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DETOUR EDGE PEBBLING NUMBER IN GRAPHS

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Abstract

Assume G is a connected graph with distributing pebbles over its edges. An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and one pebble will be added to an adjacent edge, while the other pebble will be discarded from the play. In this paper, we introduce the concept of detour edge pebbling number and find out the detour edge pebbling number for some standard graphs. We carry out the edge pebbling move in the concept of detour pebbling to arrive a new graph invariant called the detour edge pebbling number. The detour edge pebbling number of an edge e of a graph G is the minimum number of pebbles such that these pebbles are placed on the edges of G, we can move a pebble to e by making a sequence of pebble moves regardless of the initial configuration using the edge detour path. The detour edge pebbling number of a graph G, $f_e^*(G)$, is the maximum $f_e^*(G,\ e)$ over all the edges of G.
2020 Mathematical Sciences Classification: 05C12, 05C57, 05C38. **Keywords and Phrases:** Edge pebbling move, Edge detour path, Edge detour distance, Detour edge pebbling number.

1 Introduction

Graph pebbling is a game that can be applied to any connected graph. The concept of pebbling in graphs was first suggested by Lagarias and Saks to give an alternative demonstration of a theorem of Kleitman and Lemke in number theory and it was first mentioned by Chung [3] in 1989. According to Hurlbert and Kenter [5], graph pebbling is a blend of Graph theory, Number theory and Optimization and they provide a clear view of graph pebbling. A series of pebbling moves is what makes the game.

To pebble a graph, we need to choose a vertex with at least two pebbles, get rid of two pebbles from it, and add one to an adjacent vertex, and the second pebble that is removed is taken out of play. The pebbling number [3] is the minimum number of pebbles that are sufficient to reach any target vertex irrespective of the initial configuration of the pebbles. It is denoted by $f(G)$ for a graph G. The possibility exists for graph pebbling to serve as a model for distributing and transporting consumable resources.

Some invariants in pebbling are optimal pebbling number, t-pebbling number, cover pebbling number, monophonic pebbling number, etc, and which may be referred to [2, 6, 9, 12]. In 2020, Paul [11] extended the concept of pebbling by finding the edge pebbling number of certain graph classes. Chartrand *et al.* [4] presented and elaborated the fundamental concepts of detour distance in graphs. In 2023, Zhan [13] worked on the minimum number of detours in graphs and this paper paves the elegant ways of finding the detours in graphs. Lourdusamy *et al.* investigated and worked on detour pebbling number [1, 7, 8, 10].

In this paper, we define the detour edge pebbling number and compute the detour edge pebbling number of certain standard graphs and the square of some standard graph structures.

2 Detour edge pebbling number

Definition 2.1 ([11]). *An edge pebbling move on a graph G is defined to be the removal of two pebbles from one edge and one pebble will be added to an adjacent edge, while the other pebble will be discarded from the play.*

Definition 2.2. *The detour edge pebbling number of an edge e of a graph G is the minimum number of pebbles such that these pebbles are placed on the edges of G, we can move a pebble to e by making a sequence* *of pebble moves regardless of the initial configuration using the edge detour path. The detour edge pebbling number of a graph G, $f_e^*(G)$, is the maximum $f_e^*(G, e)$ over all the edges of G.*

Definition 2.3. *An edge detour path between e_x and e_y is a sequence of edges $P_e^* = \{e_x, e_1, e_2, \ldots, e_k, e_y\}$ such that $e_i \cap e_{i+1} \neq \emptyset$, no edge appears more than once in the sequence and the path allows repeated vertices. Therefore, the edge detour distance between the edges e_x and e_y, $d^*(e_x, e_y) = |P_e^*|$-1 where $|P_e^*|$ is the total number of edges in the edge detour path.*

Theorem 2.1. *The detour edge pebbling number for path P_n graph is $f_e^*(P_n) = 2^{n-2}$; $n \geq 2$.*

Proof. Let P_n be a path graph with n vertices. Let the edge set of P_n be $E(P_n) = \{e_1, e_2, \ldots, e_{n-1}\}$.
Take 2^{n-2}- 1 pebbles for distribution. Let e_{n-1} be the target edge. Placing 2^{n-2}- 1 pebbles on the edge e_1, the edge e_{n-1} cannot be reached. So, $f_e^*(P_n) \geq 2^{n-2}$.
For proving the sufficient part, consider the target edge to be either e_1 or e_{n-1}. The detour distance from e_k to e_1 is at most *k-1* where $1 < k < n$-1. By using 2^{k-1} pebbles, reaching the target edge either e_1 or e_{n-1} is possible. Now, consider e_k as the target edge. The detour distance from e_k to e_j is at most n-2 where $k < j \leq n$-1. Hence, the edge set $\{e_k, e_{k+1}, \ldots, e_{n-1}\}$ contains at least 2^{n-3} pebbles, we can reach the target edge e_k. Therefore, 2^{n-2} pebbles are sufficient to reach the target edge because the detour distance from e_k to e_i is at most k-1 where $1 \leq i < k$.
Therefore, $f_e^*(P_n) = 2^{n-2}$. □

Theorem 2.2. *The detour edge pebbling number for the Wheel graph is $f_e^*(W_n) = 2^{n+\lceil \frac{n}{3} \rceil}$; when n is odd and $n \geq 3$ and $f_e^*(W_n) = 2^{n+\frac{n}{2}}$; when n is even and $n \geq 4$.*

Proof. **Case (1).** n is odd. Consider the edge set of W_n be $E(W_n) = \{e_1, e_2, \ldots, e_{2n}\}$.
The edge detour path of W_n does not consist of all the edges i.e., for $n = 3$, the edge detour path lacks one edge from the total number of edges of W_3 since the edge detour path has only 5 edges. And the lacking of edges in the edge detour path keeps on increasing by one edge for $n \geq 3$. Take $2^{n+\lceil \frac{n}{3} \rceil}$-1 pebbles for distribution. Consider the target edge to be any one of the spokes of W_n and let it be e_1 and assume there are zero number of pebbles on it. Placing $2^{n+\lceil \frac{n}{3} \rceil}$-1 pebbles on any one of the cycle edges which is adjacent to e_1, the target edge cannot be reached using the edge detour path since the detour distance is $2n$-$\lceil \frac{n}{3} \rceil$-1.
Therefore, $f_e^*(W_n) \geq 2^{n+\lceil \frac{n}{3} \rceil}$.
Now we prove the sufficient part by distributing $2^{n+\lceil \frac{n}{3} \rceil}$ pebbles on the edges of W_n. □

Subcase 1.1: Consider the target edge to be any one of the spokes of W_n and let it be e_1 and assume there are zero number of pebbles on it.
If we place all the pebbles on any one of the cycle edges that is adjacent to the edge e_1, we can reach the target edge with $2^{n+\lceil \frac{n}{3} \rceil}$ pebbles since the length of the edge detour path is $2n$-$\lceil \frac{n}{3} \rceil$. Also, if we alter the configurations of pebbles on the edges, by Theorem 2.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the spokes of the graph W_n.
Subcase 1.2: Consider the target edge to be any one of the cycle edges of W_n and let it be e_1 and assume there are zero number of pebbles on it.
If we place all the pebbles on any one of the spokes that is adjacent to the edge e_1, we can reach the target edge with $2^{n+\lceil \frac{n}{3} \rceil}$ pebbles since the length of the edge detour path is $2n$-$\lceil \frac{n}{3} \rceil$. Also, if we alter the configurations of pebbles on the edges, by Theorem 2.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the cycle edges of the graph W_n.
Thus, $f_e^*(W_n) = 2^{n+\lceil \frac{n}{3} \rceil}$, when n is odd.
Case (2). n is even.
The edge detour path of W_n does not consist of all the edges of W_n i.e., for $n = 4$, the edge detour path lacks one edge from the total number of edges of W_4 since the edge detour path has only 7 edges. And the lacking of edges in the edge detour path keeps on increasing by one edge for $n \geq 4$.
Take $2^{n+\frac{n}{2}}$-1 pebbles for distribution. Consider the target edge to be any one of the cycle edges of W_n and let it be e_1 and assume there are zero number of pebbles on it. Placing $2^{n+\frac{n}{2}}$-1 pebbles on any one of the cycle edges that is adjacent to e_1, the target edge cannot be reached using the edge detour path since the length of the edge detour path is n+$\frac{n}{2}$+1. Therefore, $f_e^*(W_n) \geq 2^{n+\frac{n}{2}}$.
Now we prove that $f_e^*(W_n) \leq 2^{n+\frac{n}{2}}$. The proof follows from case 1.

Theorem 2.3 *The detour edge pebbling number for the triangular snake graph is $f_e^*(TS_n) = 2^{3n-4}$; $n \geq 2$.*
Proof. Consider a path v_1, v_2, ..., v_n. The triangular snake graph is obtained by joining v_i and v_{i+1} to a new vertex u_i for $1 \leq i \leq n$-1. Consider the edge set of TS_n be $E(TS_n) = \{e_1, e_2, ..., e_{3(n-1)}\}$.
Take 2^{3n-4}-1 pebbles for distribution. Consider the target edge to be v_1v_2 and assume it has zero number of pebbles. Place 2^{3n-4}-1 pebbles on the edge v_2v_3, using the edge detour path we cannot move a pebble to v_1v_2 since the length of the edge detour path is 3(n-1) and the edge detour path of TS_n consists of all the edges.
Therefore,$f_e^*(TS_n) \geq 2^{3n-4}$.
Now we prove the sufficient part by distributing 2^{3n-4} pebbles on the edges of TS_n.
Consider the target edge to be any edge of TS_n. Without loss of generality, let it be v_1v_2 and assume it has zero number of pebbles. Consider the pebble allotment in an unusual way of placing 2^{3n-5} pebbles on the edge v_2u_1 and one pebble on the edge v_1u_1. Now, reaching the target edge is possible with 2^{3n-5}+1 pebbles since the edge detour distance from v_1v_2 to v_2u_1 is 3n-5. And by placing 2^{3n-4} pebbles on the edge v_2u_2, using the edge detour path, reaching the target edge is possible. Also, if we alter the configurations of pebbles on the edges, by Theorem 2.1, shifting a pebble to the destination edge is possible. By symmetry we can reach all the edges of the graph TS_n.
Thus, the detour edge pebbling number for the triangular snake graph is $f_e^*(TS_n) = 2^{3n-4}$.

3. Detour edge pebbling number of the square of some standard graphs

Theorem 3.1. *The detour edge pebbling number for the square of path P_n^2 graph is $f_e^*(P_n^2) = 2^{2n-4}$; $n \geq 3$.*
Proof. Let P_n^2 be a square of path graph with n vertices and 2n-3 edges.
Let the edge set $S_1 = \{e_1, e_2, ..., e_{n-1}\}$ be the edges of P_n. Let the edge set $S_2 = \{e_n, e_{n+1}, ..., e_{2n-3}\}$ be the new edges added to the edges of P_n to form P_n^2. Let e_n be the edge adjacent to e_1, e_2 and e_3, let e_{n+1} be the edge adjacent to e_1, e_2, e_3 and e_4, let e_{n+2} be the edge adjacent to e_2, e_3, e_4 and e_5, ..., let e_{2n-4} be the edge adjacent to e_{n-4}, e_{n-3}, e_{n-2} and e_{n-1}, let e_{2n-3} be the edge adjacent to e_{n-1}, e_{n-2} and e_{n-3}.
The edge detour path from e_1 to e_{n-1} has all the edges of P_n^2. Thus, the edge detour distance from e_1 to e_{n-1} is 2n-4. Let e_{n-1} be the destination edge. Take 2^{2n-4}-1 pebbles for distribution. Place 2^{2n-4}-1 pebbles on the edge e_1. Now, using the edge detour path, reaching the destination edge e_{n-1} is not achievable since 2^{2n-4}-1 pebbles are sufficient only to reach the edges which are at a distance of at most 2n-5 from e_1.
So, $f_e^*(P_n^2) \geq 2^{2n-4}$.
Let D be any distribution of 2^{2n-4} pebbles on the edges of P_n^2 to demonstrate the sufficient part.
Case (1). Let e_l be the destination edge where $l \in S_1$.
Subcase 1.1. Let e_1 or e_{n-1} be the destination edge.
Without loss of generality, let e_{n-1} be the destination edge. The edge detour distance from any one of the edges of set S_1 to e_{n-1} is at most 2n-4. Now, reaching the destination edge e_{n-1} using the edge detour path is achievable by distributing at most 2^{2n-4} pebbles.
The edge detour distance from any one of the edges of set S_2 to e_{n-1} is at most 2n-5. Now, reaching the destination edge e_{n-1} using the edge detour path is achievable by distributing at most 2^{2n-5} pebbles. By symmetry, we can prove for e_1.
Subcase 1.2. Let e_2 or e_{n-2} be the destination edge.
Without loss of generality, let e_{n-2} be the destination edge. The detour distance from any one of the edges of set S_1 to e_{n-2} is at most 2n-4. Now, reaching the destination edge e_{n-2} using the edge detour path is achievable by distributing at most 2^{2n-4} pebbles.
The edge detour distance from any one of the edges of set S_2 to e_{n-2} is at most 2n-5. Now, reaching the destination edge e_{n-2} using the edge detour path is achievable by distributing at most 2^{2n-5} pebbles. By symmetry, we can prove for e_2.
Subcase 1.3. Let e_3 or e_{n-3} be the destination edge.
Without loss of generality, let e_{n-3} be the destination edge. The edge detour distance from any one of the edges of set S_1 to e_{n-3} is at most 2n-5. Now, reaching the destination edge e_{n-3} using the edge detour path is achievable by distributing at most 2^{2n-5} pebbles.
The edge detour distance from any one of the edges of set S_2 to e_{n-3} is at most 2n-6. Now, reaching the

destination edge e_{n-3} using the edge detour path is achievable by distributing at most 2^{2n-6} pebbles. By symmetry, we can prove for e_3.

Subcase 1.4. Let $S_3 = \{e_4, e_5, ..., e_{n-5}, e_{n-4}\}$ be the destination edge.

The edge detour distance from any one of the edges of set S_3 to any one of the edges of set S_1 is at most $2n$-6. Now, reaching the destination edge using the edge detour path is achievable by distributing at most 2^{2n-6} pebbles.

The edge detour distance from any one of the edges of set S_3 to any one of the edges of set S_2 is at most $2n$-8. Now, reaching the destination edge using the edge detour path is achievable by distributing at most 2^{2n-8}+n-5 pebbles.

Case (2). Let e_m be the destination edge where $m \in S_2$.

Subcase 2.1. Let e_n or e_{2n-3} be the destination edge.

Without loss of generality, let e_{2n-3} be the destination edge. The edge detour distance from any one of the edges of set S_1 to e_{2n-3} is at most $2n$-5. Now, reaching the destination edge e_{2n-3} using the edge detour path is achievable by distributing at most 2^{2n-5} pebbles.

The edge detour distance from any one of the edges of set S_2 to e_{2n-3} is at most $2n$-6. Now, reaching the destination edge e_{2n-3} using the edge detour path is achievable by distributing at most 2^{2n-6} pebbles. By symmetry, we can prove for e_n.

Subcase 2.2. Let e_{n+1} or e_{2n-4} be the destination edge.

Without loss of generality, let e_{2n-4} be the destination edge. The edge detour distance from any one of the edges of set S_1 to e_{2n-4} is at most $2n$-4. Now, reaching the destination edge e_{2n-4} using the edge detour path is achievable by distributing at most 2^{2n-4} pebbles.

The edge detour distance from any one of the edges of set S_2 to e_{2n-4} is at most $2n$-5. Now, reaching the destination edge e_{2n-4} using the edge detour path is achievable by distributing at most 2^{2n-5} pebbles. By symmetry, we can prove for e_{n+1}.

Subcase 2.3. Let e_{n+2} or e_{2n-5} be the destination edge.

Without loss of generality, let e_{2n-5} be the destination edge. The edge detour distance from any one of the edges of set S_1 to e_{2n-5} is at most $2n$-5. Now, reaching the destination edge e_{2n-5} using the edge detour path is achievable by distributing at most 2^{2n-5}+1 pebbles.

The edge detour distance from any one of the edges of set S_2 to e_{2n-5} is at most $2n$-6. Now, reaching the destination edge e_{2n-5} using the edge detour path is achievable by distributing at most 2^{2n-6}+1 pebbles. By symmetry, we can prove for e_{n+2}.

Subcase 2.4. Let $S_4 = \{e_{n+3}, e_{n+4}, ..., e_{2n-7}, e_{2n-6}\}$ be the destination edge.

The edge detour distance from any one of the edges of set S_4 to any one of the edges of set S_1 is at most $2n$-5. Now, reaching the destination edge using the edge detour path is achievable by distributing at most 2^{2n-5}+1 pebbles.

The edge detour distance from any one of the edges of set S_4 to any one of the edges of set S_2 is at most $2n$-6. Now, reaching the destination edge using the edge detour path is achievable by distributing at most 2^{2n-6}+1 pebbles.

Thus, the detour edge pebbling number for the square of path P_n^2 graph is $f_e^*(P_n^2) = 2^{2n-4}$; $n \geq 3$.

Theorem 2.3. *The detour edge pebbling number for the square of cycle C_n^2 graph is $f_e^*(C_n^2) = 2^{2n-1}$; $n \geq 5$.*

Proof. Let C_n^2 be a square of cycle graph with n vertices and $2n$ edges.

Let the edge set $S_1 = \{e_1, e_2, ..., e_n\}$ be the edges of C_n. Let the edge set $S_2 = \{e_{n+1}, e_{n+2}, ..., e_{2n}\}$ be the new edges added to the edges of C_n to form C_n^2. Let e^* be any one of the adjacent edges of e_{n+1} in S_1. The edge detour path from e_{n+1} to e^* has all the edges of C_n^2. Thus, the edge detour distance from e_{n+1} to e^* is $2n$-1. Let e_{n+1} be the destination edge. Take 2^{2n-1}-1 pebbles for distribution. Place 2^{2n-1}-1 pebbles on the edge e^*. Now, using the edge detour path, reaching the destination edge e_{n+1} is not achievable since 2^{2n-1}-1 pebbles are sufficient only to reach the edges which are at a distance of at most $2n$-2 from e^*.

So, $f_e^*(C_n^2) \geq 2^{2n-1}$.

Let D be any distribution of 2^{2n-1} pebbles on the edges of C_n^2 to demonstrate the sufficient part.

Case (1). Let e_l be the destination edge where $l \in S_1$.

Without loss of generality, let e_n be the destination edge. The edge detour distance from e_n to e_{n-1} is $2n$-1. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing 2^{2n-1}

pebbles.
The edge detour distance from e_n to e_{n-2} is $2n$-2. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing 2^{2n-2} pebbles.
The edge detour distance from e_n to any one of the remaining edges in S_1 is at most $2n$-3. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most 2^{2n-3} pebbles.
The edge detour distance from e_n to any one of the edges in S_2 which is adjacent to e_n is $2n$-1. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing 2^{2n-1} pebbles.
The edge detour distance from e_n to any one of the edges in S_2 which is not adjacent to e_n is at most $2n$-2. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most 2^{2n-2} pebbles. By symmetry, we can prove for all e_l.
Case (2). Let e_m be the destination edge where $m \in S_2$.
Without loss of generality, let e_{n+1} be the destination edge. The edge detour distance from e_{n+1} to any one of its adjacent edges in S_1 is $2n$-1. Now, reaching the destination edge e_{n+1} using the edge detour path is achievable by distributing at most 2^{2n-1} pebbles.
The edge detour distance from e_{n+1} to any one of its non-adjacent edges in S_1 is at most $2n$-2. Now, reaching the destination edge e_{n+1} using the edge detour path is achievable by distributing at most 2^{2n-2} pebbles.
The edge detour distance from e_{n+1} to any one of its adjacent edges in S_2 is at most $2n$-1. Now, reaching the destination edge e_{n+1} using the edge detour path is achievable by distributing at most 2^{2n-1} pebbles.
The edge detour distance from e_{n+1} to any one of its non-adjacent edges in S_2 is at most $2n$-2. Now, reaching the destination edge e_{n+1} using the edge detour path is achievable by distributing at most 2^{2n-2} pebbles. By symmetry, we can prove for all e_m.
Thus, the detour edge pebbling number for the square of cycle C_n^2 graph is $f_e^*(C_n^2) = 2^{2n-1}$; $n \geq 5$. ☐

Theorem 3.3. *The detour edge pebbling number for the square of star S_n^2 graph is $f_e^*(S_n^2) = 2^{2n+\lceil \frac{n}{3} \rceil}$; $n \geq 5$ when n is odd and $f_e^*(S_n^2) = 2^{\frac{5n}{2}}$; $n \geq 6$ when n is even.*
Proof. Let S_n^2 be a square of star graph with n+1 vertices and $3n$ edges.
Let the edge set $S_1 = \{e_1, e_2, ..., e_n\}$ be the edges of S_n. Let the edge set $S_2 = \{e_{n+1}, e_{n+2}, ..., e_{3n}\}$ be the new edges added to the edges of S_n to form S_n^2 i.e., let the edge set $S_3 = \{e_{n+1}, e_{n+2}, ..., e_{2n}\}$ be the edges of cycle C_n added to the edges of star graph S_n to form a wheel graph W_n $S_4 = \{e_1, e_2, ..., e_n, e_{n+1}, e_{n+2}, ..., e_{2n}\}$ as a subgraph of S_n^2 and let the edge set $S_5 = \{e_{2n+1}, e_{2n+2}, ..., e_{3n}\}$ be the remaining edges incident with the vertices of cycle C_n.
Case (1). When n is odd.
The edge detour path from e_{2n+1} to any one of the edges of the set S_4 does not consist of all the edges of S_n^2 since for $n = 5$, the edge detour path lacks two edges from the total number of edges of S_5^2, for $n = 7$, the edge detour path lacks three edges from the total number of edges of S_7^2 and the lacking of edges in the edge detour path keeps on increasing by one edge for $n \geq 5$.
Thus, the edge detour distance from e_{2n+1} to any one of the edges of the set S_4 is $\frac{5n-1}{2}$. Let e^* be any one of the edges of the set S_4 and let it be the destination edge. Take $2^{2n+\lceil \frac{n}{3} \rceil}$-1 pebbles for distribution. Place $2^{2n+\lceil \frac{n}{3} \rceil}$-1 pebbles on the edge e_{2n+1}. Now, using the edge detour path, reaching the destination edge e^* is not achievable since $2^{2n+\lceil \frac{n}{3} \rceil}$-1 pebbles are sufficient only to reach the edges which are at a distance of at most $\frac{5n-3}{2}$ from e_{2n+1}.
So, $f_e^*(S_n^2) \geq 2^{2n+\lceil \frac{n}{3} \rceil}$.
Let D be any distribution of $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles on the edges of S_n^2 to demonstrate the sufficient part.
Subcase 1.1. Let e_u be the destination edge where $u \in$ the set S_1.
Without loss of generality, let e_n be the destination edge. The edge detour distance from e_n to e_v is at most $\frac{5n-1}{2}$ where $v \in$ the set S_3. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_n to e_w is at most $\frac{5n-1}{2}$ where $w \in$ the set S_5. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_n to any other edges of the set S_1 is at most $\frac{5n-1}{2}$. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles. By symmetry, we can prove for all e_u.
Subcase 1.2. Let e_v be the destination edge where $v \in$ the set S_3.

Without loss of generality, let e_{2n} be the destination edge. The edge detour distance from e_{2n} to e_u is at most $\frac{5n-1}{2}$ where $u \in$ the set S_1. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_{2n} to e_w is at most $\frac{5n-1}{2}$ where $w \in$ the set S_5. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_{2n} to any other edges of the set S_3 is at most $\frac{5n-1}{2}$. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles. By symmetry, we can prove for all e_v.
Subcase 1.3. Let e_w be the destination edge where $w \in$ the set S_5.
Without loss of generality, let e_{3n} be the destination edge. The edge detour distance from e_{3n} to e_u is at most $\frac{5n-1}{2}$ where $u \in$ the set S_1. Now, reaching the destination edge e_{3n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_{3n} to e_v is at most $\frac{5n-1}{2}$ where $v \in$ the set S_3. Now, reaching the destination edge e_{3n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles.
The edge detour distance from e_{3n} to any other edges of the set S_5 is at most $\frac{5n-1}{2}$. Now, reaching the destination edge e_{3n} using the edge detour path is achievable by distributing at most $2^{2n+\lceil \frac{n}{3} \rceil}$ pebbles. By symmetry, we can prove for all e_w.
Thus, the detour edge pebbling number for the square of star S_n^2 graph is $f_e^*(S_n^2) = 2^{2n+\lceil \frac{n}{3} \rceil}$; $n \geq 5$ when n is odd.
Case (2). When n is even.
The edge detour path from e_{2n+1} to any one of the edges of the set S_3 does not consist of all the edges of S_n^2 since for $n = 6$, the edge detour path lacks two edges from the total number of edges of S_6^2, for $n = 8$, the edge detour path lacks three edges from the total number of edges of S_8^2 and the lacking of edges in the edge detour path keeps on increasing by one edge for $n \geq 6$.
Thus, the edge detour distance from e_{2n+1} to any one of the edges of the set S_3 is $\frac{5n}{2}$. Let e^* be any one of the edges of the set S_3 and let it be the destination edge. Take $2^{\frac{5n}{2}}$-1 pebbles for distribution. Place $2^{\frac{5n}{2}}$-1 pebbles on the edge e_{2n+1}. Now, using the edge detour path, reaching the destination edge e^* is not achievable since $2^{\frac{5n}{2}}$-1 pebbles are sufficient only to reach the edges which are at a distance of at most $\frac{5n-2}{2}$ from e_{2n+1}.
So, $f_e^*(S_n^2) \geq 2^{\frac{5n}{2}}$.
Let D be any distribution of $2^{\frac{5n}{2}}$ pebbles on the edges of S_n^2 to demonstrate the sufficient part.
Subcase 2.1. Let e_u be the destination edge where $u \in$ the set S_1.
Without loss of generality, let e_n be the destination edge. The edge detour distance from e_n to e_v is at most $\frac{5n}{2}$ where $v \in$ the set S_3. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_n to e_w is at most $\frac{5n}{2}$ where $w \in$ the set S_5. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_n to any other edges of the set S_1 is at most $\frac{5n}{2}$. Now, reaching the destination edge e_n using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles. By symmetry, we can prove for all e_u.
Subcase 2.2. Let e_v be the destination edge where $v \in$ the set S_3.
Without loss of generality, let e_{2n} be the destination edge. The edge detour distance from e_{2n} to e_u is at most $\frac{5n}{2}$ where $u \in$ the set S_1. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_{2n} to e_w is at most $\frac{5n}{2}$ where $w \in$ the set S_5. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_{2n} to any other edges of the set S_3 is at most $\frac{5n}{2}$. Now, reaching the destination edge e_{2n} using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles. By symmetry, we can prove for all e_v.
Subcase 2.3. Let e_w be the destination edge where $w \in$ the set S_5.
Without loss of generality, let e_{3n} be the destination edge. The edge detour distance from e_{3n} to e_u is at most $\frac{5n}{2}$ where $u \in$ the set S_1. Now, reaching the destination edge e_{3n} using the edge detour path is achievable

by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_{3n} to e_v is at most $\frac{5n}{2}$ where $v \in$ the set S_3. Now, reaching the destination edge e_{3n} using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles.
The edge detour distance from e_{3n} to any other edges of the set S_5 is at most $\frac{5n}{2}$. Now, reaching the destination edge e_{3n} using the edge detour path is achievable by distributing at most $2^{\frac{5n}{2}}$ pebbles. By symmetry, we can prove for all e_w.
Thus, the detour edge pebbling number for the square of star S_n^2 graph is $f_e^*(S_n^2) = 2^{\frac{5n}{2}}$; $n \geq 6$ when n is even.

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