet polynomials.

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