IMPLEMENTATION AND ASSESSMENT OF THE SIMPLE EQUATION TECHNIQUE FOR SOLVING NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract
In this paper, the simple equation method is especially used to solve two Nonlinear Partial Differential Partial Equations NLPDEs, the Kodomstev-Petviashvili (KP) equation and the (2+1)-dimensional breaking soliton equation. The modified Benjamin-Bona-Mahony equation and the Klein-Gordon equation in (1+2) dimensions are two illustrations of second order nonlinear equations that can benefit from using this approach. The Bernoulli equation acts as the trial condition and aids in the mathematical description of the nonlinear wave equation in the simple equation.

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1 Introduction
The Nonlinear Partial Differential Partial Equations (NLPDEs) plays an important role in the study of various problems in physical sciences, Bio-engineering mechanics and Chemical phenomena. Finding the exact solution of NLPDEs is important in the study of most non-linear physical phenomena [3,7]. In the recent years, many new efficient methods to obtain the exact solution has been proposed such as inverse scattering method [5], Hirota’s direct method [6], Tanh method [9], multiple exp-function method [4], first integral method [4], simple equation method [12], modified simple equation method [1,11,15-20] and so on.

In present paper, we analyse some of these technique to obtain the exact solution to NLPDEs using simple equation method and modified simple equation method

for the second order nonlinear partial differential equation \((au''(\tau) + bu(\tau) + cu^2(\tau) = 0)\). In the Section 2, we have given the brief algorithm for the simple method (using Bernoulli equation). In the Section 3, we have used these techniques to find the exact solution to Kodomstev-Petviashvili (KP) equation, (2+1)-dimensional breaking soliton equation, Klein-Gordon equation in (1+2) dimension and the modified Benjamin-Bona-Mahony equation.

2 Computational procedures

2.1 Algorithm of the simple equation method

In this section, the direct method called simple equation method for finding the traveling wave solution of nonlinear evolution equation. Let the nonlinear partial equation in two independent variable \(\alpha\) and \(\gamma\) is given by

\[ Q \{u, u_\gamma, u_\alpha, u_{\gamma\gamma}, u_{\alpha\alpha}, u_{\alpha\gamma}, \ldots\} = 0, \tag{2.1} \]

where \(u(\alpha, \gamma)\) is the unknown function, and \(Q\) is a polynomial function \(u(\alpha, \gamma)\) and its partial derivatives.
**Step 1:**
Combine the independent variable $\alpha$ and $\gamma$ into one variable $\tau = \alpha - c\gamma$, such that
\[
\alpha = \underbrace{u(\tau)}_{\text{function in } \tau}, \quad \tau = \alpha - c\gamma. \tag{2.2}
\]

Then from equation (2.2), equation (2.1) reduces to following Ordinary Differential Equation (ODE)
\[
R(u, u', u'', \ldots) = 0, \tag{2.3}
\]
where $R$ is a polynomial function of $u(\tau)$ and its derivatives.

**Step 2:**
The solution of equation (2.2) is of the form
\[
u(\tau) = \sum_{i=0}^{n} a_i F^{i}(\tau). \tag{2.4}
\]
where $a_i (i = 0, 1, 2, \ldots, n)$ are constants and $F(\tau)$ are functions that satisfy the simple equation (ODE). The simple equation we are using in this paper is Bernoulli Equation which is nonlinear ODE and its solution can be expressed by elementary function.

Consider the Bernoulli equation
\[
F'(\tau) = cF(\tau) + dF^{2}(\tau). \tag{2.5}
\]

**Step 3:**
On balancing the highest order derivative and nonlinear terms in equation (2.2), $n = 2$.

**Step 4:**
The general solution of simple equation (2.5) is
\[
F(\tau) = \frac{c e^{[c(\tau + \tau_0)]}}{1 - de^{[c(\tau + \tau_0)]}}. \tag{2.6}
\]

For case $b < 0$, $a > 0$, and $\tau_0$ is a constant of the integration, and
\[
F(\tau) = - \frac{c e^{[c(\tau + \tau_0)]}}{1 + de^{[c(\tau + \tau_0)]}}. \tag{2.7}
\]

### 2.2 Modified simple equation method

Let the equation (2.3) have solution in the form given below
\[
u(\tau) = \sum_{i=0}^{n} a_i \left( \frac{\varphi^{'}(\tau)}{\varphi(\tau)} \right)^{i}, \tag{2.8}
\]
where $a_i (i = 0, 1, 2, 3, \ldots)$ are arbitrary constants and $\varphi(\tau)$ is an unknown function to be determined.

We determine $n$ in equation (2.8) by balancing the highest order derivative and nonlinear terms in equation (2.3).

Let us consider the following ODE
\[
a u''(\tau) + b u'(\tau) + cu^{3}(\tau) = 0. \tag{2.9}
\]

Balancing the highest order derivative and highest degree we get $n = 1$. Hence the solution is of form
\[
u(\tau) = a_0 + a_1 \left( \frac{\varphi^{'}(\tau)}{\varphi(\tau)} \right), \tag{2.10}
\]
where $a_0 \& a_1 \neq 0$ are constants. On making an appeal to (2.10) into (2.9), we get
\[
ba_0 + ca_0^3 + ba_1\varphi^{'}(\tau) + aa_1\varphi^{'''}(\tau) + 3ca_0a_1^{2}\varphi^{'}(\tau) + \frac{3ca_0a_1^2(\varphi^{'}(\tau))^2 - 3aa_1\varphi^{'}(\tau)^2 - 2aa_1(\varphi^{'}(\tau))^3 + ca_1^3(\varphi^{'}(\tau))^3}{\varphi(\tau)^3} \varphi^{'}(\tau) = 0. \tag{2.11}
\]

Equating $\varphi^{0}, \varphi^{-1}, \varphi^{-2}$ and $\varphi^{-3}$ to zero, we have following equations:
\[
ba_0 + ca_0^3 = 0; \tag{2.12}
\]
\[
ba_1\varphi^{'}(\tau) + aa_1\varphi^{'''}(\tau) + 3ca_0^2a_1\varphi^{'}(\tau) = 0, \tag{2.13}
\]
\[
3ca_0a_1^2(\varphi^{'}(\tau))^2 - 3aa_1\varphi^{'}(\tau)^2 - 2aa_1(\varphi^{'}(\tau))^3 + ca_1^3(\varphi^{'}(\tau))^3 = 0. \tag{2.14}
\]

Equating $\varphi^{0}, \varphi^{-1}, \varphi^{-2}$ and $\varphi^{-3}$ to zero, we have following equations:
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ba_0 + ca_0^3 = 0; \tag{2.12}
\]
\[
ba_1\varphi^{'}(\tau) + aa_1\varphi^{'''}(\tau) + 3ca_0^2a_1\varphi^{'}(\tau) = 0, \tag{2.13}
\]
\[
3ca_0a_1^2(\varphi^{'}(\tau))^2 - 3aa_1\varphi^{'}(\tau)^2 - 2aa_1(\varphi^{'}(\tau))^3 + ca_1^3(\varphi^{'}(\tau))^3 = 0. \tag{2.14}
\]

From equation (2.12), we have
\[
a_0 = 0, \pm \sqrt{-\frac{b}{c}}. \tag{2.15}
\]
Case I:
If $a_0 = 0$, then we can obtain the trivial solution, which is rejected.

Case II:
If $a_0 = \pm \sqrt{-\frac{b}{c}}$, then

$$a\varphi'''(\tau) - 2b\varphi'(\tau) = 0. \quad (2.15)$$

and

$$-a\varphi''(\tau) + \sqrt{2ab}\varphi'(\tau) = 0. \quad (2.16)$$

By making an appeal to (2.15) and (2.16), we get

$$\pm \sqrt{\frac{a}{2b}}\varphi'''(\tau) - \varphi''(\tau) = 0. \quad (2.17)$$

Hence the general solution of equation (2.17) is given by

$$\varphi(\tau) = a_0 + a_1\tau + a_2e^{\pm\sqrt{\frac{2b}{a}}\tau}. \quad (2.18)$$

Thus the exact solution of ODE is of form as given below

$$\varphi(\tau) = a_0 + a_1\tau + a_2e^{\pm\sqrt{\frac{2b}{a}}\tau}. \quad (2.19)$$

3 Application of non-linear simple equation

3.1 KdV-Petviashvili (KP) equation

The KdV-Petviashvili (KP) equation [8, 13] is named after Boris Borisovich Kadomtsev and Vladimir Iosifovich Petviashvili as a partial differential equation to describe nonlinear wave motion. The KP equation is usually written as:

$$\delta_\alpha \left( \delta_\gamma u + u\delta_\alpha u + \epsilon^2 \delta_{\alpha\alpha\alpha} u + \omega \delta_{\beta\beta} u \right) = 0. \quad (3.1)$$

Let us consider the KP equation as

$$u_{\alpha\gamma} + 6uu_{\alpha} - 6(u_{\alpha})^2 + u_{\alpha\alpha\alpha} + 3\omega^2 u_{\beta\beta} = 0 \quad (3.2)$$

or

$$u_{\alpha\gamma} + 6uu_{\alpha} - 6(u_{\alpha})^2 + u_{\alpha\alpha\alpha} + 3\omega^2 u_{\beta\beta} = 0,$$

which is a 2-dimensional generalization of the KdV equation introduced this equation to describe slowly varying nonlinear waves in a dispersive used for finding exact solution of ODEs in mathematical physics [6,9]. The equation with $\omega^2 = -1$ arise in acoustics and admits unstable soliton solution, whereas $\omega^2 = 1$ is stable solution.

Suppose travelling wave transformation equation is

$$u(\tau) = u(\alpha, \beta, \gamma), \text{ where } \tau = \alpha + \beta - v\gamma. \quad (3.3)$$

Making an appeal to (3.2), (3.3) reduces into following ODE

$$(-vu' - 6uu' + u''')' + 3\omega^2 u'' = 0. \quad (3.4)$$

On integrating equation (3.4) twice with respect to $\tau$, and setting the integration constant equal to zero, we have

$$u'' + (3\omega^2 - v)u - 3u^2 = 0. \quad (3.5)$$

On balancing the number $n$ which is positive integer with the highest order derivative terms $u''$ in equation (2.4), i.e. $n = 2$ $2n$, hence $n = 2$.

Therefore the solution of equation (3.5) can be expressed as:

$$u = \sum_{i=0}^{2} a_i(F(\tau))^i = a_0 + a_1F + a_2F^2, \quad (3.6)$$

where $F$ satisfies equation (2.18), consequently we have:

$$u' = a_1F' + 2a_2FF' = a_1cF + (2a_2c + da_1) F^2 + 2a_2dF^3, \quad (3.7)$$
\[ u'' = a_1 F'' + 2a_2 \left( FF'' + (F')^2 \right) = a_1 c^2 F + (4a_2 c^2 + 3a_1 d) F^2 + (10a_2 d c + 2a_1 d^2) F^3 + 6a_2 d^2 F^4, \]
\[ u^2 = a_0^2 + 2a_0 a_1 F + (2a_0 a_2 + a_1^2) F^2 + 2a_1 a_2 F^3 + a_2^2 F^4. \]

On substituting equation (3.6) and (3.7) in equation (3.5) and equating the coefficient of \( F^i \) to zero, where \( i \geq 0 \), we get
\[ a_0 \left( 3\omega^2 - v \right) - 3a_0^2 = 0, \]
\[ a_1 \left( c^2 + (3\omega^2 - v) - 6a_0 \right) = 0, \]
\[ 4a_2 c^2 + 3a_1 c d + a_2 \left( 3\omega^2 - v \right) - 6a_0 a_2 - 3a_1^2 = 0, \]
\[ 10a_2 d c + 2a_1 d^2 - 6a_1 a_2 = 0, \]
\[ 6a_2 d^2 - 3a_1^2 = 0. \] (3.8)

Solving equations (3.8), we find that solution of equation (3.1) exists only in the following two cases:

**Case I:**
\[ a_0 = \frac{c^2}{3}, a_1 = 2cd, a_2 = 2d^2, v = 3\omega^2 - c^2, \quad cd \neq 0. \] (3.9)

**Case II:**
\[ a_0 = 0, a_1 = 2cd, a_2 = 2d^2, v = 3\omega^2 + c^2, \quad cd \neq 0. \] (3.10)

In Case I, when \( d < 0 \) and \( c > 0 \), the solution of equation (3.2) is given by
\[ u_1(\alpha, \beta, \gamma) = \frac{c^2}{3} + \frac{2c^2 d \exp \left[ c \left( (\alpha + \beta) + (c^2 - 3\omega^2) \gamma \right) \right]}{(1 - d \exp [c(\alpha + \beta) + (c^2 + 3\omega^2)\gamma])^2}. \] (3.11)

In Case II, the solution of equation (3.2) is
\[ u_2(\alpha, \beta, \gamma) = \frac{2c^2 d \exp \left[ c \left( (\alpha + \beta) + (c^2 + 3\omega^2) \gamma \right) \right]}{(1 - d \exp [c(\alpha + \beta) + (c^2 + 3\omega^2)\gamma])^2}. \] (3.12)

In Case I, when \( d > 0 \) and \( c < 0 \), the solution of equation (3.2) is
\[ u_3(\alpha, \beta, \gamma) = \frac{c^2}{3} + \frac{2c^2 d \exp \left[ c \left( (\alpha + \beta) + (c^2 - 3\omega^2) \gamma \right) \right]}{(1 + d \exp [c(\alpha + \beta) + (c^2 - 3\omega^2)\gamma])^2}. \] (3.13)

In Case II, the solution of equation (3.2) is
\[ u_4(\alpha, \beta, \gamma) = \frac{2c^2 d \exp \left[ c \left( (\alpha + \beta) + (c^2 + 3\omega^2) \gamma \right) \right]}{(1 + d \exp [c(\alpha + \beta) + (c^2 + 3\omega^2)\gamma])^2}. \] (3.14)

### 3.2 3.2(2+1) dimensional breaking soliton equation

Let us consider the (2+1) dimensional breaking soliton equation[20] as:
\[ u_\gamma + ku_{\alpha \alpha \gamma} + 4(uv)_\alpha = 0, \quad u_\beta = v_\alpha, \] (3.15)
where \( k \) is an arbitrary constant.

Let the wave transformation \( u(\alpha, \beta, \gamma) = u(\tau) \),
\[ \tau = \alpha + \beta - w_\gamma, \] to reduce equation (3.15) into following ODE :
\[ -wu' + ku''' + 4(u^2)' = 0, \] (3.16)
where \( u = v \). Integrating equation (3.15) with respect to \( \tau \), we get
\[ u'' - \frac{wu}{k} + \frac{4u^2}{k} = 0, \] (3.17)
with zero constant of integration. On balancing the highest order derivative \( u''' \) and nonlinear term \( u^2 \), we get \( n = 2 \), hence the solution of equation (3.15) has the form
\[ u = \sum_{i=0}^{2} a_i (F(\tau))^i = a_0 + a_1 F(\tau) + a_2 (F(\tau))^2, \] (3.18)
where \( a_0, a_1, \) & \( a_2 \) are constants such that \( a_2 \neq 0 \), while \( c,d \) are arbitrary constants. Making an appeal to eqs. (3.6) and (3.7) and setting the coefficients of \( F(\tau) \) to be zero, we get
\[ 6a_2 d^2 + \frac{4a_1^2}{k} = 0, \]
\[ 2a_1 d^2 + 10a_2 cd + \frac{8a_1 a_2}{k} = 0, \]
\[ 3a_1 cd - \frac{ca_2}{k} + 4a_2 c^2 + \frac{8a_0 a_2}{k} + \frac{4a_1^2}{k} = 0, \]
\[ - \frac{ca_1}{k} + a_1 c^2 + \frac{8a_0 a_1}{k} = 0, \]
\[ \frac{4a_0^2}{k} - \frac{ca_0}{k} = 0. \] (3.19)

Solving equations (3.19), we find that solution of equation (3.17) exists only in following two cases:

**Case I:**
\[ a_0 = -\frac{1}{4k}, a_1 = \frac{3}{2}d, a_2 = -\frac{3}{2}d^2k, c = -\frac{1}{k}. \] (3.20)

**Case II:**
\[ a_0 = 0, a_1 = -\frac{3}{2}d, a_2 = -\frac{3}{2}d^2k, c = \frac{1}{k}. \] (3.21)

In Case I, when \( d < 0 \) and \( c > 0 \), the solution of equation (3.17) is
\[ u_1(\alpha, \beta, \gamma) = -\frac{1}{4k} + \frac{3d \exp \left[ -\frac{1}{k} \left( (\alpha + \beta) + \frac{1}{k} \gamma \right) \right]}{2k (1 - d \exp \left[ -\frac{1}{k} \left( (\alpha + \beta) + \frac{1}{k} \gamma \right) \right])^2}. \] (3.22)

Using Case II
\[ u_2(\alpha, \beta, \gamma) = \frac{-3d \exp \left[ \frac{1}{k} \left( (\alpha + \beta) - \frac{1}{k} \gamma \right) \right]}{2k (1 - d \exp \left[ \frac{1}{k} \left( (\alpha + \beta) - \frac{1}{k} \gamma \right) \right])^2}. \] (3.23)

In Case I, when \( d > 0 \) and \( c < 0 \), the solution of equation (3.16) is given by
\[ u_3(\alpha, \beta, \gamma) = -\frac{1}{4k} + \frac{3d \exp \left[ -\frac{1}{k} \left( (\alpha + \beta) + \frac{1}{k} \gamma \right) \right]}{2k (1 + d \exp \left[ -\frac{1}{k} \left( (\alpha + \beta) + \frac{1}{k} \gamma \right) \right])^2}. \] (3.24)

Using Case II
\[ u_4(\alpha, \beta, \gamma) = \frac{3d \exp \left[ \frac{1}{k} \left( (\alpha + \beta) - \frac{1}{k} \gamma \right) \right]}{2k (1 + d \exp \left[ \frac{1}{k} \left( (\alpha + \beta) - \frac{1}{k} \gamma \right) \right])^2}. \] (3.25)

### 3.3 Klein-Gordon equation in (1+2) dimension

Consider the Klein-Gordon equation in (1+2) dimension [13]
\[ w_{\gamma \gamma} - k^2 (w_{\alpha \alpha} + w_{\beta \beta}) + au - bu^3 = 0, \] (3.26)
where \( k, a\&b \) are real constants.

Let \( u(\alpha, \beta, \gamma) = u(\tau) \), where \( \tau = b_1 \alpha + b_2 \beta - w \gamma \).

Substituting these values in equation (3.26), we get
\[ (w^2 - k^2 (b_1^2 + b_2^2)) u''(\tau) + au(\tau) - bu^3(\tau) = 0. \] (3.27)

By employing Section 2.2, we can find the following exact solution of equation (3.26)
\[ u_1 = \pm \sqrt{\frac{a}{b}} \left[ -1 + \sqrt{\frac{2 \left( w^2 - k^2 (b_1^2 + b_2^2) \right)}{a}} \frac{a_1 \pm a_2 e \sqrt{\frac{2a}{w^2 - k^2 (b_1^2 + b_2^2)}}}{a_0 + a_1 \tau + a_2 e \sqrt{\frac{2a}{w^2 - k^2 (b_1^2 + b_2^2)}}} \right]. \] (3.28)

The exact soliton solution of equation (3.26) for \( a (w^2 \pm k^2 (b_1^2 + b_2^2)) > 0 \):
\[ u_2 = \pm \sqrt{\frac{a}{b}} \times \tanh \left( \sqrt{\frac{a}{2 \left( w^2 - k^2 (b_1^2 + b_2^2) \right)}} (b_1 \alpha + b_2 \beta - w \gamma) \right), \] (3.29)
\[ u_3 = \pm \sqrt{\frac{a}{b}} \times \coth \left( \sqrt{\frac{a}{2 \left( w^2 - k^2 (b_1^2 + b_2^2) \right)}} (b_1 \alpha + b_2 \beta - w \gamma) \right). \] (3.30)
3.4 Modified Benjamin-Bona-Mahony Equation

Consider the modified Benjamin-Bona-Mahony equation (mBBM) \[ 2 \]

\[ u_{\gamma} + u_{\alpha} + u^{2}u_{\alpha} + u_{\alpha\alpha\gamma} = 0. \] (3.31)

Suppose the wave transformation is given below

\[ u(\alpha, \gamma) = u(\tau), \quad \tau = \alpha - \omega\gamma, \] (3.32)

where \( \omega \) is a constant.

Making an appeal to (3.32) in (3.31), we obtain ordinary differential equation

\[ (1 - \omega)u'(\tau) + u^{2}(\tau)u(\tau) - \omega u'''(\tau) = 0. \] (3.33)

Integrating (3.33) with respect to \( \tau \) and considering the zero constants for integration, we get

\[ (1 - \omega)u(\tau) + \frac{1}{3}u^{3}(\tau) - \omega u''(\tau) = 0. \] (3.34)

Employing Section 2.2, the new exact solution of equation (3.31) is

\[ u_{1} = \pm \sqrt{3(\omega - 1)(1 + \sqrt{\frac{2\omega}{\omega - 1} \times \frac{a_{1} \pm a_{2} \sqrt{\frac{2(\omega - 1)}{\omega - 1}} (\alpha - \omega\gamma)}{a_{0} + a_{1}(\alpha - \omega\gamma) + a_{2} e^{\pm \sqrt{\frac{2(\omega - 1)}{\omega - 1}} (\alpha - \omega\gamma)}}}, \]

\[ \omega(1 - \omega) < 0. \] (3.35)

The exact soliton solution of equation (3.31) is

\[ u_{2} = \pm \sqrt{3(\omega - 1)} \tanh \left( \frac{\sqrt{\frac{\omega - 1}{2\omega}} (\alpha - \omega\gamma) \right), \] (3.36)

\[ u_{3} = \pm \sqrt{3(\omega - 1)} \coth \left( \frac{\sqrt{\frac{\omega - 1}{2\omega}} (\alpha - \omega\gamma) \right). \] (3.37)

4 Conclusion

In the present study of this paper, we analysis the simple equation method is used to investigate the exact solution of some Nonlinear Partial Differential equations such as KP equation, (2+1)-dimensional breaking soliton equation. In the simple equation the trial condition is Bernoulli equation and help to solve the nonlinear wave equation in mathematical concepts. It is found that this method works well for certain types of nonlinear PDEs, particularly those that are first-order and can be transformed into a system of ODEs using characteristics. However, for more complex cases or different types of nonlinearities, other methods like numerical techniques or perturbation methods may be necessary. The techniques presented here are efficient, strong, and beneficial to generate accurate solutions to nonlinear partial differential equations (NPDE). They can be employed for solving a wide range of NLPDEs, including many that have fractional orders. In summary, we concluded that the study’s results are novel and that, to the best of our knowledge, these mathematical tools’ application to the proposed model hasn’t been recorded in any previous literature. As a result, we believe these tools are likely to be helpful in learning about the dynamics of physical behaviors. The most effective way for emphasizing the actual physical behavior of situations seen in real life is through visualization.

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