NEW REGION FOR ZEROS OF A POLAR DERIVATIVE OF POLYNOMIAL
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Abstract
Let \( P(z) := \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). Then by Cauchy’s Classical result, all zeros of \( P(z) \) lie in
\[
|z| \leq 1 + \max_{0 \leq j \leq n-1} \left( \left| \frac{a_j}{a_n} \right| \right).
\]

In this paper, we shall extend such results to polar derivative of an algebraic polynomial.

Keywords and Phrases: Polar derivative, Polynomial, Zero, Holder’s inequality.

1 Introduction
Let \( P(z) := \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) over \( \mathbb{C} \). Then Polar derivative [2,3,4,5,7] of \( P(z) \) with respect to \( \alpha \), denoted by \( D_\alpha P(z) \) is defined by
\[
D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \quad \text{for } \alpha \text{ real or complex number.}
\]

The polynomial \( D_\alpha P(z) \) is of degree at most \( n - 1 \) and it generalizes the ordinary derivative, where \( \alpha \) is real or complex number, in the sense that
\[
\lim_{\alpha \to \infty} D_\alpha P(z) = P'(z).
\]

Here we first mention the result of Cauchy [1], concerning the bounds for the moduli of zeros [6] of algebraic polynomial.

Theorem 1.1. Let \( P(z) := \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). Then all zeros of \( P(z) \) lie in
\[
|z| \leq 1 + \max_{0 \leq j \leq n-1} \left( \left| \frac{a_j}{a_n} \right| \right).
\]
This result is well known in the theory of zero distribution of polynomials.

2 Main Result
We establish a result by using Holder’s inequality and Polar derivative in improvement of the bound.

Theorem 2.1. Let \( P(z) := \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \). Then for any \( p \) and \( q \) such that \( p > 1, q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), all zeros of \( D_\alpha P(z) \) lie in
\[
|z| \leq \left[ 1 + \left( \sum_{j=0}^{n-2} \left| \frac{k_j}{k_n} \right|^p \right)^\frac{1}{p} \right]^\frac{1}{q},
\]
\[
|z| \leq \left[ 1 + (n - 1)^\frac{q}{2} M^q \right]^\frac{1}{q},
\]
where \( k_j = (n-j)a_{n-j} + (j+1)a_{n-j-1}, j = 1, 2, ..., n-1, \), \( k_n = n\alpha a_n + a_{n-1} \) and \( M = \max_{0 \leq j \leq n-2} \left| \frac{k_j}{k_n} \right| \).

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Proof. Suppose $P(z) := \sum_{j=0}^{n} a_j z^j$. Then

$$D_\alpha P(z) = (n_\alpha a_n + a_{n-1})z^{n-1} + ((n-1) \alpha a_{n-1} + 2a_{n-2})z^{n-2} + ((n-2) \alpha a_{n-2} + 3a_{n-3})z^{n-3}$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \ldots + (3\alpha a_3 + (n-2)a_2)z + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0). \tag{2.3}$$

Therefore

$$|D_\alpha P(z)| = |(n_\alpha a_n + a_{n-1})|z^{n-1} + ((n-1) \alpha a_{n-1} + 2a_{n-2})z^{n-2} + ((n-2) \alpha a_{n-2} + 3a_{n-3})z^{n-3}$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \ldots + (3\alpha a_3 + (n-2)a_2)z^2 + (2\alpha a_2 + (n-1)a_1)z + (\alpha a_1 + na_0)|. \tag{2.4}$$

Equivalently,

$$|D_\alpha P(z)| = \left| (n_\alpha a_n + a_{n-1})z^{n-1} \left[ 1 + \left( \frac{(n-1)\alpha a_{n-1} + 2a_{n-2}}{n_\alpha a_n + a_{n-1}} \right) \frac{1}{z} + \left( \frac{(n-2)\alpha a_{n-2} + 3a_{n-3}}{n_\alpha a_n + a_{n-1}} \right) \frac{1}{z^2} \right] \right|$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad + \ldots + \left( \frac{3\alpha a_3 + (n-2)a_2}{n_\alpha a_n + a_{n-1}} \right) \frac{1}{z^{n-3}} + \left( \frac{(2\alpha a_2 + (n-1)a_1)}{n_\alpha a_n + a_{n-1}} \right) \frac{1}{z^{n-2}} + \left( \frac{\alpha a_1 + na_0}{n_\alpha a_n + a_{n-1}} \right) \frac{1}{z^{n-1}} \right| \right| \geq |n_\alpha a_n + a_{n-1}|z^{n-1} \left[ 1 - \left( \sum_{j=1}^{n-1} |\zeta_{n-j}| \frac{1}{|z|^{n-j}} \right) \right], \tag{2.5}$$

where

$$\zeta_{n-j} = \frac{(n-j)\alpha a_{n-j} + (j+1)a_{n-j-1}}{n_\alpha a_n + a_{n-1}}, j = 1, 2, ..., n-1.$$

Using Holder’s inequality, we get

$$|D_\alpha P(z)| \geq |n_\alpha a_n + a_{n-1}||z^{n-1}| \left[ 1 - \left( \sum_{j=1}^{n-1} |\zeta_{n-j}|^p \right)^{\frac{1}{p}} \right] \left( \sum_{j=1}^{n-1} \frac{1}{|z|^{(n-j)q}} \right)^{\frac{1}{q}}. \tag{2.6}$$

Let

$$H_p = \left( \sum_{j=1}^{n-1} |\zeta_{n-j}|^p \right)^{\frac{1}{p}}. \tag{2.7}$$

Now for $|z| > 1$, we have

$$\sum_{j=1}^{n-1} \frac{1}{|z|^{(n-j)q}} < \sum_{j=1}^{\infty} \frac{1}{|z|^q} = \frac{1}{|z|^q - 1}. \tag{2.8}$$

From (2.6), we get

$$|D_\alpha P(z)| \geq |n_\alpha a_n + a_{n-1}||z^{n-1}| \left[ 1 - \frac{H_p}{(|z|^q - 1)^{\frac{1}{q}}} \right]. \tag{2.9}$$

This implies

$$|D_\alpha P(z)| \geq 0, \tag{2.10}$$

iff

$$1 - \frac{H_p}{(|z|^q - 1)^{\frac{1}{q}}} \geq 0, \tag{2.11}$$

that is, if

$$|z| \geq \left[ 1 + (H_p)^q \right]^\frac{1}{q}. \tag{2.12}$$

This shows that those zeros of $D_\alpha P(z)$ whose modulus is greater than one lie in

$$|z| < \left[ 1 + (H_p)^q \right]^\frac{1}{q}. \tag{2.13}$$

But those zeros of $D_\alpha P(z)$ whose modulus is less than one already satisfy the inequality

$$|z| < \left[ 1 + (H_p)^q \right]^\frac{1}{q}. \tag{2.14}$$
Thus all the zeros of $D_\alpha P(z)$ lie in

$$|z| < \left(1 + \left(\sum_{j=1}^{n-1} |\zeta_{n-j}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$  \hfill (2.15)

Equivalently

$$|z| \leq \left[1 + \left(\sum_{j=0}^{n-2} \frac{k_j}{k_n} |z| \right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$  \hfill (2.16)

Further

$$M = \max_{1 \leq j \leq n-1} |\zeta_{n-j}|.$$  \hfill (2.17)

Therefore

$$|\zeta_{n-j}|^p \leq M^p, \forall j = 1, 2, ..., n-1.$$  \hfill (2.18)

This implies

$$\sum_{j=1}^{n-1} |\zeta_{n-j}|^p \leq (n-1)M^p, \forall j = 1, 2, ..., n-1.$$  \hfill (2.19)

This gives

$$\left[\sum_{j=1}^{n-1} |\zeta_{n-j}|^p\right]^{\frac{q}{p}} \leq (n-1)^{\frac{q}{p}}M^q, \forall j = 1, 2, ..., n-1.$$  \hfill (2.20)

Therefore we have

$$|z| \leq \left(1 + (n-1)^{\frac{q}{p}}M^q\right)^{\frac{1}{q}},$$  \hfill (2.21)

which is (2.2). This establishes the theorem.

\begin{remark}
If we take $p = q = 2$ in (2.1), then $\frac{1}{p} + \frac{1}{q} = 1$ and it follows from Theorem 2.1 that all zeros of $D_\alpha P(z)$ lie in

$$|z| \leq \left[1 + \left(\sum_{j=1}^{n-1} \frac{k_j}{k_n} |z| \right)^{\frac{q}{p}}\right]^{\frac{1}{q}}.$$  \hfill (2.22)

If we let $p \to \infty$ so that $q \to 1$, we get

\begin{corollary}
Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$. Then all zeros of $D_\alpha P(z)$ lie in

$$|z| \leq \left[1 + \sum_{j=1}^{n-1} \frac{k_j}{k_n} |z| \right]^{\frac{1}{2}}.$$  \hfill (2.23)

where $M = \max_{1 \leq j \leq n-1} |\zeta_{n-j}|$, which is an extension of Theorem 1.1 to polar derivative.
\end{corollary}

3 Conclusion
The main portion of this manuscript is dedicated to polar derivative of a polynomial via bounds for the moduli of zeros of algebraic polynomial. We introduced this operator and find out relation of zeros of the polar derivative with the coefficients of the corresponding polynomial as well as with concerned region. We can further investigate such type of inequalities and relate the new results with the results in trending in this area.

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References


