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MODELLING ECO-EVOLUTIONARY DYNAMICS OF AN INVASIVE PLANT UNDER DUAL PRESSURE OF NATIVE HERBIVORES AND SOIL MICROBIAL COMPETITION

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Abstract

Invasive plant species pose a significant threat to biodiversity, ecosystem stability and agricultural productivity. This study present a novel mathematical model to explore the Eco-evolutionary dynamics of an invasive plant under simultaneous pressures from native herbivores and soil microbial competitors. The model incorporates an adaptation trait that evolves in herbivores based on invasive biomass, along with competition from native soil microbes that suppress the invader's root system. The system is analyzed using tools from dynamical system theory, including equilibrium analysis, boundedness, stability through jacobian eigenvalue analysis, and bifurcation due to trait thresholds. Result reveal the existence of two biological meaningful equilibrium states and highlight the presence of a transcritical bifurcation triggered by the invasive biomass crossing a critical threshold. Numerical simulations show how Eco-evolutionary feedback loop can suppress invasions and stabilize community dynamics over time. This study underscores the importance of integrating adaptive behavior and below-ground interactions in ecological modeling to devise more effective invasive species management strategies.

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Keywords and Phrases: Invasive plant, Plant-Herbivore, Boundedness, Stability.

1 Introduction

Biological invasions are among the pressing ecological challenges of the 21st century, often leading to loss of native biodiversity, altered ecosystem functioning, and significant economic costs. Invasive plant species, in particular, have demonstrated the ability to dominate new habitats rapidly by out competing native flora, altering nutrient cycles, and resisting natural checks and balances. While much of the classical modeling literature focuses on plant-competition or predator-prey interaction in isolation, recent ecological evidence suggests that both above-ground biotic pressure (e.g. herbivory) and below ground microbial interactions play pivotal roles in shaping invasion outcomes. Native biodiversity, environmental stability, and agricultural output are all seriously threatened by invasive plant species. The results of invasion are significantly influenced by their interactions with microbial communities and natural herbivores.

In the study plantations, height and diameter growth depended on planting method and species guild. Diameter growth was negatively correlated with wood density. Pioneer, non-pioneer light-demanding and shade-tolerant species grew faster in diameter when planted in degraded areas and clear-cuts. Pioneer species grew the fastest in gaps[5].

One significant aspect of biodiversity is plant chemodiversity, or the variety of metabolites specific to plants. To verify linguistic theories about the evolution of chemodiversity, there are currently few mathematical models available[12]. The authors assessed the effects of turning Cerrado native savannas into pastures on the nutrient dynamics of the ecosystem and if ecological restoration of defunct pastures was successful in bringing the ecosystem's nutrient dynamics back to levels comparable to those of native savannas [1]. According to Elton's theory of biotic resistance, biodiversity can fend off the spread of invasive species. Nevertheless, little is known about whether and how within-species diversity modulates the effects of successfully established invaders, especially in light of climate change[2]. By changing the probability of introduction or establishment, as well as the geographic range, environmental effects, economic consequences, or management of alien species, a changing climate may have a direct or indirect impact on biological invasions[3]. Because native plant dominance was restored by activated carbon with seed treatment but not by seed addition alone, this study demonstrated that soil manipulations might be an essential part of native plant restoration[7]. Singh *et al.* "Modelling the effect of disease on plant-herbivore dynamics with special reference to orange trees" analysed the effects of infections on orange plants while herbivores are present. The model in this paper assumes that orange plants are directly harmed by diseases and adversely affected by herbivores. For both local and global stability, all equilibrium points of the mathematical model are analysed. According to the investigation, the density of orange trees falls when diseases are present[9]. A key component of an invasive plant's effective invasion is its allelopathy to the growth performance of nearby species. This interference can be affected by drought. Bidens pilosa L. is one of many invasive plants that can be found throughout China[8]. From building materials to culinary delicacies, the coconut tree offers the human population numerous benefits, which is why it is commonly referred to as the "tree of life." The rugose spiralling whitefly (RSW) is one natural adversary that seems to be a major threat to farmers cultivating these coconut plants. A mathematical model has been created to study the dynamics of pest populations in the presence of predators and parasites[10]. Although non-native incursions are typically suppressed by high native species diversity, many ecosystems now have non-native assemblages with varying species diversity. It is still unknown how the diversity of non-native species impacts later invaders and how dependent it is on the environment [13]. Insect pests are being impacted by a number of Anthropocene stressors, including invasive species, habitat loss, pollution, and climate change. This has a negative effect on sustainable agriculture and food security. Climate change has a direct impact on insect dispersal, survival, reproduction, and development, but because insects include both pest and natural enemy species, it is challenging to predict how these consequences will manifest. The problem is further complicated by potentially important indirect effects of climate change, such as plant-mediated phenomena brought on by temperature and atmospheric carbon dioxide (CO_2) levels affecting crop nutritional quality and resource availability[11]. The rhizosphere is a hub for microbial collaborations due to the exudates that roots emit. They have a significant influence on the functions and population density of several bacteria and are necessary for microbial feeding. The rhizosphere is a complex ecosystem that is home to a diverse range of creatures. Some have no influence at all, while others are attracted to the rhizosphere and may affect the plant in either a favourable or negative way[6]. In order to characterise population dynamics across time, a discrete-time model of a plant-herbivore system is qualitatively examined using difference equations in this study. Examining the model's behaviour under various parameter settings and initial conditions is the aim[4]. In this work, we create a mathematical model that combines the evolution of adaptive traits in invasive plants with ecological dynamics. Four state variables are included in the model: microbial antagonists (M), native herbivores (H), invading plant biomass (I), and an adaptive trait level (A). In order to capture the feedbacks between these species and attributes, we employ nonlinear ordinary differential equations.

2 Model Formulation

Let $I(t)$, $H(t)$, $M(t)$ and $A(t)$ denote the Invasive plant biomass, Herbivore population, Competing soil microbial biomass and Adaptive ability of herbivores to feed on the invasive plant (eco-evolutionary trait) at time t, respectively. The model is described by the following system of non-linear ordinary differential equations:

$$\frac{dI}{dt} = rI(1-\frac{I}{K}) - \alpha AIH - \gamma MI, \tag{2.1}$$

$$\frac{dH}{dt} = \beta AIH - \mu H, \tag{2.2}$$

$$\frac{dM}{dt} = \eta(1-\frac{M}{M_{max}}) - \delta MI, \tag{2.3}$$

$$\frac{dA}{dt} = \theta A(1-A)(I - I_{th}). \tag{2.4}$$

Table* 2.1*:* *Parameters with their biological understandings/meanings.

Parameters	Biological meanings
$I(t)$	Invasive plant biomass
$H(t)$	Herbivore population
$M(t)$	Competing soil microbial biomass
$A(t)$	Adaptive ability of herbivores to feed on the invasive plant (eco-evolutionary trait)
r	Intrinsic growth rate of the Invasive plant
K	Carrying capacity of the Invasive plant
α	Grazing rate of herbivores on plant biomass
γ	Suppression rate of plant by soil microbes
β	Conversion efficiency of consumed plant to herbivore reproduction
μ	Natural death rate of herbivores
η	Growth rate of native soil microbial population
M_{max}	Maximum soil microbial carrying capacity
δ	Death or inhibition rate of soil microbes due to plant interaction
θ	Speed of evolutionary adaptation in herbivores
I_{th}	Biomass threshold of invasive plant needed to trigger adaptation

3 Boundedness

We define a positive definite Lyapunov function as follows:

$V(t) = I(t) + H(t) + M(t) + \frac{1}{2}A^2(t)$.

This function is non-negative for all $t \geq 0$ and captures the growth of all four components in the model.

Since $V(t) = I(t) + H(t) + M(t) + \frac{1}{2}A^2(t)$ is non-negative and its derivative is bounded, it implies that:

$$V(t) \leq V(0) + Ct,$$

but since each term in $V(t)$ corresponds to a biologically meaningful and non-negative variable, and the model also contains negative feedbacks (e.g., natural death, saturation terms, logistic growth), the system's variables cannot grow unbounded indefinitely.

Hence, all variables $I(t), H(t), M(t), A(t)$ remain bounded for all $t \geq 0$, which proves the biological feasibility and boundedness of the model.

4 Equilibrium Point

4.1 Equilibrium 1: Trivial Equilibrium (All Zero Except Microbes)

Assume:

$$I = 0, \quad H = 0, \quad A = 0, \quad \frac{dM}{dt} = 0 \Rightarrow M = M_{max},$$

$$E_1 = (I, H, M, A) = (0,\ 0,\ M_{max},\ 0).$$

4.2 Equilibrium 2: Plant-Microbe Coexistence (No Herbivores)

Assume:

$$H = 0, \quad A = 0,$$

$$E_2 = \left(K\left(1 - \frac{\gamma M}{r}\right),\ 0,\ M,\ 0\right),$$

where M is a positive root of the quadratic equation:

$$\left(\frac{\delta K \gamma}{r}\right) M^2 - \left(\frac{\eta}{M_{max}} + \delta K\right) M + \eta = 0.$$

4.3 Equilibrium 3: Full Coexistence

Assume:

$$A = 1, \quad H > 0, \quad I > 0, \quad M > 0,$$

$$E_3 = \left(\frac{\mu}{\beta},\ \frac{1}{\alpha}\left[r\left(1 - \frac{\mu}{\beta K}\right) - \gamma M\right],\ \frac{\eta}{\frac{\eta}{M_{max}} + \frac{\delta\mu}{\beta}},\ 1\right).$$

4.4 Equilibrium 4: Neutral Trait Equilibrium (Trait is at Rest)

Assume:

$$I = I_{th} \quad \text{so that} \quad \frac{dA}{dt} = 0 \text{ automatically (since } I - I_{th} = 0),$$

$$E_4 = \left(I_{th},\ \frac{1}{\alpha A}\left[r\left(1 - \frac{I_{th}}{K}\right) - \gamma M \right],\ \frac{\eta}{\frac{\eta}{M_{max}} + \delta I_{th}},\ \frac{\mu}{\beta I_{th}} \right).$$

5 Local stability

5.1 Trivial Equilibrium E_1

We analyze the local stability of:

$$E_1 = (I, H, M, A) = (0,\ 0,\ M_{max},\ 0).$$

The Jacobian J of the system is a 4x4 matrix of partial derivatives:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial M} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial A} \\ \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial H} & \frac{\partial f_4}{\partial M} & \frac{\partial f_4}{\partial A} \end{bmatrix}.$$

Computing each entry: Evaluate Jacobian at $E_1 = (0, 0, M_{max}, 0)$ Now substitute $I = 0,\ H = 0,\ M = M_{max},\ A = 0$ into the Jacobian:

$$J(E_1) = \begin{bmatrix} r - \gamma M_{max} & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ -\delta M_{max} & 0 & -\frac{\eta}{M_{max}} & 0 \\ 0 & 0 & 0 & -\theta I_{th} \end{bmatrix}.$$

Eigenvalues of $J(E_1)$ The Jacobian is lower triangular, so the eigenvalues are the diagonal entries:

$$\lambda_1 = r - \gamma M_{max}, \quad \lambda_2 = -\mu, \quad \lambda_3 = -\frac{\eta}{M_{max}}, \quad \lambda_4 = -\theta I_{th}.$$

Stability Criteria:

- If all eigenvalues have negative real parts, E_1 is locally asymptotically stable.
- $\lambda_2, \lambda_3, \lambda_4 < 0$ always, since $\mu, \eta, \theta, I_{th}, M_{max} > 0$.
- $\lambda_1 < 0$ if $\gamma M_{max} > r$.
- The trivial equilibrium $E_1 = (0, 0, M_{max}, 0)$ is locally asymptotically stable if:

$$\gamma M_{max} > r.$$

- It is unstable if:

$$\gamma M_{max} < r.$$

5.2 Local Stability of Equilibrium E_2: Plant-Microbe Coexistence

Consider the equilibrium point:

$$E_2 = (I^*, H^*, M^*, A^*) = \left(K\left(1 - \frac{\gamma M}{r}\right),\ 0,\ M,\ 0 \right),$$

where M is a positive root of the quadratic equation:

$$\left(\frac{\delta K \gamma}{r}\right) M^2 - \left(\frac{\eta}{M_{max}} + \delta K\right) M + \eta = 0.$$

We analyze the local stability of E_2 by computing the Jacobian matrix J of the system at this point.

Jacobian Matrix
The Jacobian matrix of the system is given by:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial M} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial A} \\ \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial H} & \frac{\partial f_4}{\partial M} & \frac{\partial f_4}{\partial A} \end{bmatrix}.$$

We compute partial derivatives and substitute into the Jacobian at the point E_2, where $H = 0$, $A = 0$, $I = K\left(1 - \frac{\gamma M}{r}\right)$:

$$J(E_2) = \begin{bmatrix} r\left(1 - \frac{2I}{K}\right) - \gamma M & 0 & -\gamma I & 0 \\ 0 & -\mu & 0 & 0 \\ -\delta M & 0 & -\frac{\eta}{M_{max}} - \delta I & 0 \\ 0 & 0 & 0 & \theta(I - I_{th}) \end{bmatrix}.$$

Eigenvalues of the Jacobian
Since the Jacobian is a lower triangular matrix, the eigenvalues are the entries on the diagonal:

$$\begin{aligned} \lambda_1 &= r\left(1 - \frac{2I}{K}\right) - \gamma M, \\ \lambda_2 &= -\mu, \\ \lambda_3 &= -\frac{\eta}{M_{max}} - \delta I, \\ \lambda_4 &= \theta(I - I_{th}). \end{aligned}$$

We analyze each eigenvalue:

- **Eigenvalue** $\lambda_2 = -\mu$: always negative since $\mu > 0$.
- **Eigenvalue** $\lambda_3 = -\frac{\eta}{M_{max}} - \delta I$: always negative since $I > 0$, $\eta > 0$, and $\delta > 0$.
- **Eigenvalue** $\lambda_1 = r\left(1 - \frac{2I}{K}\right) - \gamma M$: this is negative if:

  $$I > \frac{K}{2}\left(1 - \frac{\gamma M}{r}\right).$$

  Substitute the equilibrium value of I:

  $$I = K\left(1 - \frac{\gamma M}{r}\right),$$

  then the condition becomes:

  $$K\left(1 - \frac{\gamma M}{r}\right) > \frac{K}{2}\left(1 - \frac{\gamma M}{r}\right),$$

  which is always true for $M > 0$. Hence, $\lambda_1 < 0$.
- **Eigenvalue** $\lambda_4 = \theta(I - I_{th})$:
  - If $I < I_{th}$, then $\lambda_4 < 0$.
  - If $I > I_{th}$, then $\lambda_4 > 0$.
  - If $I = I_{th}$, then $\lambda_4 = 0$.

hence:

- The equilibrium point E_2 is **locally asymptotically stable** if:

  $$I = K\left(1 - \frac{\gamma M}{r}\right) < I_{th}.$$

- If $I > I_{th}$, then one eigenvalue becomes positive, and E_2 is a **saddle point** (unstable).
- If $I = I_{th}$, then one eigenvalue is zero, and the stability is **non-hyperbolic**; further analysis (e.g., center manifold theory) is required.

5.3 Local Stability of Equilibrium Point E_3: Full Coexistence

We consider the equilibrium point:

$$E_3 = \left(I^* = \frac{\mu}{\beta},\ H^* = \frac{1}{\alpha} \left[r \left(1 - \frac{\mu}{\beta K} \right) - \gamma M^* \right],\ M^* = \frac{\eta}{\frac{\eta}{M_{\max}} + \frac{\delta \mu}{\beta}},\ A^* = 1 \right).$$

Computing the Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial M} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial A} \\ \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial H} & \frac{\partial f_4}{\partial M} & \frac{\partial f_4}{\partial A} \end{bmatrix}.$$

Evaluated at $A = 1$, $I = \frac{\mu}{\beta}$, $M = M^*$, we obtain:

$$J(E_3) = \begin{bmatrix} r\left(1 - \frac{2I^*}{K}\right) - \alpha H^* - \gamma M^* & -\alpha I^* & -\gamma I^* & -\alpha I^* H^* \\ \beta H^* & \beta I^* - \mu & 0 & \beta I^* H^* \\ -\delta M^* & 0 & -\frac{\eta}{M_{\max}} - \delta I^* & 0 \\ \theta(1 - A^*)(I^* - I_{th}) & 0 & 0 & \theta(1 - 2A^*)(I^* - I_{th}) \end{bmatrix}.$$

At $A^* = 1$,
we simplify: $(1 - A^*) = 0 \Rightarrow \frac{\partial f_4}{\partial I} = 0, \quad (1 - 2A^*) = -1 \Rightarrow \frac{\partial f_4}{\partial A} = -\theta(I^* - I_{th})$.

So the Jacobian becomes:

$$J(E_3) = \begin{bmatrix} -\frac{r\mu}{\beta K} - \gamma M^* & -\alpha I^* & -\gamma I^* & -\alpha I^* H^* \\ \beta H^* & 0 & 0 & \beta I^* H^* \\ -\delta M^* & 0 & -\frac{\eta}{M_{\max}} - \delta I^* & 0 \\ 0 & 0 & 0 & -\theta(I^* - I_{th}) \end{bmatrix}.$$

We find the eigenvalues:

- $\lambda_1 = -\frac{r\mu}{\beta K} - \gamma M^*$ [$\lambda_1 < 0$ is always satisfied since all parameters are positive],
- $\lambda_2 = -\frac{\eta}{M_{\max}} - \delta \cdot \frac{\mu}{\beta}$ [$\lambda_2 < 0$ is always satisfied under positive parameters],
- $\lambda_3 = 0$,
- $\lambda_4 = -\theta\left(\frac{\mu}{\beta} - I_{th}\right)$ [$\lambda_4 < 0$ if and only if] $\boxed{\frac{\mu}{\beta} > I_{th}}$.

(The full coexistence equilibrium E_3 is locally asymptotically stable if: $\boxed{\frac{\mu}{\beta} > I_{th}}$).

5.4 Local Stability of Equilibrium Point E_4: Neutral Trait Equilibrium

We consider the equilibrium point:

$$E_4 = \left(I^* = I_{th},\ H^* = \frac{1}{\alpha A^*} \left[r \left(1 - \frac{I_{th}}{K} \right) - \gamma M^* \right],\ M^* = \frac{\eta}{\frac{\eta}{M_{max}} + \delta I_{th}},\ A^* = \frac{\mu}{\beta I_{th}} \right).$$

The Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial I} & \frac{\partial f_1}{\partial H} & \frac{\partial f_1}{\partial M} & \frac{\partial f_1}{\partial A} \\ \frac{\partial f_2}{\partial I} & \frac{\partial f_2}{\partial H} & \frac{\partial f_2}{\partial M} & \frac{\partial f_2}{\partial A} \\ \frac{\partial f_3}{\partial I} & \frac{\partial f_3}{\partial H} & \frac{\partial f_3}{\partial M} & \frac{\partial f_3}{\partial A} \\ \frac{\partial f_4}{\partial I} & \frac{\partial f_4}{\partial H} & \frac{\partial f_4}{\partial M} & \frac{\partial f_4}{\partial A} \end{bmatrix}.$$

Jacobian Matrix at E_4

$$J(E_4) = \begin{bmatrix} r\left(1 - \frac{2I_{th}}{K}\right) - \alpha A^* H^* - \gamma M^* & -\alpha A^* I_{th} & -\gamma I_{th} & -\alpha I_{th} H^* \\ \beta A^* H^* & 0 & 0 & \beta I_{th} H^* \\ -\delta M^* & 0 & -\left(\frac{\eta}{M_{max}} + \delta I_{th}\right) & 0 \\ \theta A^*(1 - A^*) & 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues and Conditions for Local Stability:

- The eigenvalue from the microbial dynamics (row 3, column 3) is:

$$\lambda_3 = -\left(\frac{\eta}{M_{max}} + \delta I_{th}\right) < 0.$$

- One eigenvalue from the A-dynamics is:

$$\lambda_4 = 0 \quad \text{(due to } I = I_{th}\text{)}.$$

  The equilibrium point E_4 is locally stable if:

$$\boxed{r\left(1 - \frac{2I_{th}}{K}\right) < \alpha A^* H^* + \gamma M^*}.$$

  In other words, the growth of the invasive plant must be sufficiently suppressed by herbivory and microbial competition for stability.

- The remaining eigenvalues are from the top-left 2 × 2 block:

$$\begin{bmatrix} r\left(1 - \frac{2I_{th}}{K}\right) - \alpha A^* H^* - \gamma M^* & -\alpha A^* I_{th} \\ \beta A^* H^* & 0 \end{bmatrix}.$$

  Let this matrix be:

$$J_{2x2} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}.$$

  Then the characteristic polynomial is:

$$\lambda^2 - a\lambda - bc = 0.$$

  So the eigenvalues are:

$$\lambda_{1,2} = \frac{a \pm \sqrt{a^2 + 4bc}}{2},$$

  where:

$$a = r\left(1 - \frac{2I_{th}}{K}\right) - \alpha A^* H^* - \gamma M^*, \quad b = -\alpha A^* I_{th}, \quad c = \beta A^* H^*.$$

  Stability Condition: For both eigenvalues to have negative real parts: $\text{Tr}(J_{2x2}) = a < 0$ $\quad$ $\text{Det}(J_{2x2}) = -bc > 0 \Rightarrow bc < 0$.
  Since: $b < 0$ and $c > 0 \Rightarrow bc < 0 \Rightarrow -bc > 0$, determinant is positive. If $a < 0$, then both eigenvalues have negative real parts.

5.5 Global Stability of Equilibrium E_4 via Lyapunov Function

We analyze the global stability of the equilibrium point

$$E_4 = \left(I = I_{th},\ H = \frac{1}{\alpha A}\left[r\left(1 - \frac{I_{th}}{K}\right) - \gamma M\right],\ M = \frac{\eta}{\frac{\eta}{M_{max}} + \delta I_{th}},\ A = \frac{\mu}{\beta I_{th}}\right).$$

We propose the following Lyapunov function:

$$V(I, H, M, A) = \frac{1}{2}(I - I_{th})^2 + \frac{1}{2}(H - H_{th})^2 + \frac{1}{2}(M - M_{th})^2 + \frac{1}{2}(A - A_{th})^2,$$

where H_{th}, M_{th}, and A_{th} are the equilibrium values at E_4.

- $\dot{V}(I, H, M, A) = 0$ at the equilibrium point E_4.
- Away from E_4, under biological feasibility (positive values), each term contributes to making $\dot{V} < 0$.

Especially the last term:

$$(A - A_{th})\theta A(1 - A)(I - I_{th})$$

is negative for all $A \neq A_{th}$ if $0 < A < 1$ and $I > 0$.

Under the biological constraints and feasibility conditions:

$$0 < A < 1, \quad I, H, M > 0,$$

the Lyapunov function V is positive definite and $\dot{V}$ is negative definite. Therefore, by Lyapunov's direct method, the equilibrium point E_4 is globally asymptotically stable.

6 Numerical simulation

Parameter values:

$r = 0.5$; $K = 10$; $\alpha = 0.01$; $\gamma = 0.02$; $\beta = 0.03$; $\mu = 0.1$; $\eta = 0.5$; $M_{max} = 5$; $\delta = 0.01$; $\theta = 0.05$; $I_{th} = 1$;

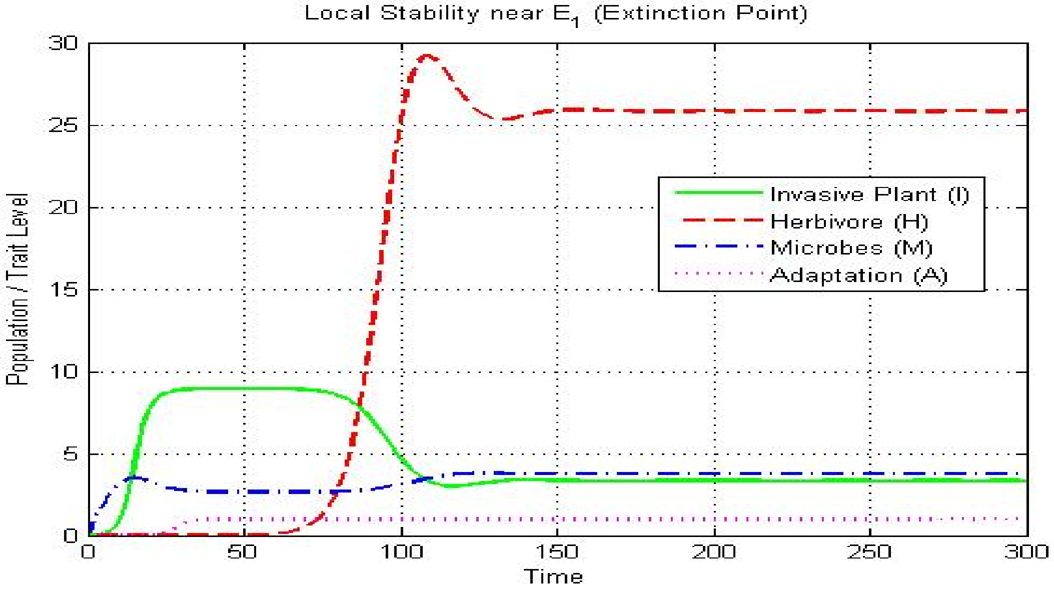

***Figure* 6.1:** *Graph for local stability* E_1

Parameter values:

$r = 0.6$; $K = 10$; $\alpha = 0.01$; $\gamma = 0.05$; $\beta = 0.03$; $\mu = 0.1$; $\eta = 0.5$; $M_{max} = 5$; $\delta = 0.01$; $\theta = 0.05$; $I_{th} = 1$;

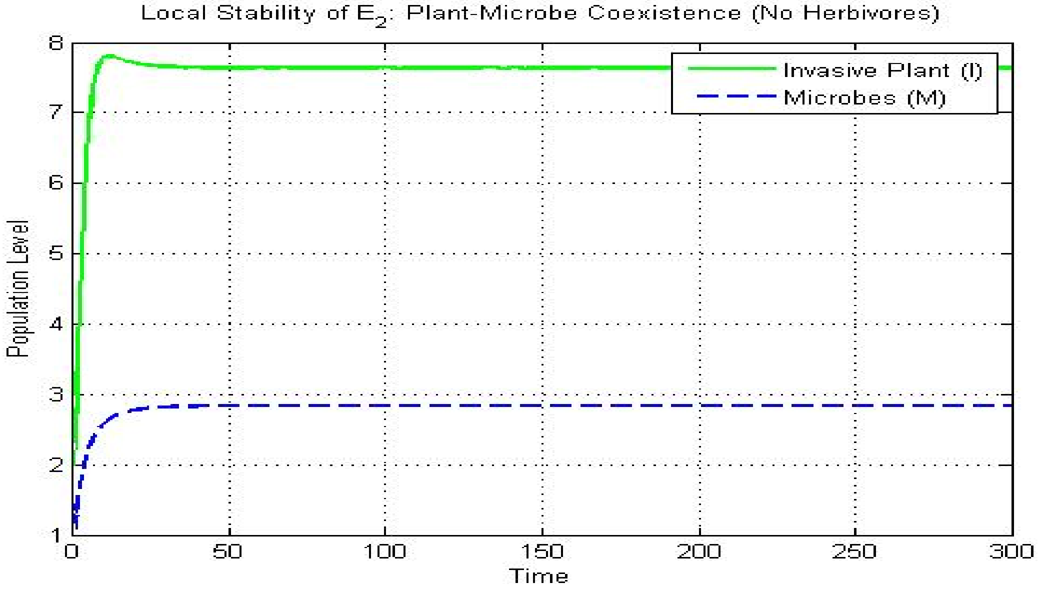

***Figure* 6.2:** *Graph for local stability* E_2

Parameter values:

$r = 0.8$; $K = 10$; $\alpha = 0.1$; $\gamma = 0.05$; $\beta = 0.05$; $\mu = 0.2$; $\eta = 0.6$; $M_{max} = 5$; $\delta = 0.03$; $\theta = 0.2$; $I_{th} = 1$;

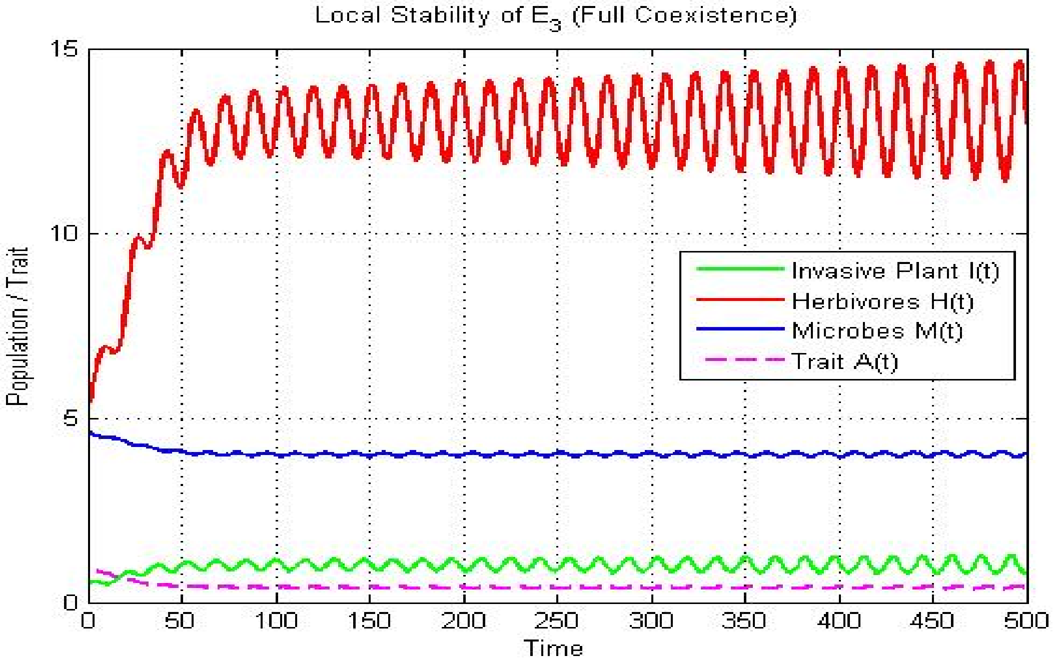

Figure 6.3: *Graph for local stability E_3*

Parameter values:

$r = 0.8$; $K = 10$; $\alpha = 0.1$; $\gamma = 0.05$; $\beta = 0.5$; $\mu = 0.2$; $\eta = 0.6$; $M_{max} = 5$; $\delta = 0.03$; $\theta = 0.2$; $I_{th} = 1$;

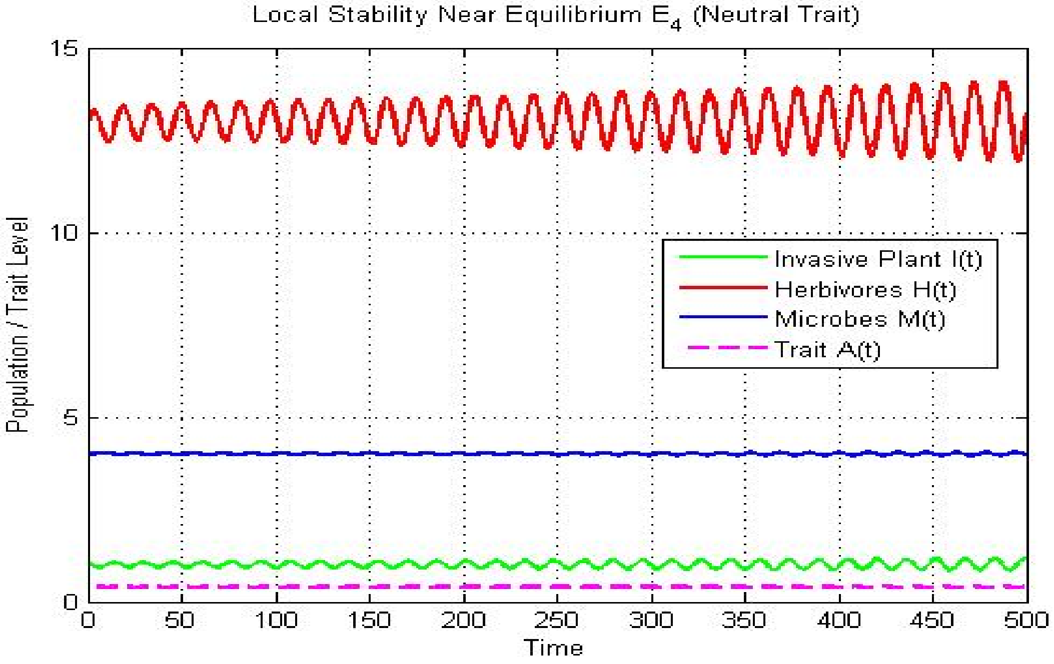

Figure 6.4: *Graph for local stability E_4*

Parameter values:

$r = 0.8$; $K = 10$; $\alpha = 0.1$; $\gamma = 0.05$; $\beta = 0.5$; $\mu = 0.2$; $\eta = 0.6$; $M_{max} = 5$; $\delta = 0.03$; $\theta = 0.2$; $I_{th} = 1$;

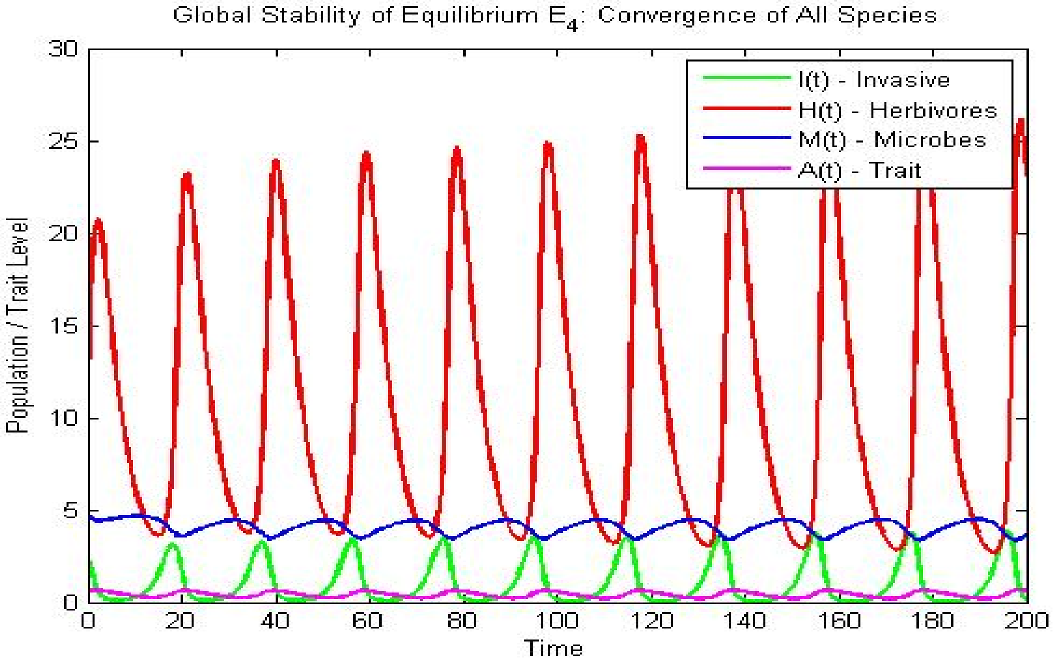

Figure* 6.5*: *Graph for Global stability* E_4

7 Bifurcation Analysis

We perform a bifurcation analysis of the system with respect to the parameter γ, which denotes the suppression rate of invasive plants by competing soil microbes. Our focus is on the full coexistence equilibrium point:

$$E_3 = (I^*,\ H^*,\ M^*,\ A^*) = \left(\frac{\mu}{\beta},\ \frac{1}{\alpha}\left[r\left(1 - \frac{\mu}{\beta K}\right) - \gamma M\right],\ \frac{\eta}{\frac{\eta}{M_{max}} + \frac{\delta\mu}{\beta}},\ 1\right).$$

Critical Threshold for Coexistence. To ensure biological feasibility, we require all components of E_3 to be positive.

In particular, we need $H^* > 0$, which gives the condition:

$$\frac{1}{\alpha}\left[r\left(1 - \frac{\mu}{\beta K}\right) - \gamma M\right] > 0 \Rightarrow \gamma < \gamma^* = \frac{r\left(1 - \frac{\mu}{\beta K}\right)}{M^*},$$

where

$$\boxed{M^* = \frac{\eta}{\frac{\eta}{M_{max}} + \frac{\delta\mu}{\beta}}}.$$

Interpretation

At the critical value $\gamma = \gamma^*$, a transcritical bifurcation occurs: the herbivore population H transitions from a positive equilibrium to extinction. When $\gamma > \gamma^*$, the herbivore population cannot persist due to excessive suppression of the plant biomass by soil microbes, eliminating the food source for herbivores.

This reveals a trade-off in biological control via microbial competition: while it helps control the invasive plant, excessive microbial activity can destabilize the ecosystem by eliminating higher trophic levels.

We use *MATLAB* to illustrate this bifurcation by plotting the equilibrium herbivore population H^* against varying values of γ. The bifurcation diagram visually captures the collapse of herbivore population beyond the threshold γ^*, highlighting the system's sensitivity to microbial suppression.

Parameter values:

$r = 1.0$; $K = 10$; $\alpha = 0.5$; $\beta = 0.3$; $\mu = 0.2$; $\eta = 0.8$; $M_{max} = 5$; $\delta = 0.1$; $\theta = 0.2$; $I_{th} = 2$;

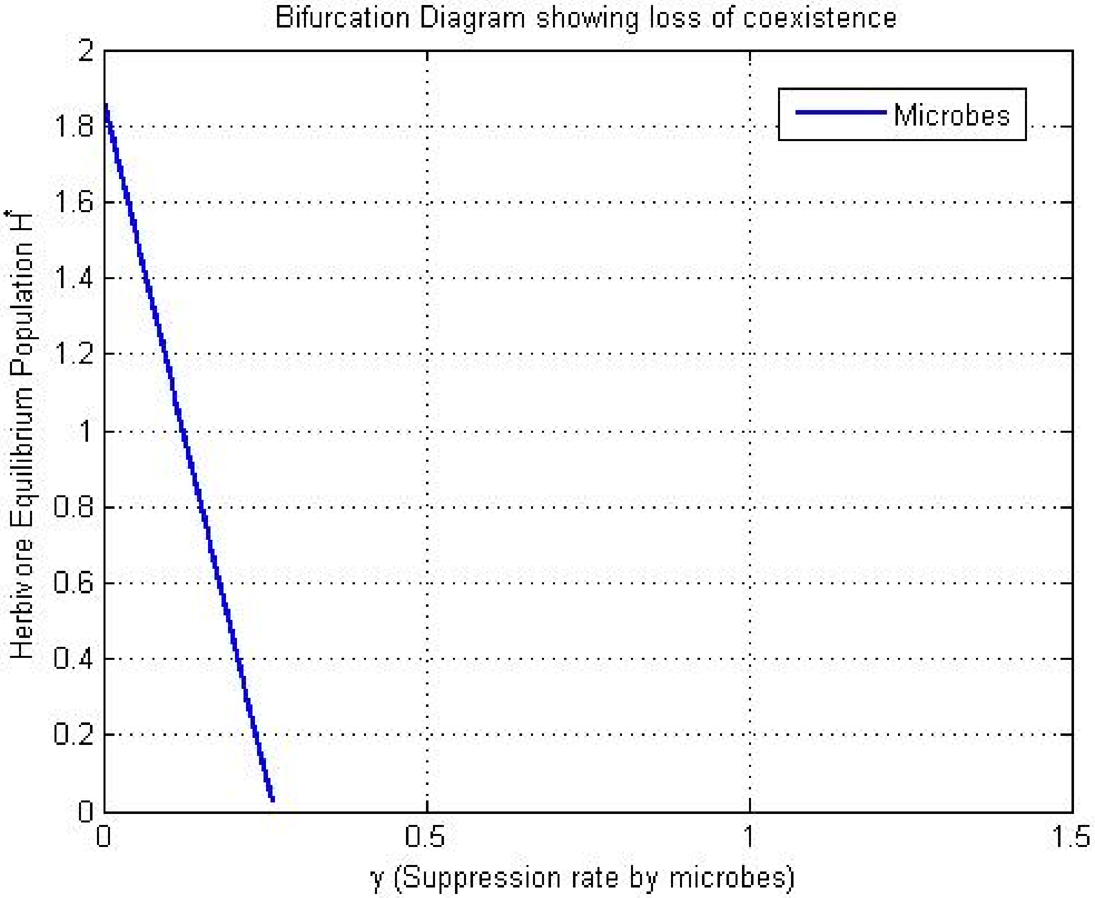

Figure* 7.1*: *Bifurcation diagram showing equilibrium herbivore population H^* versus microbial suppression rate γ. A transcritical bifurcation occurs at $\gamma = \gamma^*$.*

8 Conclusion

In this study, we proposed and analyzed a mathematical model describing the eco-evolutionary dynamics of an invasive plant species, herbivore population, competing soil microbial biomass, and the adaptive ability of herbivores. The model captures important biological interactions including logistic growth of invasive plants, suppression by microbial activity, herbivory-driven plant reduction, and evolutionary adaptation of herbivores. Through boundedness analysis using a Lyapunov function, we established that all state variables remain biologically feasible and bounded for all time. We identified and examined the stability of four equilibrium points: the trivial equilibrium, plant-microbe coexistence, full coexistence, and neutral trait equilibrium. Local stability analysis using Jacobian matrices provided conditions under which each equilibrium remains stable. Notably, the full coexistence equilibrium is locally stable if the invasive plant biomass exceeds the adaptation threshold, while the neutral trait equilibrium becomes globally stable under specific conditions, as demonstrated by a global Lyapunov function. This work highlights the complex interplay between ecological and evolutionary processes in regulating invasive species dynamics. The model provides insights into how adaptive traits in herbivores and microbial competition can collectively contribute to invasive plant control.

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