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LORENTZ TRANSFORMATIONS AND DIRAC SPINOR

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Abstract

This work constructs Lorentz transformations and Dirac spinor mappings using two complex 2×2 matrices under a unimodular condition. The approach leverages Pauli matrices and Hermitian conjugates to derive the spinor transformation matrix. It demonstrates equivalence with Macfarlanes formula and Veblens construction. The framework clarifies the role of InfeldVan der Waerden symbols in linking spacetime and spinor structures. This compact method highlights the symmetry principles underlying relativistic quantum theory.

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1 Introduction

Lorentz transformations and Dirac spinors are central to the formulation of relativistic quantum mechanics and field theory. These mathematical structures describe how physical quantities transform under changes in inertial reference frames, preserving the fundamental symmetries of spacetime. Dirac spinors, in particular, are essential in modelling spin-$\frac{1}{2}$ particles such as electrons, and their behaviour is governed by the Dirac equation using gamma matrices.

Traditionally, the construction of Lorentz transformations and spinor mappings has relied on tensor calculus and group-theoretic methods, which, while powerful, can be abstract and computationally intensive. This paper introduces a more accessible and elegant matrix-based approach using complex 2×2 matrices that satisfy a unimodular condition. These matrices allow for the direct construction of Lorentz mappings and the associated spinor transformation matrix S in the Dirac-Pauli representation.

The primary goal of this work is to provide a compact algebraic framework that connects classical tensorial formulations with modern matrix techniques. By employing Pauli matrices and Hermitian conjugates, the method reveals deep symmetry principles and clarifies the role of Infeld–van der Waerden symbols in linking spacetime geometry with spinor algebra. This approach not only reproduces known results such as Macfarlanes and Veblens constructions but also offers a streamlined alternative that is computationally tractable and conceptually transparent.

Although the mathematical formalism is advanced, the underlying ideas are broadly relevant to physicists and mathematicians interested in the foundations of relativistic theory. The paper aims to make these connections clearer and more intuitive, even for readers who may not specialize in spinor analysis or quantum field theory.

The arbitrary complex quantities $\alpha, \beta, \gamma, \delta$ verifying the condition $\alpha\delta - \beta\gamma = 1$, generate a Lorentz transformation $L = (L^{\mu}{}_{\nu})$ via the expressions:

$$\begin{aligned}
L_0^0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}), & L_0^1 &= \frac{1}{2}(\bar{\alpha}\gamma + \bar{\beta}\delta) + cc, & L_0^2 &= -\frac{i}{2}(\alpha\bar{\gamma} - \bar{\beta}\delta) + cc, \\
L_1^0 &= \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L_1^1 &= \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L_1^2 &= -\frac{i}{2}(\alpha\bar{\delta} + \beta\bar{\gamma}) + cc, \\
L_2^0 &= -\frac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L_2^1 &= -\frac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L_2^2 &= \frac{1}{2}(\bar{\alpha}\delta - \bar{\beta}\gamma) + cc, \\
L_3^0 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}), & L_3^1 &= \frac{1}{2}(\bar{\alpha}\gamma - \bar{\beta}\delta) + cc, & L_3^2 &= -\frac{i}{2}(\alpha\bar{\gamma} + \bar{\beta}\delta) + cc, \\
L_3^3 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}), & L_3^3 &= \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, & L_3^3 &= -\frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \\
L_3^3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}), & \alpha\delta - \beta\gamma &= 1,
\end{aligned} \tag{1.1}$$

where cc means the complex conjugate of all the previous terms (see [18, 1, 20, 17, 11, 2, 3, 12, 4, 14]).

On the other hand, the Dirac 4-spinor obeys the transformation law [13, 16]:

$$\tilde{\Psi}\left(\tilde{x}^{\mu}\right) = S(L)\Psi\left(x^{\mu}\right), \tag{1.2}$$

for a non-singular matrix S such that [in terms of the gamma matrices]:

$$L_{\nu}^{\mu}\gamma^{\nu} = S^{-1}\gamma^{\mu}S, \tag{1.3}$$

and we must determine a solution of (1.3) for a given Lorentz mapping. From (1.3) are immediate the relations [16, 15]:

$$L^{\mu}{}_{0} = \frac{1}{4}\operatorname{tr}\left(\gamma^{0}S^{-1}\gamma^{\mu}S\right), \quad L^{\mu}{}_{k} = -\frac{1}{4}\operatorname{tr}\left(\gamma^{k}S^{-1}\gamma^{\mu}S\right), \quad \mu = 0, \ldots, 3, \quad k = 1, 2, 3, \tag{1.4}$$

that is, if we know S then with (1.4) we can determine the Lorentz matrix. However, here we have the inverse problem, that is, to obtain S for a given Lorentz transformation, which will depend on the representation used for the γ^{μ} [14].

In this work we exhibit that the following complex matrices:

$$A = \frac{1}{2}\begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \quad E = \frac{1}{2}\begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \bar{\gamma} + \beta & \bar{\delta} - \alpha \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1 \tag{1.5}$$

allow $(L^{\mu}{}_{\nu})$ and S to be constructed.

2 Dirac representation

We have the Dirac equation for spin-1/2 particles [13, 16, 19] $[(x^{\mu}) = (t, x, y, z), \hbar = c = 1]$:

$$\left(i\gamma^{\mu}\partial_{\mu} - m_0\right)\Psi = 0, \quad i = \sqrt{-1}, \quad \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}, \tag{2.1}$$

where Ψ is a 4-spinor with the γ^{μ} matrices verifying the anticommutators:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}I_{4\times 4}, \quad (g^{\mu\nu}) = \mathrm{Diag}(1, -1, -1, -1)\}\,. \tag{2.2}$$

In the Dirac-Pauli (or standard) representation:

$$\gamma^{0} = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}, \quad \gamma^{j} = \begin{pmatrix} O & \sigma_j \\ -\sigma_j & O \end{pmatrix}, \quad j = 1, 2, 3, \tag{2.3}$$

with the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

the transformation law (1.2) for Ψ, under the orthochronic and proper Lorentz group $\tilde{x}^{\mu} = L^{\mu}{}_{\nu}x^{\nu}$, is governed by the matrix:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} A^{\dagger} & -E^{\dagger} \\ -E^{\dagger} & A^{\dagger} \end{pmatrix}. \tag{2.5}$$

in terms of (1.5) and their corresponding Hermitian matrices. Indeed, if in (1.3) or (1.4) we use (1.5), (2.3) and (2.5) we recover the expressions (1.1). Our result (2.5) is an alternative to the following explicit general formula obtained by Macfarlane [15, 5, 23]:

$$S=\frac{1}{4\sqrt{G}}\left[GI+\frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}L^{\mu\nu}L^{\alpha\beta}\gamma^{5}+i\Gamma\left(L^{2}\right)-i(2+\operatorname{tr}L)\Gamma(L)\right] \tag{2.6}$$

such that:

$$G=2(1+\operatorname{tr}L)+\frac{1}{2}\left[(\operatorname{tr}L)^{2}-\operatorname{tr}L^{2}\right],\quad \operatorname{tr}L=\sum_{\mu=0}^{3}L^{\mu}{}_{\mu},\operatorname{tr}L^{2}=\sum_{\nu,\alpha=0}^{3}L^{\nu}_{\alpha}L^{\alpha}_{\nu},$$

$$\Gamma(L)=\sum_{\mu,\nu=0}^{3}L^{\mu\nu}\sigma^{\mu\nu},\qquad \Gamma\left(L^{2}\right)=\sum_{\alpha,\mu,\nu=0}^{3}L_{\mu\alpha}L^{\alpha}{}_{\nu}\sigma^{\mu\nu}. \tag{2.7}$$

3 Lorentz transformations

Now, we exhibit that the matrices (1.5) and (2.4) allow obtain the Lorentz matrix, in fact:

$$L^{0}_{0}I=AA^{\dagger}+EE^{\dagger},\quad L^{j}{}_{0}I=\left(E^{\dagger}\sigma_{j}A\right)+\left(E^{\dagger}\sigma_{j}A\right)^{\dagger},\quad j=1,2,3$$

$$\begin{pmatrix} L^{0}{}_{3} & L^{0}{}_{1}-iL^{0}{}_{2}\\ L^{0}{}_{1}+iL^{0}{}_{2} & -L^{0}{}_{3}\end{pmatrix}=2E^{\dagger}A,$$

$$\begin{pmatrix} L^{j}{}_{3} & L^{j}{}_{1}-iL^{j}{}_{2}\\ L^{j}{}_{1}+iL^{j}{}_{2} & -L^{j}{}_{3}\end{pmatrix}=A^{\dagger}\sigma_{j}A+E^{\dagger}\sigma_{j}E, \tag{3.1}$$

which imply the relations (1.1). We may indicate some additional properties:

$$A^{\dagger}E=E^{\dagger}A,$$
$$AA^{\dagger}-EE^{\dagger}=I,$$
$$AA^{\dagger}=A^{\dagger}A,$$
$$EE^{\dagger}=E^{\dagger}E,$$
$$AE^{\dagger}-E^{\dagger}A=AE-EA,$$
$$\begin{pmatrix} L^{3}{}_{0} & L^{1}{}_{0}-iL^{2}{}_{0}\\ L^{1}{}_{0}+iL^{2}{}_{0} & -L^{3}{}_{0}\end{pmatrix}=2AE^{\dagger}=2EA^{\dagger},$$
$$\det A=\frac{1}{2}\left(L^{0}{}_{0}+1\right),$$
$$\det E=\frac{1}{2}\left(1-L^{0}{}_{0}\right),$$
$$\begin{pmatrix} \frac{1}{2}\left(L^{1}{}_{2}-L^{2}{}_{1}\right) & L^{2}{}_{3}-L^{3}{}_{2}-i\left(L^{3}{}_{1}-L^{1}{}_{3}\right)\\ L^{2}{}_{3}-L^{3}{}_{2}+i\left(L^{3}{}_{1}-L^{1}{}_{3}\right) & -\frac{1}{2}\left(L^{1}{}_{2}-L^{2}{}_{1}\right)\end{pmatrix}=i\left(A^{\dagger 2-A^{2}+E^{2}-E^{\dagger 2}}\right). \tag{3.2}$$

Remark: We consider that (3.1) are equivalent to the Veblen's formula [3, 21, 6, 8]:

$$L^{\mu}_{\nu}=\sigma^{\mu}{}_{A\dot{R}}B^{A}_{C}\sigma^{C\dot{E}}_{\nu}B^{\dagger\dot{R}}_{\dot{E}},\quad B=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad B^{\dagger}=\begin{pmatrix}\alpha^{*} & \gamma^{*}\\ \beta^{*} & \delta^{*}\end{pmatrix},\quad \alpha\delta-\beta\gamma=1, \tag{3.3}$$

in terms of the Infeld-van der Waerden symbols [22, 9, 10, 7].

4 Conclusion

In this work, we have presented a compact and algebraically rich method for constructing Lorentz transformations and Dirac spinor mappings using complex 2×2 matrices satisfying a unimodular condition. The derived spinor transformation matrix S, expressed in terms of Pauli matrices and Hermitian conjugates, aligns with established formulations such as Macfarlanes and Veblens constructions. This framework not only reproduces the Lorentz matrix components but also highlights the symmetry principles inherent in relativistic quantum theory. The use of Infeld–van der Waerden symbols further reinforces the connection between spacetime and spinor structures. Overall, the approach offers a streamlined alternative to traditional tensorial methods and opens avenues for further exploration in spinor geometry, quantum field theory, and computational physics.

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