ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha

Jñānābha, **Vol. 55(2) (2025), 147-156**

GROWTH PROPERTIES OF AN ANALYTIC FUNCTION OF SEVERAL COMPLEX VARIABLES IN THE UNIT POLYDISC ON THE BASIS OF RELATIVE ORDER

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(Received: November 06, 2024*; In format: January* 08, 2025*; Revised: August* 17, 2025*; Accepted: October* 22, 2025*)*

DOI: https://doi.org/10.58250/jnanabha.2025.55216

Abstract

In this paper, we study some comparative growth properties of composite analytic function of several complex variables, on the basis of relative order and relative lower order of an entire function with respect to an analytic function in the unit polydisc. Here we introduce some definitions related to relative order and relative lower order in terms of central index and study them critically.

2020 Mathematical Sciences Classification: 32A05, 32C15

Keywords and Phrases: Analytic function in the unit polydisc, maximum term, several complex variables, maximum modulus, composition, generalized relative order, generalized relative L order, generalized relative L^* order, central index.

1 Introduction

Let $f(z_1, z_2, ..., z_n)$ be an analytic function of the n complex variables defined as

$$f(z_1, z_2, ..., z_n) = \sum_{m_j=0, j=1,2,\cdots,n}^{\infty} a_{m_j} z_j^{m_j}, \tag{1.1}$$

where $z_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R}$.

Consider the maximum modulus principle of f, maximum term $\mu_f(r)$, central index $\nu_f(r)$ and unit polydisc U are defined in the n complex variables in the following way respectivily:

$$M_f(r_1, r_2, \cdots, r_n) := \max_{\{|z_j|=r_j; j=1,2,\cdots,n\}} |f(z_j)|, \tag{1.2}$$

$$\mu_f(r_1, r_2, \cdots, r_n) := \max_{m_j \geq 0, j=1,2,\cdots,n} |a_{m_j}| r_j^{m_j}, \tag{1.3}$$

$$\nu_f(r_1, r_2, \cdots, r_n) := \{m_k \mid \mu_f(r_1, r_2, \cdots, r_n) = |a_{m_k}| r^{m_k}\}, \tag{1.4}$$

$$U = \{(z_1, z_2, ...z_n) : |z_j| \leq 1, j = 1, 2...n\}. \tag{1.5}$$

The central index $\nu_f(r_1, r_2, \cdots, r_n) := \{m_k \mid \mu_f(r_1, r_2, \cdots, r_n) = |a_{m_k}| r^{m_k}\}$ so $|a_{\nu_f(r_1,r_2,\cdots,r_n)}| r^{\nu_f(r_1,r_2,\cdots,r_n)} = \mu_f(r_1, r_2, \cdots, r_n)$. Clearly $\mu_f(r_1, r_2, \cdots, r_n)$ is a non decreasing function and $\mu_f(r_1, r_2, \cdots, r_n) \leq M_f(r_1, r_2, \cdots, r_n)$.

Let g be an analytic function in the unit polydisc. Then the ratio $\frac{M_f(r_1,r_2,\cdots,r_n)}{M_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to 1$, $k = 1, 2, \cdots, n$ is called the growth of f with respect to g in term of maximum moduli. In fact $\mu_f(r_1, r_2, \cdots, r_n)$ is much weaker than $M_f(r_1, r_2, \cdots, r_n)$ in some sense. So from another angle of view $\frac{\mu_f(r_1,r_2,\cdots,r_n)}{\mu_g(r_1,r_2,\cdots,r_n)}$, where $r_k \to 1$, $k = 1, 2, \cdots, n$ is called growth of f with respect to g, in term of maximum terms, now in similar way we get growth of f with respect to g in terms of central index, $\frac{\nu_f(r_1,r_2,\cdots,r_n)}{\nu_g(r_1,r_2,\cdots,r_n)}$ where $r_k \to 1$, $k = 1, 2, \cdots, n$.

Rastogi [9], Biswas [3] and Pramanik [12] worked on central index. A non-constant analytic function of n-complex variables $f(z_1, z_2, ..., z_n)$ is said to have index-pair (p, q) if $b < v_n\rho^{(p,q)}(f) < \infty$ and $v_n\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number,where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < v_n\rho^{(p,q)}(f) < \infty$, then

$$\begin{cases} v_n\rho^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ v_n\rho^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ v_n\rho^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, ... \cdot \end{cases}$$

Similarly for $0 < v_n\lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} v_n\lambda^{(p-n,q)}(f) = \infty & \text{for } n < p, \\ v_n\lambda^{(p,q-n)}(f) = 0 & \text{for } n < q, \\ v_n\lambda^{(p+n,q+n)}(f) = 1 & \text{for } n = 1, 2, \ldots \cdot \end{cases}$$

The function $f(z_1, z_2, ...z_n)$ is said to be of regular (p, q) growth when (p, q)-th order and (p, q)-th lower order of $f(z_1, z_2, ...z_n)$ are the same. Functions which are not of regular (p, q) growth are said to be of irregular (p, q) growth.

Somasundaram and Thamizharasi [14] introduced the notions of L-order (L-lower order) for entire functions of single variable where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e., $L(ar) \sim L(r)$ as $r \to 1$ for every positive constant a. In the line of Somasundaram and Thamizharasi [14] one may introduce the definition of $(p, q, t)L$-th order and $(p, q, t)L$-th lower order for functions of n complex variables holomorphic in a unit polydisc in the following way

The details of the notions of maximum modulus , entire functions, growth, maximum term, central index for one variable appear in [1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 15].

To start our paper we just recall the following definitions:

2 Definitions

Definition 2.1 ([14]). *Let f and g be analytic functions in the unit polydisc U. The relative order of f with respect to g is defined by*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r)}{\log^{[q]} \frac{1}{1-r}},$$

and relative lower order is defined by

$$\lambda_g^{(p,q)}(f) = \underline{\lim_{r\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r)}{\log^{[q]} \frac{1}{1-r}}.$$

Definition 2.2 ([14]). *L order and L lower order for an analytic function in the unit polydisc U, where $L \equiv L(r)$ is a positive continuous function such that $L(ar) \sim L(r)$ as $r \to 1$, for every positive constant a, on the basis of maximum modulus $M(r, f)$. The relative L order of an analytic function f with respect to g, in terms of central index is defined by*

$$\rho_g^{(p,q)L}(f) = \overline{\lim_{r\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r)}{\log^{[q]}[\frac{1}{1-r}L(\frac{1}{1-r})]},$$

and relative L lower order is defined by

$$\lambda_g^{(p,q)L}(f) = \underline{\lim_{r\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r)}{\log^{[q]}[\frac{1}{1-r}L(\frac{1}{1-r})]}.$$

In the light of Definition 2.2, and from the concept of several complex variables [7] we would like to introduce the following definitions for several complex variables:

Definition 2.3. *Let f and g be analytic functions of n complex variables in the unit polydisc U. The relative order and relative lower order of f with respect to g are defined by*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)})},$$

and

$$\lambda_g^{(p,q)}(f) = \underline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)})},$$

respectively.

Definition 2.4. *Let f and g be analytic functions of n complex variables in the unit polydisc U. The relative L order and relative L lower order of f with respect to g, are defined by*

$$\rho_g^{(p,q)L}(f) = \overline{\lim_{(r_1,r_2,\cdots r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1,r_2\cdots,r_n)}{\log^{[q]}[(\frac{1}{1-r_1}\frac{1}{1-r_2}\cdots\frac{1}{1-r_n})L(\frac{1}{1-r_1}\frac{1}{1-r_2}\cdots\frac{1}{1-r_n})]},$$

and

$$\lambda_g^{(p,q)L}(f) = \underline{\lim_{(r_1,r_2,\cdots r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1,r_2\cdots,r_n)}{\log^{[q]}[(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)})L(\frac{1}{1-r_1}\frac{1}{1-r_2}\cdots\frac{1}{1-r_n})]},$$

respectively. Here idea of L order (respectively, L lower order) of an analytic function is defined in [12, 14].

Definition 2.5. *Let f and g be analytic functions of n complex variables in the unit polydisc U. The relative L^* order and relative L^* lower order of f with respect to g is defined by*

$$\rho_g^{(p,q)L^*}(f) = \overline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1,r_2,\cdots,r_n)}{\log^{[q]}[(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)})\exp^{L(\frac{1}{1-r_1}\frac{1}{1-r_2}\cdots\frac{1}{1-r_n})}]}.$$

Here idea of L^* order (respectively, L^* lower order) of an analytic function where L^* is nothing but a weaker assumption of L [12]. *The relative L^* lower order is defined by*

$$\lambda_g^{(p,q)L^*}(f) = \underline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_g^{-1}\nu_f(r_1,r_2,\cdots,r_n)}{\log^{[q]}[(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)})\exp^{L(\frac{1}{1-r_1}\frac{1}{1-r_2}\cdots\frac{1}{1-r_n})}]}.$$

3 Results

The results of the paper are following:

Theorem 3.1. *Let f, g and h be analytic functions in the unit polydisc U such that $0 < {\lambda_h}^{(p,q)}(f\circ g) \leq {\rho_h}^{(p,q)}(f\circ g) < \infty$ and $0 < {\lambda_h}^{(p,q)}(f) \leq {\rho_h}^{(p,q)}(f) < \infty$.*
Then

$$\begin{aligned}
\frac{{\lambda_h}^{(p,q)}(f\circ g)}{{\rho_h}^{(p,q)}f} &\leq \underline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \\
&\leq \min\left\{\frac{{\lambda_h}^{(p,q)}(f\circ g)}{{\lambda_h}^{(p,q)}f}, \frac{{\rho_h}^{(p,q)}(f\circ g)}{{\rho_h}^{(p,q)}f}\right\} \\
&\leq \max\left\{\frac{{\lambda_h}^{(p,q)}(f\circ g)}{{\lambda_h}^{(p,q)}f}, \frac{{\rho_h}^{(p,q)}(f\circ g)}{{\rho_h}^{(p,q)}f}\right\} \\
&\leq \overline{\lim_{(r_1,r_2,\cdots,r_n)\to 1}} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]} {\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)} \\
&\leq \frac{{\rho_h}^{(p,q)}(f\circ g)}{{\lambda_h}^{(p,q)}f}.
\end{aligned}$$

Proof. From the definition of relative order defined in (2.3), for arbitrary $\epsilon > 0$, we get the following

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n) \leq (\rho_h^{(p,q)}(f\circ g)+\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}), \tag{3.1}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n) \geq (\lambda_h^{(p,q)}(f\circ g)-\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}). \tag{3.2}$$

When $r_k \to 1$, where $k = 1,2,\cdots,n$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n) \leq (\lambda_h^{(p,q)}(f\circ g)+\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}), \tag{3.3}$$

and

$$\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)\geq(\rho_h^{(p,q)}(f\circ g)-\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}). \tag{3.4}$$

Similarly when we replace $f\circ g$ by f in the above equation, we get the following

$$\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)\leq(\rho_h^{(p,q)}(f)+\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}), \tag{3.5}$$

and

$$\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)\geq(\lambda_h^{(p,q)}(f)-\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}). \tag{3.6}$$

When $r_k\to 1$, where $k=1,2,\cdots,n$

$$\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)\leq(\lambda_h^{(p,q)}(f)+\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}), \tag{3.7}$$

and

$$\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)\geq(\rho_h^{(p,q)}(f)-\epsilon)\log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}). \tag{3.8}$$

From (3.2) and (3.5) it follows for sufficiently large value of $\frac{1}{1-r_1},\frac{1}{1-r_2},\cdots\frac{1}{1-r_n}$ that

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\geq\frac{\lambda_h^{(p,q)}(f\circ g)-\epsilon}{\rho_h^{(p,q)}(f)+\epsilon}.$$

Since ϵ is arbitrary

$$\underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\geq\frac{\lambda_h^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}(f)}. \tag{3.9}$$

From (3.3) and (3.6), we obtain

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\leq\frac{\lambda_h^{(p,q)}(f\circ g)+\epsilon}{\lambda_h^{(p,q)}(f)-\epsilon}.$$

Since ϵ is arbitrary

$$\underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots r_n)}\leq\frac{\lambda_h^{(p,q)}(f\circ g)}{\lambda_h^{(p,q)}(f)}, \tag{3.10}$$

From (3.1) and (3.8)

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\leq\frac{\rho_h^{(p,q)}(f\circ g)+\epsilon}{\rho_h^{(p,q)}(f)-\epsilon}.$$

Since ϵ is arbitrary

$$\underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\leq\frac{\rho_h^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}(f)}. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}\frac{\lambda_h^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}(f)}&\leq\underline{\lim}_{(r_1,r_2,\cdots r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\\&\leq\min\left\{\frac{\lambda_h^{(p,q)}(f\circ g)}{\lambda_h^{(p,q)}(f)},\frac{\rho_h^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}(f)}\right\}.\end{aligned} \tag{3.12}$$

From (3.2) and (3.7) for $r_k\to 1$, where $k=1,2,\cdots,n$, we get

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\geq\frac{\lambda_h^{(p,q)}(f\circ g)-\epsilon}{\lambda_h^{(p,q)}(f)+\epsilon}.$$

As $\epsilon>0$ is arbitrary, we obtain

$$\overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\geq\frac{\lambda_h^{(p,q)}(f\circ g))}{\lambda_h^{(p,q)}(f))}. \tag{3.13}$$

From (3.1) and (3.6), we obtain the following

$$\frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h^{(p,q)}(f\circ g)+\epsilon}{\lambda_h^{(p,q)}(f)-\epsilon}.$$

As $\epsilon > 0$, we obtain

$$\overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots r_n)} \leq \frac{\rho_h^{(p,q)}(f\circ g)}{\lambda_h^{(p,q)}(f)}. \tag{3.14}$$

Similarly from (3.4) and (3.5)

$$\frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^{(p,q)}(f\circ g)-\epsilon}{\rho_h^{(p,q)}(f)+\epsilon}.$$

As ϵ is arbitrary

$$\overline{\lim}_{(r_1,r_2,\cdots r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}(f)}. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we obtain

$$\begin{aligned}\max\left\{\frac{\lambda_h{}^{p,q}(f\circ g)}{\lambda_h{}^{(p,q)}f}, \frac{\rho_h{}^{(p,q)}(f\circ g)}{\rho_h{}^{(p,q)}f}\right\} &\leq \overline{\lim}_{(r_1,r_2,\cdots r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}\\ &\leq \frac{\rho_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}(f)}.\end{aligned} \tag{3.16}$$

Hence, from (3.12) and (3.16), we obtain

$$\begin{aligned}\frac{\lambda_h{}^{(p,q)}(f\circ g)}{\rho_h{}^{(p,q)}(f)} &\leq \underline{\lim}_{(r_1,r_2,\cdots r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}\\ &\leq \min\left\{\frac{\lambda_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}f}, \frac{\rho_h{}^{(p,q)}(f\circ g)}{\rho_h{}^{(p,q)}f}\right\}\\ &\leq \max\left\{\frac{\lambda_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}f}, \frac{\rho_h{}^{(p,q)}(f\circ g)}{\rho_h{}^{(p,q)}f}\right\}\\ &\leq \overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots, r_n)}\\ &\leq \frac{\rho_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}(f)}.\end{aligned}$$

□

Theorem 3.2. *Let f, g and h be analytic functions of n complex variables in the unit polydisc U such that $0 < \lambda_h{}^{(p,q)}(f\circ g) \leq \rho_h{}^{(p,q)}(f\circ g) < \infty$ and $0 < \lambda_h{}^{(p,q)}(f) \leq \rho_h{}^{(p,q)}(f) < \infty$.*
Then

$$\begin{aligned}\frac{\lambda_h{}^{(p,q)}(f\circ g)}{\rho_h{}^{(p,q)}(f)} &\leq \underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\lambda_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}f}\\ &\leq \overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^{(p,q)}(f\circ g)}{\lambda_h{}^{(p,q)}(f)}.\end{aligned}$$

Proof. The above theorem follows from (3.9), (3.10), (3.13) and (3.16). □

Theorem 3.3. *Let f, g and h be analytic functions of n complex variables in the unit polydisc U. such that $0 < \lambda_h{}^{(p,q)}(f\circ g) \leq \rho_h(f\circ g) < \infty$ and $0 < \lambda_h(f) \leq \rho_h{}^{(p,q)}(f) < \infty$.*
Then

$$\begin{aligned}\underline{\lim}_{(r_1,r_2,\cdots r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots r_n)} &\leq \frac{\rho_h{}^{(p,q)}(f\circ g)}{\rho_h^{(p,q)}f}\\ &\leq \overline{\lim}_{(r_1,r_2,\cdots r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]} \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots r_n)}.\end{aligned}$$

Proof. From the definition of relative order of f for $\epsilon > 0$ and $r_k \to 1$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n) \geq (\rho_h^{(p,q)}(f) - \epsilon) \log^{[p]}(r_1 r_2 \cdots r_n), \tag{3.17}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n) \leq (\rho_h^{(p,q)}(f \circ g) + \epsilon) \log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}) \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^{(p,q)}(f \circ g) + \epsilon}{\rho_h{}^{(p,q)} f - \epsilon}$$

As $\epsilon > 0$

$$\underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \leq \frac{\rho_h{}^{(p,q)}(f \circ g)}{\rho_h{}^{(p,q)} f} \tag{3.19}$$

Since $r_k \to 1$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n) \geq (\rho_h^{(p,q)}(f \circ g) - \epsilon) \log^{[q]}(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}) \tag{3.20}$$

Now combining form of (3.5) and (3.20) is the following:

$$\frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \geq \frac{\rho_h{}^{(p,q)}(f \circ g) + \epsilon}{\rho_h{}^{(p,q)} f - \epsilon}$$

As $\epsilon > 0$ is arbitrary

$$\overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots r_n)} \geq \frac{\rho_h^{(p,q)}(f \circ g)}{\rho_h^{(p,q)}(f)} \tag{3.21}$$

Hence, from (3.21) and (3.19), we obtain

$$\begin{aligned}\underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} &\leq \frac{\rho_h{}^{(p,q)}(f \circ g)}{\rho_h{}^{(p,q)} f} \\ &\leq \overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_{f\circ g}^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots r_n)}.\end{aligned}$$

□

Theorem 3.4. *Let f, g and h be analytic functions in the unit polydisc U, such that $0 < \lambda_h{}^{(p,q)L}(f \circ g) \leq \rho_h{}^{(p,q)L}(f \circ g) < \infty$ and $0 < \lambda_h{}^{(p,q)L}(f) \leq \rho_h{}^{(p,q)L}(f) < \infty$.*
Then

$$\begin{aligned}\frac{\lambda_h{}^{(p,q)L}(f \circ g)}{\rho_h{}^{(p,q)L}(f)} &\leq \underline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \\ &\leq \min\left\{\frac{\lambda_h{}^{(p,q)L}(f \circ g)}{\lambda_h{}^{(p,q)L} f}, \frac{\rho_h{}^{(p,q)L}(f \circ g)}{\rho_h{}^{(p,q)L} f}\right\} \\ &\leq \max\left\{\frac{\lambda_h{}^{(p,q)L}(f \circ g)}{\lambda_h{}^{(p,q)L} f}, \frac{\rho_h{}^{(p,q)L}(f \circ g)}{\rho_h{}^{(p,q)L} f}\right\} \\ &\leq \overline{\lim}_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h{}^{-1}\nu_f(r_1, r_2, \cdots r_n)} \\ &\leq \frac{\rho_h{}^{(p,q)L}(f \circ g)}{\lambda_h{}^{(p,q)L}(f)}.\end{aligned}$$

Proof. From the definition (2.4) and arbitrary $\epsilon > 0$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq \left(\rho_h^{(p,q)L}(f\circ g)+\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right), \tag{3.22}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \geq \left(\lambda_h^{(p,q)L}(f\circ g)-\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right). \tag{3.23}$$

Since $r_k \to 1$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n) \leq \left(\lambda_h^{(p,q)L}(f\circ g)+\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right), \tag{3.24}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n) \geq \left(\rho_h^{(p,q)L}(f\circ g)-\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right). \tag{3.25}$$

Similarly for the function f, we obtain

$$\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \cdots, r_n) \leq \left(\rho_h^{(p,q)L}(f)+\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right), \tag{3.26}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \ldots, r_n) \geq \left(\lambda_h^{(p,q)L}(f)-\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right). \tag{3.27}$$

Since $r_k \to 1$, where $k = 1, 2, \cdots, n$

$$\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \cdots, r_n) \leq \left(\lambda_h^{(p,q)L}(f)+\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right), \tag{3.28}$$

and

$$\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \cdots, r_n) \geq \left(\rho_h^{(p,q)L}(f)-\epsilon\right)$$
$$\log^{[q]}\left(\frac{1}{(1-r_1)(1-r_2).....(1-r_n)}L\left(\frac{1}{(1-r_1)}\frac{1}{(1-r_2)}\cdots\frac{1}{(1-r_n)}\right)\right). \tag{3.29}$$

From (3.23) and (3.26) it follows for sufficiently large value of $(r_1, r_2, \cdots, r_n)$

$$\frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \cdots, r_n)} \geq \frac{\lambda_h^{(p,q)L}(f\circ g)-\epsilon}{\rho_h^{(p,q)L}(f)+\epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\varliminf_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]} \nu_h^{-1}\nu_{f}(r_1, r_2, \cdots r_n)} \geq \frac{\lambda_h^{(p,q)L}(f\circ g)}{\rho_h^{(p,q)L}(f)}. \tag{3.30}$$

From (3.24) and (3.27), we obtain

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \dots r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\lambda_h^{(p,q)L}(f\circ g)+\epsilon}{\lambda_h^{(p,q)L}(f)-\epsilon}.$$

Since $\epsilon > 0$ is arbitrary

$$\varliminf_{r_1, r_2, \cdots r_n \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\lambda_h^{(p,q)L}(f\circ g)}{\lambda_h^{(p,q)L}(f)}. \tag{3.31}$$

From (3.22) and (3.29), we obtain

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h^{(p,q)L}(f\circ g)+\epsilon}{\rho_h^{(p,q)L}(f)-\epsilon}.$$

As $\epsilon > 0$

$$\varliminf_{(r_1, r_2, \cdots, r_n) \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h^{(p,q)L}(f\circ g)}{\rho_h^{(p,q)L}(f)}. \tag{3.32}$$

From (3.30), (3.31) and (3.32), we get the following

$$\begin{aligned}\frac{\lambda_h^{(p,q)L}(f\circ g)}{\rho_h^{(p,q)L}(f)} &\le \varliminf_{r_1, r_2, \cdots, r_n \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \\ &\le \min\left\{\frac{\lambda_h^{(p,q)L}(f\circ g)}{\lambda_h^{(p,q)L}(f)}, \frac{\rho_h^{(p,q)L}(f\circ g)}{\rho_h^{(p,q)L}(f)}\right\}.\end{aligned} \tag{3.33}$$

From (3.23) and (3.28) for $r_k \to 1$, where $k = 1, 2, \cdots, n$, then

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\lambda_h^{(p,q)L}(f\circ g)-\epsilon}{\lambda_h^{(p,q)L}(f)+\epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\varlimsup_{(r_1, r_2, \cdots, r_n) \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \dots, r_n)} \ge \frac{\lambda_h^{(p,q)L}(f\circ g)}{\lambda_h^{(p,q)L}(f)}. \tag{3.34}$$

From (3.22) and (3.27)

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h^{(p,q)L}(f\circ g)+\epsilon}{\lambda_h^{(p,q)L}(f)-\epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\varlimsup_{(r_1, r_2, \cdots, r_n) \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h^{(p,q)L}(f\circ g)}{\lambda_h^{(p,q)L}(f)}. \tag{3.35}$$

Similarly from (3.25) and (3.26), we get the following:

$$\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\rho_h^{(p,q)L}(f\circ g)-\epsilon}{\rho_h^{(p,q)L}(f)+\epsilon}.$$

As $\epsilon > 0$ is arbitrary

$$\varlimsup_{(r_1, r_2, \cdots, r_n) \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)} \ge \frac{\rho_h^{(p,q)L}(f\circ g)}{\rho_h^{(p,q)L}(f)}. \tag{3.36}$$

From (3.34), (3.35) and (3.36) we obtain

$$\max\left\{\frac{\lambda_h{}^{(p,q)L}(f\circ g)}{\lambda_h{}^{(p,q)L} f}, \frac{\rho_h{}^{(p,q)L}(f\circ g)}{\rho_h{}^{(p,q)L} f}\right\} \le \varlimsup_{(r_1, r_2, \cdots, r_n) \to 1} \frac{\log^{[p]}\nu_h^{-1}\nu_f(r_1, r_2, \cdots, r_n)}{\log^{[p]}\nu_h^{-1}\nu f(r_1, r_2, \cdots, r_n)} \le \frac{\rho_h{}^{(p,q)L}(f\circ g)}{\lambda_h{}^{(p,q)L}(f)}. \tag{3.37}$$

From (3.33) and (3.37)

$$\begin{aligned}\frac{{\lambda_h}^{(p,q)L}(f\circ g)}{{\rho_h}^{(p,q)L}(f)} &\le \varliminf_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\\ &\le \min\left\{\frac{{\lambda_h}^{(p,q)L}(f\circ g)}{{\lambda_h}^{(p,q)L}f},\frac{{\rho_h}^{(p,q)L}(f\circ g)}{{\rho_h}^{(p,q)L}f}\right\}\\ &\le \max\left\{\frac{{\lambda_h}^{(p,q)L}(f\circ g)}{{\lambda_h}^{(p,q)L}f},\frac{{\rho_h}^{(p,q)L}(f\circ g)}{{\rho_h}^{(p,q)L}f}\right\}\\ &\le \varlimsup_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\ldots,r_n)}{\log^{[p]}{\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\\ &\le \frac{{\rho_h}^{(p,q)L}(f\circ g)}{{\lambda_h}^{(p,q)L}(f)}.\end{aligned}$$

□

From the above theorems we can obtain the following corollaries:

Corollary 3.1. *Let f, g and h be analytic functions in the unit polydisc U such that $0<{\lambda_h}^{(p,q)L}(f\circ g)\le {\rho_h}^{(p,q)L}(f\circ g)<\infty$ and $0<{\lambda_h}^{(p,q)L}(f)\le{\rho_h}^{(p,q)L}(f)<\infty$.*
Then

$$\begin{aligned}\frac{{\lambda_h}^{(p,q)L}(f\circ g)}{{\rho_h}^{(p,q)L}(f)} &\le \varliminf_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\le\frac{{\lambda_h}^{(p,q)L}(f\circ g)}{{\lambda_h}^{(p,q)L}f}\\ &\le \varlimsup_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}{\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\le\frac{{\rho_h}^{(p,q)L}(f\circ g)}{{\lambda_h}^{(p,q)L}(f)}.\end{aligned}$$

Proof. When we take L-lower order and L-order in Theorem 3.2 then we get the corollary. □

Corollary 3.2. *Let f, g and h be analytic functions in the unit polydisc U such that $0<{\lambda_h}^{(p,q)L}(f\circ g)\le {\rho_h}^{(p,q)L}(f\circ g)<\infty$ and $0<{\lambda_h}^{(p,q)L}(f)\le{\rho_h}^{(p,q)L}(f)<\infty$.*
Then

$$\begin{aligned}\liminf_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)} &\le \frac{{\rho_h}^{(p,q)L}(f\circ g)}{{\rho_h}^{(p,q)L}f}\\ &\le \limsup_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}{\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}.\end{aligned}$$

Proof. When we take L-order and L- lower order in Theorem 3.3 we can get the result. □

Corollary 3.3. *Let f, g and h be analytic functions in the unit polydisc U such that $0<{\lambda_h}^{(p,q)L^*}(f\circ g)\le {\rho_h}^{(p,q)L^*}(f\circ g)<\infty$ and $0<{\lambda_h}^{(p,q)L^*}(f)\le{\rho_h}^{(p,q)L^*}(f)<\infty$.*
Then

$$\begin{aligned}\frac{{\lambda_h}^{(p,q)L^*}(f\circ g)}{{\rho_h}^{(p,q)L^*}(f)} &\le \liminf_{(r_1,r_2,\cdots,r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots r_n)}\le\frac{{\lambda_h}^{(p,q)L^*}(f\circ g)}{{\lambda_h}^{(p,q)L^*}f}\\ &\le \limsup_{(r_1,r_2,\cdots r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots r_n)}{\log^{[p]}{\nu_h}^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\le\frac{{\rho_h}^{(p,q)L^*}(f\circ g)}{{\lambda_h}^{(p,q)L^*}(f)}.\end{aligned}$$

Proof. When we take L^* order and L^* lower order in Theorem 3.2 we can get the result. □

Corollary 3.4. *Let f, g and h be analytic functions in the unit polydisc U such that $0<{\lambda_h}^{(p,q)L^*}(f\circ g)\le {\rho_h}^{(p,q)L^*}(f\circ g)<\infty$ and $0<{\lambda_h}^{(p,q)L^*}(f)\le{\rho_h}^{(p,q)L^*}(f)<\infty$.*
Then

$$\liminf_{(r_1,r_2,\cdots r_n)\to 1}\frac{\log^{[p]}\nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots r_n)}{\log^{[p]}\nu_h^{-1}\nu_f(r_1,r_2,\cdots,r_n)}\le\frac{{\rho_h}^{(p,q)L^*}(f\circ g)}{{\rho_h}^{(p,q)L^*}f}$$

$$\leq \limsup_{(r_1,r_2,\cdots,r_n)\to 1} \frac{\log^{[p]} \nu_h^{-1}\nu_{f\circ g}(r_1,r_2,\cdots,r_n)}{\log^{[p]} \nu_h{}^{-1}\nu_f(r_1,r_2,\ldots,r_n)}.$$

Proof. When we take L^* order and L^* lower order in Theorem 3.3 we can get the result. □

4 Conclusion

In this paper we have established some inequalities between relative order and relative lower order of analytic functions of several complex variables in the unit polydisc in terms of central index. Further we have obtained some corollaries of the above theorems.

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