AN ALGORITHMIC APPROACH TO LOCAL SOLUTION OF THE NONLINEAR HIGHER ORDER ORDINARY HYBRID DIFFERENTIAL EQUATIONS

By

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Abstract

In this paper, we introduce a new notion of local or neighborhood solutions and establish a couple of approximation results for local existence and uniqueness of the solution of an IVP of nonlinear higher order ordinary hybrid differential equations by using the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage (2023). An approximation result for Ulam-Hyers stability of the local solution of the considered hybrid differential equation is also established. Our main abstract results of this paper are also illustrated with a couple of numerical examples. Finally, we compare our existence and uniqueness results with those existing in the literature via other operator theoretic methods from nonlinear functional analysis. We claim that the method and the results of this paper are new to the literature.

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1 Introduction

If the nonlinear function or its splitting components under different perturbations involved in the nonlinear differential or integral equations possess some mixed properties of algebra, analysis and topology, that is, qualitative characteristics from different branches of mathematics are called hybrid differential or hybrid integral equations and the study of such equations begins with the invention of hybrid fixed point theory on the lines of Krasnoselskii [25] and Dhage [5, 6, 7, 8, 9]. The iterative methods for finding the approximate solution of such nonlinear differential and integral equations are very much popular among the mathematicians all over the world. There are several iteration methods used in nonlinear analysis having different characterizations. The Picard’s iteration method used in the analysis of nonlinear differential equations employs the Lipschitz condition on the nonlinear function involved in the equations and the solution is obtained in the form of a sequence of successive approximations (see Coddington [3] and Granas and Dugundji [20]). Other iterative methods involve the use of monotone iterations together with the existence of both lower as well as upper solution of the related differential equations. See Ladde et al. [26], Heikkilä and Lakshmikantham [23] and references therein. The most recent Dhage iteration method based on earlier hybrid fixed point theorems also needs the existence of at least one of the lower and upper solutions of the related nonlinear problem (see [11, 12, 13]). Therefore, it is of interest to discuss the nonlinear higher order ordinary differential equations via Dhage iteration method under certain monotonicity condition but without the use of usual strong Lipschitz condition on the nonlinear function as well as without the existence of both lower and upper solutions of the given differential equations which is the main motivation or objective of the present paper. This will be furnished by a novel application of Dhage iteration method based on a very recent hybrid fixed point theorem proved in Dhage [10].

Given a closed and bounded interval \( J = [t_0, t_0 + a] \) in \( \mathbb{R} \) for some \( t_0, a \in \mathbb{R} \) with \( a > 0 \), we consider the IVP of nonlinear \( k \)th order hybrid ordinary differential equation (HDE),

\[
\begin{align*}
    x^{(k)}(t) &= f(t, x(t)), \quad t \in J, \\
    x^{(i)}(t_0) &= \alpha_i, \quad i = 0, 1, 2, \ldots, k-1,
\end{align*}
\] (1.1)
where \(\alpha_i, i = 0, 1, \ldots, k-1\) are real constants and the function \(f : J \times \mathbb{R} \to \mathbb{R}\) satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

**Definition 1.1.** A function \(x \in C^{k-1}(J, \mathbb{R})\) is said to be a solution of the HDE (1.1) if it satisfies the equations in (1.1) on \(J\), where \(C^{k-1}(J, \mathbb{R})\) is the space of \((k-1)\) times continuously differential real-valued functions defined on \(J\). If the solution \(x\) lies in a closed ball \(B_r(x_0)\) centered at some point \(x_0 \in C(J, \mathbb{R})\) of radius \(r > 0\), then we say it is a local solution or neighbourhood solution (in short nbhd solution) of the HDE (1.1) on \(J\).

**Remark 1.1.** The concept of local or nbhd solution of the VHIE (1.1) is different from that of usual notion of local solution as mentioned in Coddington [3]. In the terminology of Coddington [3], it is a nonlocal solution of the VHIE (1.1) defined on all of \(J\).

The HDE (1.1) is familiar in the subject of nonlinear analysis and can be studied for a variety of different aspects of the solution by using different methods form nonlinear functional analysis. The special case with \(k = 1\) has been discussed in Dhage and Dhage [15] while the special case when \(k = 2\) has been discussed in Dhage et al. [17] and so, the results of this paper generalizes with different approach the results of earlier papers in the literature. The existence of local solution for the HDE (1.1) with usual sense can be proved by using the Schauder fixed point principle, see for example, Coddington [3], Granas and Dugundji [20] and references therein. The approximation result for uniqueness of solution can be proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the area of nonlinear analysis. But to the knowledge of the present authors, the approximation results for the local existence and uniqueness theorems without using the Lipschitz condition or under its weaker form is not discussed in the literature as for the theory of nonlinear differential and integral equations. In this paper, we discuss the approximation results for local existence and uniqueness of solution for the HDE (1.1) under weaker Lipschitz condition but via construction of the algorithms based on monotone iteration method and a hybrid fixed point theorem of Dhage [8]. Also see Dhage et al. [14] and references therein.

The rest of the paper is organized as follows. Section 2 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 3. The approximation of the Ulam-Hyers stability is discussed in Section 4 and a couple of illustrative examples are presented in Section 5. Finally, some concluding remarks are mentioned in Section 6.

### 2 Auxiliary Results

We place the problem of HDE (1.1) in the function space \(C(J, \mathbb{R})\) of continuous, real-valued functions defined on \(J\). We introduce a supremum norm \(\| \cdot \|\) in \(C(J, \mathbb{R})\) defined by

\[\|x\| = \sup_{t \in J} |x(t)|,\]

(2.1)

and an order relation \(\leq\) in \(C(J, \mathbb{R})\) by the cone \(K\) given by

\[K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \forall \ t \in J\}.\]

(2.2)

Thus,

\[x \leq y \iff y - x \in K,\]

or equivalently,

\[x \leq y \iff x(t) \leq y(t) \ \forall \ t \in J.\]

(2.3)

It is known that the Banach space \(C(J, \mathbb{R})\) together with the order relations \(\leq\) becomes an ordered Banach space which we denote for convenience, by \((C(J, \mathbb{R}), K)\). We denote the open and closed spheres centered at \(x_0 \in C(J, \mathbb{R})\) of radius \(r\), for some \(r > 0\), by

\[B_r(x_0) = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| < r\},\]

and

\[B_r[x_0] = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| \leq r\},\]

(2.4)

receptively. It is clear that \(B_r[x_0] = \overline{B_r(x_0)}\).

It is well-known that the hybrid fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively. See Granas and Dugundji [20] and
the references therein. Here, we employ the Dhage monotone iteration method or simply Dhage iteration method based on the generalizations two hybrid fixed point theorems in the partially ordered abstract spaces. Generalizing the hybrid fixed point theorem of Dhage [9] and Dhage et al. [14], the present second author in [10] proved a Schauder type hybrid fixed point theorem in a partially ordered Banach space. Before stating this theorem, we give some preliminaries needed in the sequel.

Let \((E,d,\preceq)\) be a partially ordered metric space and let \(S \subseteq E\). \(E\) is called regular if a monotone nondecreasing (resp. monotone nonincreasing) sequence \(\{x_n\}\) in \(E\) converges to \(x_\star\), then \(x_n \preceq x_\star\) (resp. \(x_\star \preceq x_n\)) for all \(n \in \mathbb{N}\). The metric \(d\) and the order relation \(\preceq\) are said to be compatible in \(S\) if a monotone sequence \(\{x_n\}\) in \(S\) has a convergent subsequence, then the original sequence \(\{x_n\}\) is convergent and converges to the same limit point. \(S\) is called a Janhavi set if \(\|\|\cdot\|\|,\preceq\) are compatible in it. \(S\) is called partial bounded (resp. partially closed, partially compact) if every chain \(C\) in \(S\) is bounded (resp. closed, compact).

A mapping \(T : S \to S\) is called monotone nondecreasing (resp. monotone nonincreasing) if \(x \preceq y\) implies \(Tx \preceq Ty\) (resp. \(x \preceq y\) implies \(Tx \succeq Ty\)). \(T\) is called monotone if it is either monotone nondecreasing or monotone nonincreasing. \(T\) is called partially bounded (resp. partially totally bounded or partially precompact) if \(T(S)\) is partially bounded (resp. partially totally bounded or partially precompact for partially bounded \(S\)). \(T\) is partially continuous if \(\{x_n\} \subseteq S\) converges to \(x_\star\) with \(x_n \preceq x_\star\), then \(Tx_n \to Tx_\star\). \(T\) is called partially completely continuous if it is partially continuous and partially totally bounded.

Now we are equipped with all the necessary details to state our required hybrid fixed point theorems needed in what follows.

**Theorem 2.1.** Let \(S\) be a non-empty, partial closed and partial bounded subset of a partially ordered Banach space \((E,\|\|,\preceq)\) and let every chain \(C\) in \(S\) be Janhavi set. Suppose that \(T : S \to S\) is a partially completely continuous and monotone nondecreasing operator. If there exists an element \(x_0 \in S\) such that \(x_0 \preceq Tx_0\) or \(x_0 \preceq Tx_0\) satisfying
\[
\|x_0 - Tx_0\| \leq (1-q)r,
\]
for some real number \(r > 0\), then \(T\) has a unique comparable fixed point \(x^*\) in \(B_r[x_0]\) and the sequence \(\{x_n\}_{n=0}^\infty\) of successive iterations converges monotonically to \(x^*\). Furthermore, if every pair of elements in \(X\) has a lower or upper bound, then \(x^*\) is unique.

**Remark 2.1.** We note that every pair of elements in a partially ordered set (in short poset) \((E,\preceq)\) has a lower or upper bound if \((E,\preceq)\) is a lattice, that is, \(\preceq\) is a lattice order in \(E\). In this case the poset \((E,\|\|,\preceq)\) is called a partially lattice ordered Banach space. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. The details of the lattice structure of a Banach space appear in Birkhoff [2].

As a consequence of Remark 2.1, we obtain

**Theorem 2.2** (Dhage [9]). Let \(B_r[x]\) denote the partial closed ball centered at \(x\) of radius \(r\), in a regular partially ordered Banach space \((E,\|\|,\preceq)\) and let \(T : E \to E\) be a monotone nondecreasing and partial contraction operator with contraction constant \(q\). If there exists an element \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) or \(x_0 \preceq Tx_0\) satisfying
\[
\|x_0 - Tx_0\| \leq (1-q)r,
\]
for some real number \(r > 0\), then \(T\) has a unique comparable fixed point \(x^*\) in \(B_r[x_0]\) and the sequence \(\{x_n\}_{n=0}^\infty\) of successive iterations converges monotonically to \(x^*\). Furthermore, if every pair of elements in \(X\) has a lower or upper bound, then \(x^*\) is unique.

**Theorem 2.3.** Let \(B_r[x]\) denote the partial closed ball centered at \(x\) of radius \(r\) for some real number \(r > 0\), in a regular partially lattice ordered Banach space \((E,\|\|,\preceq)\) and let \(T : E \to E\) be a monotone nondecreasing and partial contraction operator with contraction constant \(q\). If there exists an element \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) or \(x_0 \preceq Tx_0\) satisfying \((2.5)\), then \(T\) has a unique fixed point \(\xi^*\) in \(B_r[x_0]\) and the sequence \(\{T^n x_0\}_{n=0}^\infty\) of successive iterations converges monotonically to \(\xi^*\).

If a Banach \(X\) is partially ordered by an order cone \(K\) in \(X\), then in this case we simply say \(X\) is an ordered Banach space which we denote it by \((X,K)\). Similarly, an ordered Banach space \((X,K)\), where
Lemma 2.1 (Dhage [7, 8]). Every ordered Banach space \((X, K)\) is regular.

Lemma 2.2 (Dhage [7, 8]). Every partially compact subset \(S\) of an ordered Banach space \((X, K)\) is a Janhavi set in \(X\).

As a consequence of Lemmas 2.1 and 2.2, we obtain the following applicable hybrid fixed point theorems which we need in what follows.

Theorem 2.4. Let \(S\) be a non-empty, partially closed and partially bounded subset of an ordered Banach space \((X, K)\) and let \(T : S \to S\) be a partially completely continuous and monotone nondecreasing operator. If there exists an element \(x_0 \in S\) such that \(x_0 \preceq Tx_0\) or \(x_0 \succeq Tx_0\), then \(T\) has a fixed point \(\xi^* \in S\) and the sequence \(\{T^n x_0\}_{n=0}^\infty\) of successive iterations converges monotonically to \(\xi^*\).

Theorem 2.5. Let \(B_r[x]\) denote the partial closed ball centered at \(x\) of radius \(r\) for some real number \(r > 0\), in a lattice ordered Banach space \((X, K)\) and let \(T : (X, K) \to (X, K)\) be a monotone nondecreasing and partial contraction operator with contraction constant \(q\). If there exists an element \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) or \(x_0 \succeq Tx_0\) satisfying (2.5), then \(T\) has a unique fixed point \(\xi^*\) in \(B_r[x_0]\) and the sequence \(\{T^n x_0\}_{n=0}^\infty\) of successive iterations converges monotonically to \(\xi^*\).

A few details of order spaces, hybrid fixed point theorems and related applications appear in Guo and Lakshmikantham [21], Deimling [4], Heikkilä and Lakshmikantham [23], Dhage [6, 7, 8], Dhage and Dhage [11], Dhage et al. [14], Dhage and Dhage [15, 16], Dhage et al. [17, 18, 19] and references therein.

3 Local Approximation Results

We consider the following set of hypotheses in what follows.

(H1) The function \(f\) is continuous and bounded on \(J \times R\) with bound \(M_f\).

(H2) There exists a constant \(\ell > 0\) such that

\[
0 \leq f(t, x) - f(t, y) \leq \ell(x - y)
\]

for all \(x, y \in R\) with \(x \geq y\), where \(\int_0^1 \frac{a^k}{(k-1)!} < 1\).

(H3) \(f(t, x)\) is nondecreasing in \(x\) for each \(t \in J\).

(H4) \(f(t, \alpha_0) \geq 0\) for all \(t \in J\) and \(\alpha_i \geq 0\) for all \(i = 0, 1, \ldots, k - 1\).

Now using the theory of calculus, we obtain the following useful lemma.

Lemma 3.1. If \(h \in L^1(J, R)\), then the IVP of ordinary \(k^{th}\) order linear differential equation

\[
x^{(k)}(t) = h(t), \quad t \in J, \quad x^{(i)}(t_0) = \alpha_i, \quad i = 0, \ldots, k - 1, \tag{3.1}
\]

is equivalent to the Volterra integral equation (VIE)

\[
x(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} h(s) \, ds, \quad t \in J. \tag{3.2}
\]

Theorem 3.1. Suppose that the hypotheses (H1), (H3) and (H4) hold. Furthermore, if the inequality

\[
\sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + \frac{M_f a^k}{k!} \leq r, \tag{3.3}
\]

holds, then the HDE (1.1) has a solution \(x^*\) in \(B_r[\alpha_0]\), where \(x_0 \equiv \alpha_0\), and the sequence \(\{x_n\}_{n=0}^\infty\) of successive approximations defined by

\[
\begin{cases}
  x_0(t) = \alpha_0, & t \in J, \\
  x_{n+1}(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x_n(s)) \, ds, & t \in J, 
\end{cases} \tag{3.4}
\]

where \(n = 0, 1, \ldots;\) converges monotone nondecreasingly to \(x^*\).
Proof. Set \( X = C(J, \mathbb{R}) \). Clearly, \((X, K)\) is a partially ordered Banach space. Let \( x_0 \) be a constant function on \( J \) such that \( x_0(t) = \alpha_0 \) for all \( t \in J \) and define a closed ball \( B_r[x_0] \) in \( X \) defined by (2.3), where the number \( r \) satisfies the inequality (3.3). By Lemma 3.1, the HDE (1.1) is equivalent to the integral equation (VHIE)

\[
x(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x(s)) ds, \quad t \in J.
\]

(3.5)

Now, define an operator \( T \) on \( B_r[x_0] \) into \( X \) by

\[
Tx(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x(s)) ds, \quad t \in J.
\]

(3.6)

We shall show that the operator \( T \) satisfies all the conditions of Theorem 2.1 on \( B_r[x_0] \) in the following series of steps.

**Step I:** The operator \( T \) maps \( B_r[x_0] \) into itself.

Firstly, we show that \( T \) maps \( B_r[x_0] \) into itself. Let \( x \in B_r[x_0] \) be arbitrary element. Then,

\[
|Tx(t) - x_0(t)| = \left| \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x(s)) ds \right|
\]

\[
\leq \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + M_f \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} |f(s, x(s))| ds
\]

\[
\leq \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + M_f a^k
\]

\[
\leq \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + M_f a^k
\]

for all \( t \in J \). Taking the supremum over \( t \) in the above inequality yields

\[
\|Tx - x_0\| \leq \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + \frac{M_f a^k}{k!} \leq r
\]

which implies that \( Tx \in B_r[x_0] \) for all \( x \in B_r[x_0] \).

**Step II:** \( T \) is a monotone nondecreasing operator on \( B_r[x_0] \).

Let \( x, y \in B_r[x_0] \) be any two elements such that \( x \succeq y \). Then,

\[
Tx(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x(s)) ds
\]

\[
\geq \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, y(s)) ds
\]

\[
= Ty(t),
\]

for all \( t \in J \). So, \( Tx \succeq Ty \), that is, \( T \) is monotone nondecreasing on \( B_r[x_0] \).

**Step III:** \( T \) is partially continuous operator on \( B_r[x_0] \).

Let \( C \) be a chain in \( B_r[x_0] \) and let \( \{x_n\} \) be a sequence in \( C \) converging to a point \( x \in C \). Then, by dominated convergence theorem, we have

\[
\lim_{n \to \infty} Tx_n(t) = \lim_{n \to \infty} \left[ \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x_n(s)) \right]
\]

\[
= \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \lim_{n \to \infty} \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x_n(s)) ds
\]
\[
= \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \left[ \lim_{n \to \infty} f(s,x_n(s)) \right] \, ds
\]
\[
= \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s,x(s)) \, ds
\]
\[
= \mathcal{T}x(t),
\]
for all \( t \in J \). Therefore, \( \mathcal{T}x_n \to \mathcal{T}x \) pointwise on \( J \). Below we show that \( \{ \mathcal{T}x_n \} \) is an equicontinuous sequence of points in \( X \). Next, let \( t_1, t_2 \in J \) be arbitrary. Then, for any \( x \in C \), by hypothesis (H\(_1\)), we obtain
\[
|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)| \leq \sum_{i=1}^{k-1} \left| \frac{\alpha_i}{i!} \left| (t_1 - t_0)^i - (t_2 - t_0)^i \right| \right|
\]
\[
+ \frac{1}{(k-1)!} \int_{t_0}^{t_1} \left| (t_1 - s)^{k-1} - (t_2 - s)^{k-1} \right| |f(s,x(s))| \, ds
\]
\[
+ \frac{1}{(k-1)!} \int_{t_0}^{t_1} \left| (t_2 - s)^{k-1} \right| |f(s,x(s))| \, ds
\]
\[
\leq \sum_{i=1}^{k-1} \frac{\alpha_i}{i!} \left| (t_1 - t_0)^i - (t_2 - t_0)^i \right|
\]
\[
+ \frac{M_f}{(k-1)!} \int_{t_0}^{t_1} \left| (t_1 - s)^{k-1} - (t_2 - s)^{k-1} \right| \, ds
\]
\[
+ \frac{M_f}{(k-1)!} \int_{t_0}^{t_1} \left| (t_2 - s)^{k-1} \right| \, ds
\]
\[
\leq \sum_{i=1}^{k-1} \frac{\alpha_i}{i!} \left| (t_1 - t_0)^i - (t_2 - t_0)^i \right|
\]
\[
+ \frac{M_f}{(k-1)!} \int_{t_0}^{t_1} \left| (t_1 - s)^{k-1} - (t_2 - s)^{k-1} \right| \, ds
\]
\[
+ \frac{M_f}{(k-1)!} \int_{t_0}^{t_1} \left| (t_2 - s)^{k-1} \right| \, ds
\]
\[
\to 0 \quad \text{as} \quad t_1 \to t_2,
\]
uniformly for \( x \in C \). As a result, we have that \( \mathcal{T}x_n \to \mathcal{T}x \) uniformly on \( J \). Hence \( \mathcal{T} \) is partially continuous operator on \( B_C[x_0] \).
Step IV: \( T \) is partially compact operator on \( B_r[x_0] \) into itself.

To show \( T \) is partially compact operator, it is enough to prove that \( T(B_r[x_0]) \) is a partially compact subset of \( B_r[x_0] \). Let \( C \) be a chain in \( T(B_r[x_0]) \). We show that \( T(C) \) is a relatively compact subset of \( B_r[x_0] \). We apply the Arzelá-Ascoli theorem for compactness of a set in \( C(J, \mathbb{R}) \). Firstly, let \( y \in T(C) \) be any element. Then there is an element \( x \in C \) such that \( y = T x \). Now, by hypothesis (H₁), we get

\[
\|y\| = \|T x\| \leq |\alpha_0| + \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + \frac{M f a^k}{k!}
\]

for all \( y \in T(C) \). This shows that \( T(C) \) is uniformly subset of \( B_r[x_0] \). Next, let \( t_1, t_2 \) be arbitrary. Then following the arguments given in Step III, it can be proved that

\[
|y(t_1) - y(t_2)| = |T x(t_1) - T x(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2,
\]

uniformly for \( y \in T(C) \). Now, by an application of Arzelá-Ascoli theorem, we conclude that \( T(C) \) is a relatively compact subset of \( B_r[x_0] \). Consequently \( T \) is a partially compact operator on \( B_r[x_0] \) into itself.

Step V: The element \( x_0 \in B_r[x_0] \) satisfies the relation \( x_0 \leq T x_0 \).

Since (H₄) holds, one has

\[
x_0(t) = \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x_0(s)) \, ds
\]

\[
\leq x_0(s) + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, \alpha_0) \, ds
\]

\[
= \alpha_0 + \sum_{i=1}^{k-1} \frac{\alpha_i (t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, x_0(s)) \, ds
\]

\[
= T x_0(t),
\]

for all \( t \in J \). This shows that the constant function \( x_0 \) in \( B_r[x_0] \) serves as to satisfy the operator inequality \( x_0 \leq T x_0 \).

Thus, the operator \( T \) satisfies all the conditions of Theorem 2.1, and so \( T \) has a fixed point \( x^* \) in \( B_r[x_0] \) and the sequence \( \{T^n x_0\}_{n=0}^{\infty} \) of successive iterations converges monotone nondecreasingly to \( x^* \). This further implies that the HIE (3.5) and consequently the HDE (1.1) has a local solution \( x^* \) and the sequence \( \{x_n\}_{n=0}^{\infty} \) of successive approximations defined by (3.4) is monotone nondecreasing and converges to \( x^* \). This completes the proof.

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under partial Lipschitz condition which is weaker form of usual Lipschitz condition.

**Theorem 3.2.** Suppose that the hypotheses (H₁), (H₂) and (H₄) hold. Furthermore, if

\[
\sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + \frac{M f a^k}{k!} \leq \left[ 1 - \frac{\ell a^k}{k!} \right] r, \quad \frac{\ell a^k}{k!} < 1,
\]

for some real number \( r > 0 \), then the HDE (1.1) has a unique solution \( x^* \) in \( B_r[x_0] \) defined on \( J \) and the sequence \( \{x_n\}_{n=0}^{\infty} \) of successive approximations defined by (3.3) is monotone nondecreasing and converges to \( x^* \).

**Proof.** Set \( (X, K) = (C(J, \mathbb{R}), \leq) \) which is a lattice w.r.t. the lattice join and meet operations defined by \( x \lor y = \max \{x, y\} \) and \( x \land y = \min \{x, y\} \), and so every pair of elements of \( X \) has a lower and an upper bound. Let \( r > 0 \) be a fixed real number and consider closed sphere \( B_r[x_0] \) centred at \( x_0 \) of radius \( r \) in the lattice ordered Banach space \( (X, K) \).

Define an operator \( T \) on \( X \) into \( X \) by \( (3.5) \). Clearly, \( T \) is monotone nondecreasing on \( X \). To see this, let \( x, y \in X \) be two elements such that \( x \geq y \). Then, by hypothesis (H₂),

\[
T x(t) - T y(t) = \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \left[ f(s, x(s)) - f(s, y(s)) \right] ds \geq 0,
\]
for all $t \in J$. Therefore, $T x \geq T y$ and consequently $T$ is monotone nondecreasing on $X$.

Next, we show that $T$ is a partial contraction on $X$. Let $x, y \in X$ be such that $x \geq y$. Then, by hypothesis (H\textsubscript{2}), we obtain

$$
|T x(t) - T y(t)| = \left| \int_{t_0}^{t} \frac{(t - s)^{k-1}}{(k-1)!} \left[ f(s, x(s)) - f(s, y(s)) \right] ds \right|
$$

$$
\leq \left| \int_{t_0}^{t} \frac{\ell a^k}{k!} (x(s) - y(s)) ds \right|
$$

$$
= \int_{t_0}^{t} \frac{\ell a^k}{k!} \|x - y\| ds
$$

$$
\leq \frac{\ell a^k}{k!} \|x - y\|
$$

for all $t \in J$, where $\frac{\ell a^k}{k!} < 1$. Taking the supremum over $t$ in the above inequality yields

$$
\|T x - T y\| \leq \frac{\ell a^k}{k!} \|x - y\|
$$

for all comparable elements $x, y \in X$. This shows that $T$ is a partial contraction on $X$ with contraction constant $\frac{\ell a^k}{k!}$. Furthermore, it can be shown as in the proof of Theorem 3.1 that the element $x_0 \in B_r[x_0]$ satisfies the relation $x_0 \leq T x_0$ in view of hypothesis (H\textsubscript{4}). Finally, by hypothesis (H\textsubscript{1}) and condition (3.6), one has

$$
\|x_0 - T x_0\| \leq \sup_{t \in J} \sum_{i=1}^{k-1} \frac{\alpha_i(t - t_0)^i}{i!} + \sup_{t \in J} \left| \int_{t_0}^{t} \frac{\ell a^k}{k!} f(s, \alpha_0) ds \right|
$$

$$
\leq \sum_{i=1}^{k-1} \frac{|\alpha_i| a^i}{i!} + \frac{Ma^k}{k!}
$$

$$
\leq \left[ 1 - \frac{\ell a^k}{k!} \right] r
$$

which shows that the condition (2.5) of Theorem 2.2 is satisfied. Hence $T$ has a unique fixed point $x^*$ in $B_r[x_0]$ and the sequence $\{T^n x_0\}_{n=0}^\infty$ of successive iterations converges monotone nondecreasingly to $x^*$. This further implies that the HIE (3.4) and consequently the HDE (1.1) has a unique local solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.3) is monotone nondecreasing and converges to $x^*$. This completes the proof.

\begin{remark}
The conclusion of Theorems 3.1 and 3.2 also remains true if we replace the hypothesis (H\textsubscript{4}) with the following one.

(H\textsubscript{4}) $f(t, \alpha_0) \leq 0$ for all $t \in J$ and $\alpha_i \leq 0$ for all $i = 0, 1, \ldots, k-1$.

In this case, the HDE (1.1) has a local solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.3) is monotone nonincreasing and converges to the solution $x^*$.
\end{remark}

\begin{remark}
If the initial condition in the equation (1.1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 3.1, the HDE (1.1) has a local positive solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.3) converges monotone nondecreasingly to the positive solution $x^*$. Similarly, under the conditions of Theorem 3.2, the HDE (1.1) has a unique positive local solution $x^*$ defined on $J$ and the sequence of successive approximations defined by (3.3) $\{x_n\}_{n=0}^\infty$ converges monotone nondecreasingly to the unique positive local solution $x^*$.
\end{remark}

\section{Approximation of Local Ulam-Hyers Stability}
The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Tripathy [29], Huang et al. [24] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the HDE (1.1) under the conditions of hybrid fixed point principle stated in Theorem 2.2. We need the following definition in what follows.
Definition 4.1. The HDE (1.1) is said to be locally Ulam-Hyers stable if for \( \epsilon > 0 \) and for each solution \( y \in B_r[x_0] \) of the inequality
\[
(*) \quad \begin{cases} 
|y^{(k)}(t) - f(t, y(t))| \leq \epsilon, & t \in J, \\
y^{(i)}(t_0) = \alpha_i, & i = 0, 1, 2, \ldots, k - 1,
\end{cases}
\]
there exists a constant \( K_f > 0 \) such that
\[
(**) \quad |y(t) - \xi(t)| \leq K_f \epsilon,
\]
for all \( t \in J \), where \( \xi \in B_r[x_0] \) is a local solution of the HDE (1.1) defined on \( J \). The solution \( \xi \) of the HDE (1.1) is called Ulam-Hyers stable local solution.

Theorem 4.1. Assume that all the hypotheses of Theorem 3.2 hold. Then the HDE (1.1) has a unique Ulam-Hyers stable local solution \( x^* \in B_r[x_0] \) and the sequence \( \{x_n\}_{n=0}^\infty \) of successive approximations given by (3.3) converges monotone nondecreasingly to \( x^* \).

Proof. Let \( \epsilon > 0 \) be given and let \( y \in B_r[x_0] \) be a solution of the functional inequality (4.1) on \( J \), that is, we have
\[
(4.1) \quad \begin{cases} 
|y^{(k)}(t) - f(t, y(t))| \leq \epsilon, & t \in J, \\
y^{(i)}(t_0) = \alpha_i, & i = 0, 1, 2, \ldots, k - 1,
\end{cases}
\]
By Theorem 3.2, the HDE (1.1) has a unique local solution \( \xi \in B_r[x_0] \). Then by Lemma 3.1, one has
\[
(4.2) \quad \xi(t) = \alpha_0 + \sum_{i=1}^{k-1} \alpha_i \frac{(t-t_0)^i}{i!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, \xi(s)) \, ds, \quad t \in J.
\]
Now, by integration of (4.1) yields the estimate:
\[
(4.3) \quad |y(t) - \alpha_0 - \sum_{i=1}^{k-1} \alpha_i \frac{(t-t_0)^i}{i!} - \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, y(s)) \, ds| \leq \frac{a^k}{k!} \epsilon,
\]
for all \( t \in J \).

Next, from (4.2) and (4.3) we obtain
\[
|y(t) - \xi(t)| = |y(t) - \alpha_0 - \sum_{i=1}^{k-1} \alpha_i \frac{(t-t_0)^i}{i!} - \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, \xi(s)) \, ds|
\leq |y(t) - \alpha_0 - \sum_{i=1}^{k-1} \alpha_i \frac{(t-t_0)^i}{i!} - \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, y(s)) \, ds|
+ \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, y(s)) \, ds - \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} f(s, \xi(s)) \, ds
\leq \frac{a^k}{k!} \epsilon + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} |f(s, y(s)) - f(s, \xi(s))| \, ds
\leq \frac{a^k}{k!} \epsilon + \ell \frac{a^k}{k!} \|y - \xi\|.
\]
Taking the supremum over \( t \), we obtain
\[
\|y - \xi\| \leq \frac{a^k}{k!} \epsilon + \ell \frac{a^k}{k!} \|y - \xi\|,
\]
or
\[
\|y - \xi\| \leq \left[ \frac{a^k \epsilon/k!}{1 - \ell a^k/k!} \right],
\]
where, \( \ell a^k/k! < 1 \). Letting \( K_f = \left[ \frac{a^k}{k!} \right] > 0 \), we obtain
\[
|y(t) - \xi(t)| \leq K_f \epsilon,
\]
for all \( t \in J \). As a result, \( \xi \) is a Ulam-Hyers stable local solution of the HDE (1.1) on \( J \) and the sequence \( \{x_n\}_{n=0}^\infty \) of successive approximations defined by (3.3) is monotone nondecreasing and converges to \( \xi \). Consequently the HDE (1.1) is a locally Ulam-Hyers stable on \( J \). This completes the proof.
Remark 4.1. If given initial condition in the equation (1.1) is such that \( \alpha_0 > 0 \), then under the conditions of Theorem 4.1, the HDE (1.1) has a unique Ulam-Hyers stable local positive solution \( x^* \) defined on \( J \) and the sequence \( \{x_n\}_{n=0}^{\infty} \) of successive approximations defined by (3.3) converges monotone nondecreasingly to \( x^* \).

5 The Examples

In this section we illustrate the hypotheses and main approximation result by giving a couple of numerical examples.

Example 5.1. Given a closed and bounded interval \( J = [0, 1] \) in \( \mathbb{R} \), consider the IVP of nonlinear first order HDE,

\[
\begin{align*}
x^{(k)}(t) &= \text{tanh} \, x(t), \quad t \in [0, 1]; \\
x^{(i)}(t_0) &= \frac{1}{i + 1}, \quad i = 0, 1, 2, \ldots, k - 1,
\end{align*}
\]

(5.1)

Here \( t_0 = 0, a = 1, \alpha_0 = 1, \alpha_i = \frac{1}{i + 1}, i = 0, \ldots, k - 1, \) and \( f(t, x) = \text{tanh} \, x \) for \( (t, x) \in [0, 1] \times \mathbb{R} \). We show that \( f \) satisfies all the conditions of Theorem 3.1. Clearly, \( f \) is bounded on \([0, 1] \times \mathbb{R}\) with bound \( M_f = 1 \) and so the hypothesis \((H_1)\) is satisfied. Also the function \( f(t, x) \) is nondecreasing in \( x \) for each \( t \in [0, 1] \). Therefore, hypothesis \((H_2)\) is satisfied. Moreover, \( f(t, \alpha_0) = f(t, 1) = \text{tanh}(1) \geq 0 \) for all \( t \in [0, 1] \), and \( \frac{1}{i + 1} \geq 0 \) for each \( i, i = 0, 1, \ldots, k - 1 \). So the hypothesis \((H_3)\) holds. If we take \( r = 2 \), then the condition (3.3) of Theorem 3.1 holds. Thus, all the conditions of Theorem 3.1 are satisfied. Hence, the HDE (5.1) has a local solution \( x^* \) in the closed ball \( B_2[1] \) of the Banach space \( C(J, \mathbb{R}) \) and the sequence \( \{x_n\}_{n=0}^{\infty} \) of successive approximations defined by

\[
\begin{align*}
x_0(t) &= 1, \quad t \in J, \\
x_{n+1}(t) &= 1 + \sum_{i=1}^{k-1} \frac{(t-t_0)^i}{(i+1)!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \text{tanh} \, x_n(s) \, ds, \quad t \in J,
\end{align*}
\]

converges monotone nondecreasingly to \( x^* \).

Example 5.2. Given a closed and bounded interval \( J = [0, 1] \) in \( \mathbb{R} \), consider the IVP of nonlinear first order HDE,

\[
\begin{align*}
x^{(k)}(t) &= \frac{1}{2} \tan^{-1} x(t), \quad t \in [0, 1]; \\
x(0) &= 0, \quad x^{(i)}(t_0) = \frac{1}{i}, \quad i = 1, 2, \ldots, k - 1,
\end{align*}
\]

(5.2)

Here \( t_0 = 0, a = 1, \alpha_0 = 0, \alpha_i = \frac{1}{i}, i = 1, \ldots, k - 1, \) and \( f(t, x) = \frac{1}{2} \tan^{-1} x \) for \( (t, x) \in [0, 1] \times \mathbb{R} \). We show that \( f \) satisfies all the conditions of Theorem 3.2. Clearly, \( f \) is bounded on \([0, 1] \times \mathbb{R}\) with bound \( M_f = \frac{\pi}{4} \) and so, the hypothesis \((H_1)\) is satisfied. Next, let \( x, y \in \mathbb{R} \) be such that \( x \geq y \). Then there exists a constant \( \xi \) with \( \alpha_1 < \xi < y \) satisfying

\[
0 \leq f(t, x) - f(t, y) \leq \frac{1}{2} \cdot \frac{1}{1 + \xi^2} (x - y) \leq \frac{1}{2} \cdot (x - y),
\]

for all \( t \in [0, 1] \). So the hypothesis \((H_2)\) holds with \( \ell = \frac{1}{2} \). Moreover, \( f(t, \alpha_0) = f(t, 0) = \tan^{-1}(0) = 0 \) for all \( t \in [0, 1] \), and \( \alpha_0 = 0, \alpha_i = \frac{1}{i} \geq 0 \) for each \( i, i = 1, \ldots, k - 1 \). So the hypothesis \((H_3)\) holds. If we take the number \( r \) such that \( r \geq \frac{2 \, k!}{k! - 1} \), then we have

\[
\sum_{i=1}^{k-1} \frac{|\alpha_i| \cdot a^i}{i!} \frac{M_f a^k}{k!} \leq \left[ 1 - \frac{1}{2, k!} \right] r = \left[ 1 - \frac{\ell \, a^k}{k!} \right] r,
\]

and so, the condition (3.6) is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied. Hence, the HDE (5.2) has a unique local solution \( x^* \) in the closed ball \( B_2[0] \) of \( C(J, \mathbb{R}) \). This further in view of Remark 3.2 implies that the HDE (5.2) has a unique local positive solution \( x^* \) and and the sequence \( \{x_n\}_{n=0}^{\infty} \) of successive approximations defined by

\[
\begin{align*}
x_0(t) &= 0, \quad t \in J, \\
x_{n+1}(t) &= \sum_{i=1}^{k-1} \frac{(t-t_0)^i}{(i+1)!} + \int_{t_0}^{t} \frac{(t-s)^{k-1}}{(k-1)!} \tan^{-1} x_n(s) \, ds, \quad t \in J,
\end{align*}
\]
is monotone nondecreasing and converges to \(x^*\). Moreover, the unique local solution \(x^*\) is Ulam-Hyers stable on \([0,1]\) in view of Definition 4.1. Consequently the HDE (5.2) is a locally Ulam-Hyers stable on the interval \([0,1]\).

**Remark 5.1.** The approximation results of this paper may be extended to nonlinear IVPs of higher order ordinary integrodifferential equations

\[
\begin{cases}
    x^{(k)}(t) = f\left(t, x(t), \int_0^t g(s, x(s)) \, ds\right), & t \in J, \\
    x^{(i)}(t_0) = \alpha_{(i)}, & i = 0, 1, 2, \ldots, k - 1,
\end{cases}
\]

(5.3)

by using similar arguments with appropriate modifications. A few related results for second order integro-differential equations with older hybrid fixed point theorems along with the illustrative examples appear in Dhage et al. [18] and references therein.

6 Remarks and Conclusion

Finally, while concluding this paper, we remark that unlike Schauder fixed point theorem we do not require any convexity argument as well as restricted interval of the solution in the proof of main existence theorem, Theorem 3.1 and additionally we get an algorithm that goes to the solution monotonically on all of the interval of the problem. See Agarwal and Regan [1], Lakshmikantham and Leela [27], Zeidler [30] and references therein. Again, we do not require the usual strong Lipschitz condition in order to apply Banach fixed point theorem in the proof of uniqueness theorem, Theorem 3.2, but a weaker form of one sided or partial Lipschitz condition is enough to serve the purpose. However, the analysis remains true if we use the usual Lipschitz condition, but in that case we do not get the monotone convergence of the sequence of successive approximations to the solution of the HDE (1.1). See Coddington [3], Deimling [4], Krasnoselskii [25], Agarwal and Regan [1] and references therein. Similarly, we do not need the ordered solution space of the problem to be a complete lattice for the application of lattice fixed point theorem due to Tarski [28]. The details of order structure and complete lattice may be referred to Birkhoff [2]. However, in all above three cases we are able to achieve the existence of local solution by monotone convergence of the sequence of successive approximations involved in the Dhage iteration method based on recently developed new hybrid fixed point principles. Thus the hybrid fixed point theory has much more advantages over the classical analytic, algebraic and topological fixed point theory for qualitative study of the nonlinear differential and integral equations. Moreover, in order to illustrate the underlined ideas and the procedure of finding the approximate solution, in this paper a simple form of a differential equation (1.1) is considered for the study, however other complex nonlinear IVPs of HDEs with integer or fractional orders may also be considered and the present study can also be extended to such sophisticated nonlinear differential equations with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with applications. Some of the results in this direction will be reported elsewhere.

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