

*Jñānābha*, Vol. 55(2) (2025), 114-122

## CHAOS AND COMPLEXITY IN QUADRATIC AREA PRESERVING HÉNON MAP WITH APPLICATIONS OF *DLI*

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(Received: May 29, 2025; In format: June 10, 2025; Revised: October 09, 2025;

Accepted: October 21, 2025)

DOI: <https://doi.org/10.58250/jnanabha.2025.55213>

### Abstract

Dynamic studies conducted on regular and chaotic evolution of quadratic area-preserving Hénon map. The model considered a good one for general study of dynamical systems of two degrees of freedom. Evolution comprises with formation of very interesting attractors form by phase plane portraits. Assuming some characteristic values of only parameter  $\alpha$  of the map displays remarkable variety of shapes in the phase plane. Amazing regular and chaotic attractors drawn during the process of numerical calculations. For some cases Lyapunov exponents drawn and shown by the sides of chaotic attractors. Applications of Dynamic Lyapunov Indicator (*DLI*) made here for some regular and chaotic cases. The correlation dimension of some chaotic attractors obtained and presented in a Table form. Topological entropies calculated for some ranges of values of parameter  $\alpha$  and very interesting results obtained showing presence of significant complexity in Hénon system. Most results presented through graphics are interesting and significant.

**2020 Mathematical Sciences Classification:** 34D45, 37H15, 34C23, 65P20

**Keywords and Phrases:** Chaos, Lyapunov Exponents, Bifurcation, Topological Entropy

### 1 Introduction

Some nonlinear system during evolution display property, "complexity", in addition to chaos in some parameter space. Such systems are internally composed of multi-components structure and are termed as the complex system. Detailed articles on complex system and complexity appeared through articles by Weaver [42, 1948], Walby [40, 2007], Hefferman [15, 1985] and Simon [37, 1962]. As movements internal multicomponent are independent and not follow any definite rule, their movements measured by application of rule of probability. Such measure termed as topological entropy of the complex system. Some recent articles on chaos and complexity in complex system referred as: Chialvo [7, 1995], Wang and Cao [41, 2018], Elsadany [10, 2012], Saha [27, 2020], Gribbin [13, 2004], Saha and Mandal [27, 2021], Sharma and Saha [35, 2022], Saha *et al.* [24, 2023], Saha *et al.* [30, 2023] and Saha and Sharma [28, 2018].

Chaos is a state of the evolving system when it shows sensitivity to initial conditions. At this stage system losses predictably and disorder prevails resulting in appearance of chaotic (strange) attractors. Chaos can be measured Lyapunov exponents or Lyapunov characteristic exponents (*LCEs*) such that if at any state one finds *LCEs*  $> 0$  then the system evolution there is chaotic and, if *LCEs*  $< 0$  the system evolution is regular, Elsadany [10, 2012], Saha *et al.* [25, 2023], Saha [26, 2021], Sharma and Saha [35, 2022], Saha and Mandal [27, 2021], Saha *et al.* [30, 2023], Andrecut and Kauffman [2, 2007], Benettin *et al.* [13, 1990], Bryant *et al.* [6, 1990], Grassberger and Itamar Procaccia [14, 1983], Saha *et al.* [29, 2012], Chialvo [7, 1995] Sandri [34, 1996].

The movements of multi-components present internally in complex system independent and do not follow any definite rule. Each component evolves in its own way and do no follow any definite deterministic rule. Their collective response produces complexity during evolution. Thus, for measure of complexity, probabilistic rule can only be applied and the measure of complexity provided by the increase of topological entropy of

the system supported with statement that “more increase and fluctuations in topological entropy implies the system is more complex”, Saha *et al.* [25 2023], Hefferman [15, 1985], Gribbin [12, 2004], Saha *et al.* [30, 2023], Andrecut and Kauffman [2, 2007], Adler *et al.* [1, 1965]. Gribble [13, 1995], Bowen [5, 1973], Balmforth *et al.* [4, 1994], Iwai [21, 1998], Stewart and Edward [36, 1997], Hénon [17, 1969] and, Nagashima and Baba [24, 2005].

Hénon’s quadratic area-preserving map, introduced by Hénon in [17, 1969], is extensively studied through many articles and numerous results of great significance are drawn. Hénon [17, 1969], Hénon [16, 1965], Engel [11, 1958], Moser [23, 1900], Strogatz [39, 2019]. A great number of applications of the map available in the eBook by Steven H. Strogatz. Our objective presented in this article is to perform dynamic study in Hénon’s quadratic area-preserving map. The mixture of regularity and chaos is typical in this area-preserving map. In the processes of numerical study, amazing attractors of this system are drawn, the Lyapunov exponents calculated and represented graphically for regular and chaotic states of evolution. Presence of complexity in this system explored by calculation topological entropies and presenting graphically which provides the measure of complexity in the system. Correlation dimensions, a kind of fractal dimension, obtained and presented in tabular form for some chaotic attractors.

## 2 Model Description

Hénon’s area preserving quadratic map is described by two-dimensional invertible map proposed by Hénon [17, 1969]. The system consists of two coupled equations of two variables represented by

$$\begin{aligned}x_{n+1} &= x_n \cos \alpha - (y - x^2) \sin \alpha, \\y_{n+1} &= x_n \sin \alpha + (y - x^2) \cos \alpha.\end{aligned}\tag{2.1}$$

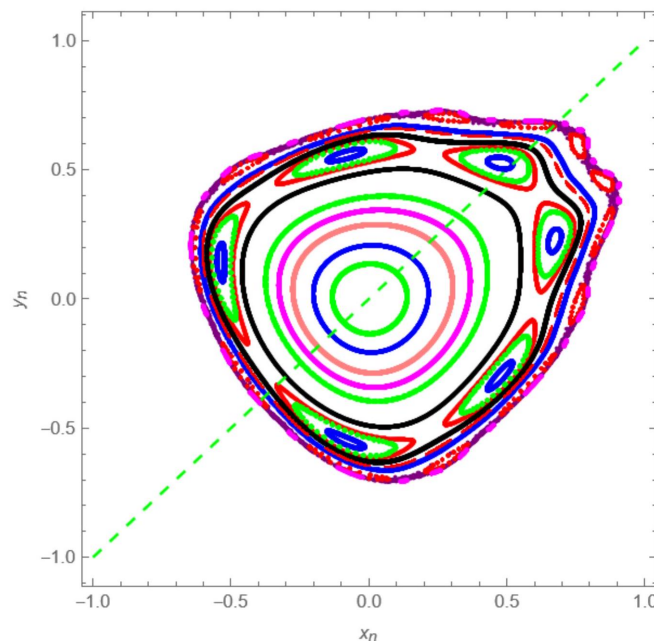
The Hénon map takes a point  $(x_n, y_n)$  in the plane and maps it to a new point  $(x_{n+1}, y_{n+1})$ . Here,  $\alpha$  is a parameter and  $0 \leq \alpha \leq \pi$ .

The Jacobian matrix of above system obtained as

$$J = \begin{pmatrix} \cos \alpha + 2x \sin \alpha & -\sin \alpha \\ \sin \alpha - 2x \cos \alpha & \cos \alpha \end{pmatrix}.\tag{2.2}$$

It is easy to verify that  $\det J = 1$  and thus, confirming that the map (2.1) is area preserving.

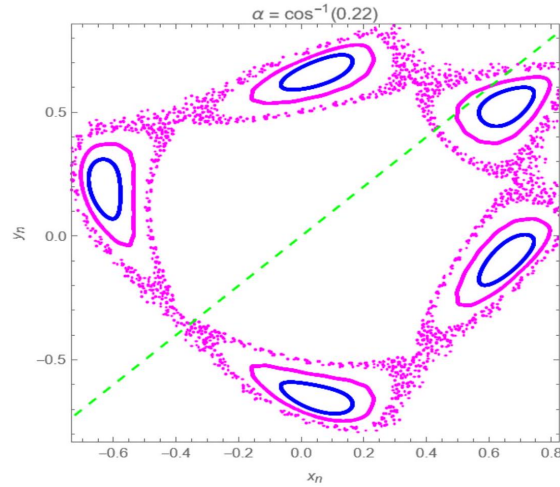
The system (2.1) has contributed a number of theoretically derived results, Ref: Henon and Heile [18, 1964], Hénon [19, 1976], Hénon [20, 1983], Engel [11, 1958], Moser [23, 1900]. As it is well known fixed points, which are invariant in system (2.1) and stable in linear approximation, play central role in the evolution. We can assume the fixed point  $(x^*, y^*) = (0, 0)$  is such a  $n$  invariant point.



**Figure 2.1:** Plots of phase plane orbits, (attractors), of Hénon map when  $\alpha = \cos^{-1} 0.4$  and with 15 different initial conditions in the neighbourhood of  $(0.5, 0.2)$ .

The Hénon system (2.1) contains only one parameter  $\alpha$  and so this is a one-parameter family of mappings. We must explore the system behaviour by considering different values of  $\alpha$  in different situations to obtain many significant interesting evolutionary phenomena.

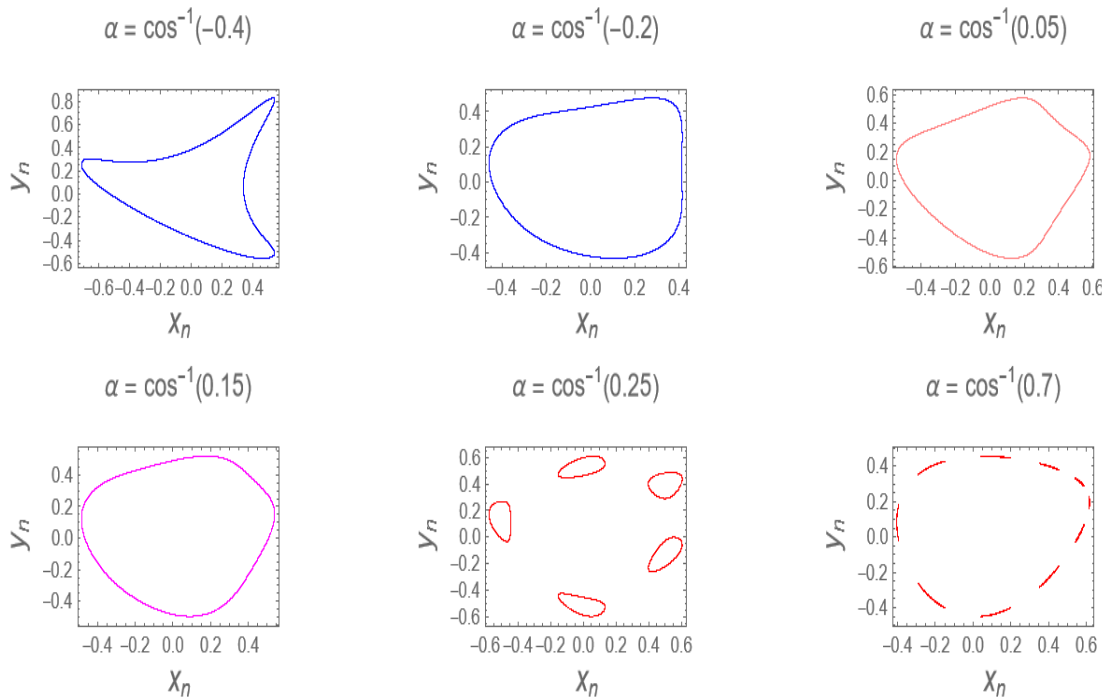
As shown in Figure 2.1 and Figure 2.2, for a fixed value of  $\alpha$  and then, with different sets of initial conditions the initial closed orbits first splits into islands of closed orbits and then to chaos. When projected together, these plots display very interesting scenario.



**Figure 2.2:** Phase plane orbits, (attractors), of Hénon map when  $\alpha = \cos^{-1} 0.22$  and with three different initial conditions nearby  $(0.7, 0.6)$ .

### 3 Numerical Simulations

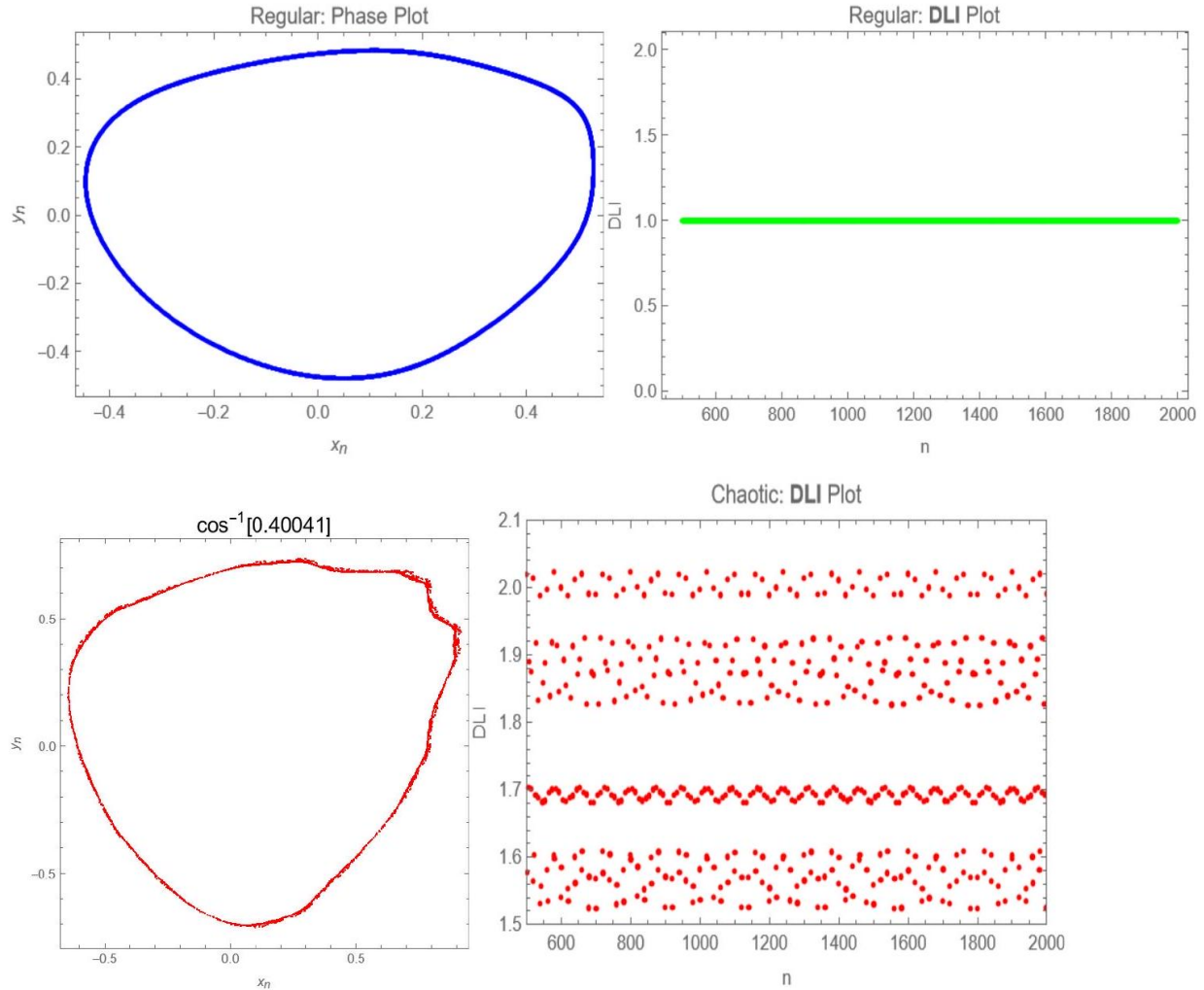
#### (a) Attractors:



**Figure 3.1:** Attractors of Hénon map for different values of parameter  $\alpha$ .

**(b): DLI Plots:**

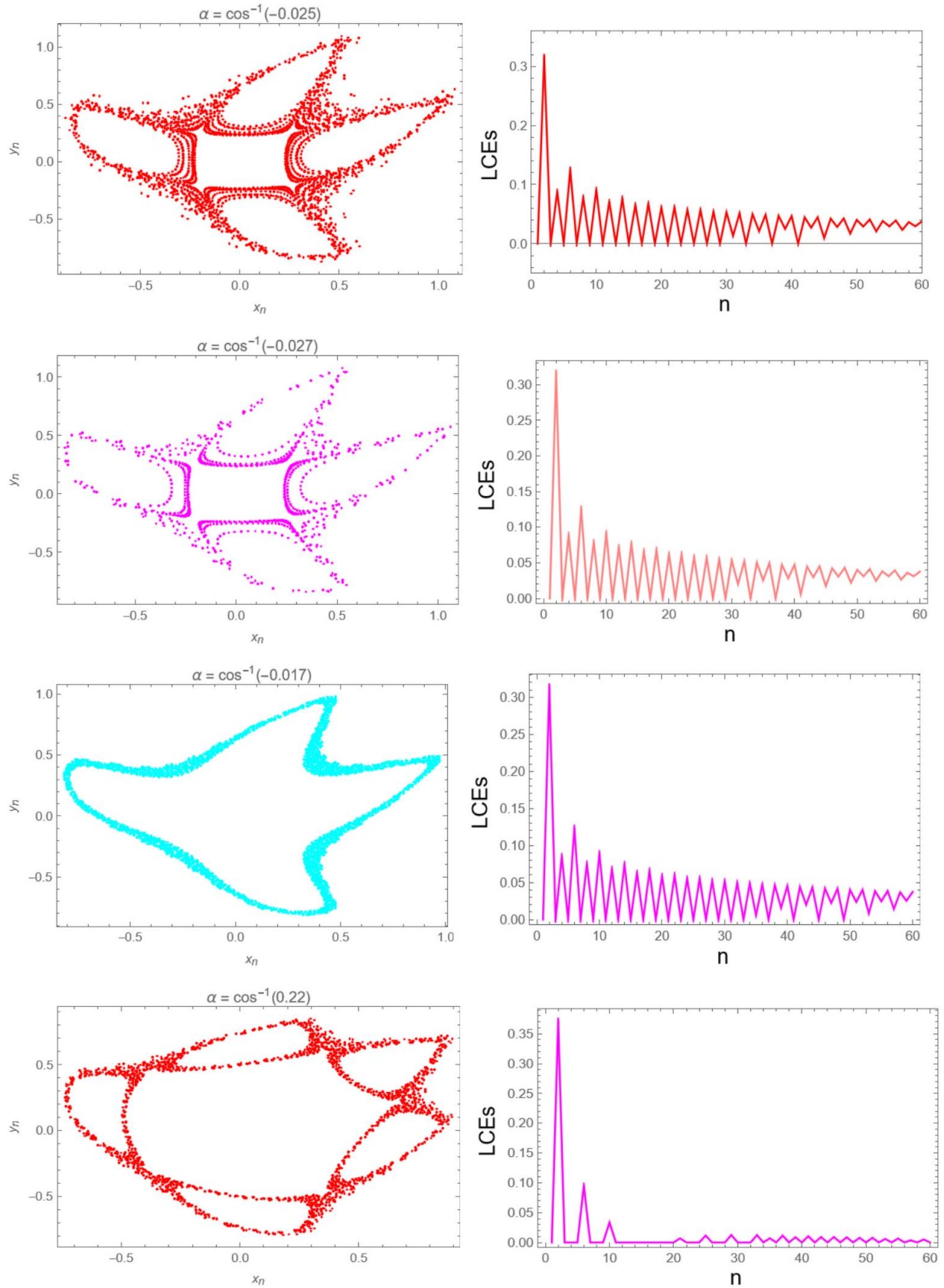
The dynamic Lyapunov indicator (*DLI*), introduced by Saha and Budhரா [31, 2007] and later applied in articles by Deleanu [9, 2011], Sahniet *al.* [33, 2016] etc., is defined by the largest value estimated from all eigenvalues  $\lambda_i$  of the Jacobian matrix  $J$  such that  $|J - \lambda_i| = 0, i = 1, 2, \dots, n$ , (for  $n$ -dimensional map), of the map to be studied for all discrete times. If these eigenvalues form a definite pattern, then the motion is regular and if they are randomly distributed randomly all over the space, (with no definite pattern), then the motion is chaotic.



**Figure 3.2:** Phase plots and DLI plots corresponding to regular and chaotic motion of Hénon map when  $\alpha = \cos^{-1} 0.4$ .

**(c) Plots of Lyapunov Exponents:**

Lyapunov exponents, or Lyapunov characteristic exponents (*LCEs*), provides measure of chaos and regularity. As in case of chaos, system shows sensitivity to initial conditions i.e. two trajectories originated extremely close to each other show divergence behaviour during long term evolution. If at any state during evolution, one finds  $LCEs > 0$  then the system evolution there is chaotic and, if  $LCEs < 0$  the system evolution is regular. In Figure 3.3, we have drawn some chaotic attractors, (left figures), and shown their corresponding plots of *LCE*.

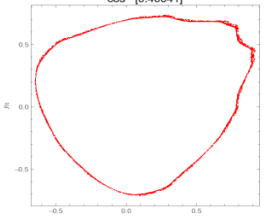
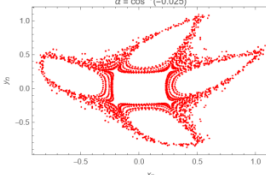
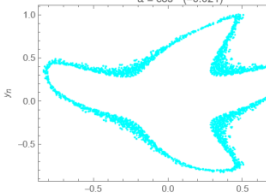
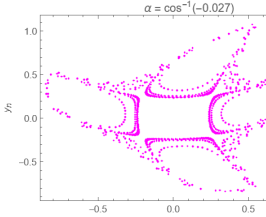
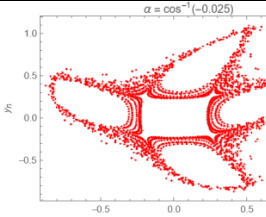


**Figure 3.3:** Chaotic attractors and corresponding LCEs of and corresponding Lyapunov exponents, (right column figures), obtained when  $\alpha = \cos^{-1}(-0.025)$ ,  $\alpha = \cos^{-1}(-0.027)$ ,  $\alpha = \cos^{-1}(-0.021)$  and  $\alpha = \cos^{-1}(-0.021)$ .

**(c) Correlation Dimensions of Chaotic Attractors:**

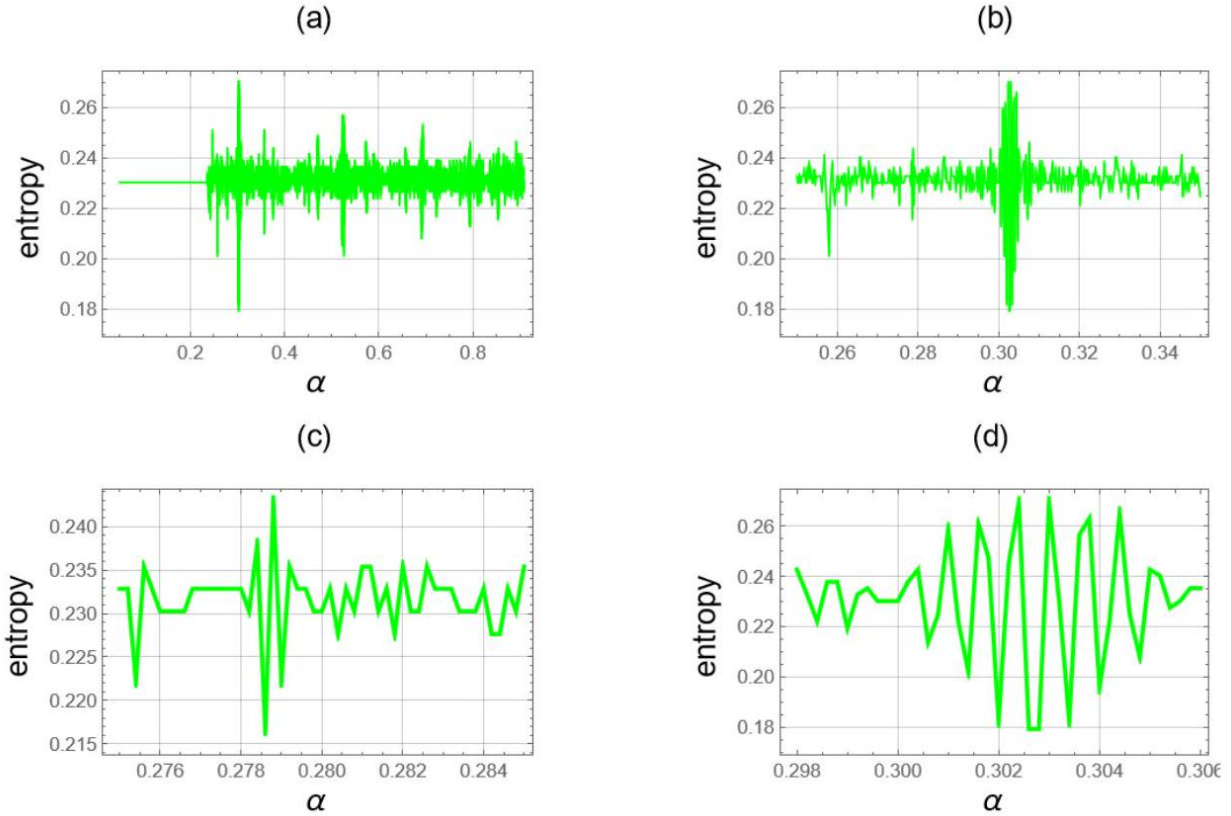
Correlation dimension provides dimensionality of a chaotic attractor. Correlation dimensions of some chaotic attractors obtained by using Mathematica Codes suggested by Martelli, [22, 1999]. Some recent articles further explained the process, be referred, Saha *et al.* [32, 2013], Saha *et al.* [25, 2023], Saha *et al.* [30, 2023], DeCoster and Mitchell [8, 1991], Sarma and Saha [36, 2024]. In the following Table 3.1, correlation dimensions for some attractors of Hénon map, (2.1), are shown:

**Table 3.1**

Attractor	$\alpha$	Correlation Dimension: $D_C$
	$\cos^{-1}(0.40041) = 1.15928$	1.405
	$\cos^{-1}(-0.022) = 1.5928$	1.139
	$\cos^{-1}(-0.017) = 1.5878$	1.251
	$\cos^{-1}(-0.027) = 1.5978$	1.5304
	$\cos^{-1}(-0.025) = 1.5958$	1.159

**(d) Topological Entropies:**

As stated, increase in topological entropy provides the measure of complexity and calculated statistically by using the law of probability. Procedure of calculation discussed in articles, Ref: Adler *et al.* [1, 1965], Saha and Sharma [28, 2018], Saha *et al.* [25, 2023], Sarma and Saha [36, 2024] etc. Here below, in Figure 3.4, increase and rapid fluctuations of topological entropy shown for Hénon map (2.1) for four different ranges of values of parameter  $\alpha$ .



**Figure 3.4:** Four different plots of topological entropies of Hénon map (2.1) are shown for different ranges of parameter  $\alpha$  : (a)  $\cos^{-1}(0.05) \leq \alpha \leq \cos^{-1}(0.95)$ , (b)  $\cos^{-1}(0.25) \leq \alpha \leq \cos^{-1}(0.35)$ , (c)  $\cos^{-1}(0.275) \leq \alpha \leq \cos^{-1}(0.285)$  and (d)  $\cos^{-1}(0.298) \leq \alpha \leq \cos^{-1}(0.306)$

#### 4 Concluding Remarks

Through this study we tried to explore some important dynamic evolutionary properties of Hénon's area preserving map (2.1). Figure 2.1 shows changing and disintegrating scenario of phase portraits, (attractors), of Hénon map when  $\alpha = \cos^{-1} 0.4$  and with 16 different initial conditions in the neighbourhood of  $(0.5, 0.2)$ . One finds regular orbits splitting into different islands and chaotic orbits. Similar phenomena also seen when  $\alpha = \cos^{-1} 0.22$  and with three different initial conditions nearby  $(0.7, 0.6)$  in Figure 2.2. Figure 3.1 shows some attractors of Hénon map for different values of parameter  $\alpha$ . Figure 3.2 indicates behaviour of regular and chaotic motions of Hénon map in accordance of *DLI* indicator. In Figure 3.3, Lyapunov exponents are drawn corresponding to some chaotic attractor of map (2.1). In Table 3.1, we shown correlation dimensions of some chaotic attractors of Hénon's attractors which are nonintegers. Four plots in Figure 3.4, show significant increase and fluctuations in topological entropies. This indicates the presence of complexity in the Hénon system (2.1).

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