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BRINKMAN EQUATION IN TANGENT SPHERE AND CARDIOID COORDINATE SYSTEMS: STREAM FUNCTION SOLUTION

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Abstract

In the present investigation, analytical stream function solutions of the Brinkman equation in cardioid coordinates (λ, σ, ζ) and tangent sphere coordinates (τ, η, ζ) are investigated. Trigonometric functions, special functions such as modified Bessel functions and Bessel functions, are contained in the analytical expressions of the stream function and velocity components. Authors also discuss about how permeability depends on the estimated stream function solution for the Brinkman equation in the cardioid and tangent sphere coordinate systems, which can be deduced as a particular scenario that corresponds with the solution which is currently available [17].
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Nomenclature

Symbols	Description	Symbols	Description
(x, y, z)	Cartesian coordinates	ρ	fluid density
(τ, η, ζ)	Tangent sphere coordinates	$\vec{v}$	fluid velocity
(λ, σ, ζ)	Cardioid coordinates	k	permeability
J_1	Bessel's function of first kind	p	fluid pressure
Y_1	Bessel's function of second kind	ψ	stream function
I_1	modified Bessel's function of first kind	E^2	Stokes operator
K_1	modified Bessel's function of second kind	∇	gradiant operator
μ	dynamic coefficient of viscosity	∇^2	Laplacian operator

1 Introduction

Stokes flow, also referred to as creeping flow, is a particular kind of fluid flow that the Stokes equations explain. Stokes flow [8] offers a condensed perspective on fluid motion by emphasizes superiority of viscosity over inertia. Stokes equation gives a connection between viscosity, fluid velocity and fluid pressure. The mathematical expression of Stokes equations of motion is given by

$$\nabla p = \mu \nabla^2 \vec{v}. \tag{1.1}$$

Dassios *et al.* [2] reported the stream function solution of the Stokes equations in the prolate spheroidal coordinates using the theory of generalised eigenfunctions. The steady state of two-dimensional Stokes flow in the annular region between two fixed concentric cylindrical walls, perturbed by an infinitesimal rotating cylinder, was investigated by Hackborn [9].

Porous medium [16] is a material consisting of a solid matrix with an interconnected voids or pores. These voids enable the passage of fluids, air, and materials with varying chemical and physical properties. Darcy's law establishes a fundamental framework for understanding fluid flow in porous media. The mathematical expression for Darcy's law, which governs fluid flow through porous media, is

$$\nabla p = -\frac{\mu}{k} \vec{v}. \tag{1.2}$$

Khuri and Wazwaz [10] reported the solution of a second order partial differential equation $E^2\psi = 0$, in the toroidal coordinates which arised in the case of irrotational fluid motion, where E^2 is Stokes operator that is mentioned in classical book [8]. In the case of irrotational fluid motion, Deo and Tiwari [3] developed the stream function solution of the equation $E^2\psi = 0$, in the bi-spherical polar coordinates and also solved for the toroidal coordinates. In the parabolic, tangent sphere, and cardioid axisymmetric coordinate systems, Protopapas [17] obtained the particular solution of $E^2\psi = 0$. Maurya and Lata [12] established the general solution of Darcy equations in three axisymmetric system of coordinates: parabolic, bi-polar cylindrical and parabolic cylindrical coordinates.

We can examine the fluid flow in porous media while taking permeability and viscosity effects into account to the Brinkman equation, which fills the gap between Darcy's law and the Stokes equation. In the case of a material with high porosity, Brinkman [1] proposed the modified version of the Darcys law for the porous medium to characterise the flow through the porous medium, and provided an expression of the form

$$\nabla p = -\frac{\mu}{k}\vec{v} + \mu_e \nabla^2 \vec{v}. \tag{1.3}$$

The above equation reduces to the Stokes equation when the permeability k approaches toward infinity. A Cartesian-tensor solution of the Brinkman equation was investigated by Qin and Kaloni [19] and they also evaluated the drag force on a porous sphere in an unbounded medium. Pop and Cheng [18] evaluated a particular solution of the Brinkman equation and a series solution for the Stokes equation in the cylindrical polar coordinates. Zlatanovski [20] investigated the stream function solution of the Stokes equations and the Brinkman equations for axisymmetric creeping flow in the prolate spheroidal coordinates and also reported the general solutions of Stokes equations in the spherical polar coordinates as a limiting case. Deo *et al.* [4] studied the behavior of fluid flow through a porous medium of variable permeability confined within two coaxial cylinders by establishing the stream function solution of the Brinkman equation. In the cylindrical polar coordinates, Deo and Maurya [5] presented the generalized stream function solution of the Brinkman equation and deduced the solutions [4, 18] as a special case using obtained generalized solution. The general stream function solution of the Brinkman and the Stokes equations in the parabolic cylindrical coordinates, obtained by Maurya and Deo [13].

Maurya *et al.* [14] investigated the Stokes flow of micropolar fluid (i.e. non-Newtonian) through porous cylinder for two types of boundary value problems and presented a comparision for both *BVPs* graphically. Deo *et al.* [6] reported the influence of external and uniform magnetic field on hydrodynamic permeability of biporous membrane relative to the flow of micropolar fluid using cell models. Deo and Maurya [7] investigated the *MHD* impacts on micropolarNewtonian fluid flow through composite porous channel and reported the numerical value of flow rate, wall shear stresses and couple stresses at respective porous interfaces. After investigating the flow of micropolar liquid, Khanukaeva and Stokes [11] concluded that neither polar nor non-polar liquids exhibit the Stokes paradox.

2 Notion of Tangent Sphere and Cardioid Coordinates System:

In this proposed research problem, authors are promised to present the stream function solution of the Brinkman equation in tangent sphere and cardioid coordinates. Here, we are proposing these coordinates as follows:

- Tangent sphere coordinates (τ, η, ζ).
- Cardioid coordinates (λ, σ, ζ).

2.1 Tangent Sphere coordinates

Points on a tangent-plane to the sphere are designated as tangent sphere coordinates. Next, visualize a sphere that has a single point of contact with a flat plane. Finding locations on that plane in relation to the sphere is described by the tangent sphere coordinates. Even though they aren't as often utilized as other coordinate systems, tangent sphere coordinates offer some intriguing uses. In particular, tangent sphere coordinates come in handy for solving spherically symmetric geometry problems. These coordinates, for instance, make it easier to describe points on the surface of a sphere while evaluating a problem involving a sphere concentric with the origin. Six tangency points are found given four mutually tangent spheres. The tangent sphere coordinate system (τ, η, ζ) and the Cartesian coordinate system (x, y, z) are connected through the following relations [15]:

$$x = \frac{\tau \cos\zeta}{\tau^2 + \eta^2},\ y = \frac{\tau \sin\zeta}{\tau^2 + \eta^2} \text{ and } z = \frac{\eta}{\tau^2 + \eta^2},\quad \text{where } \tau > 0,\ \eta \in \mathbb{R} \text{ and } \zeta \in [0, 2\pi). \tag{2.1}$$

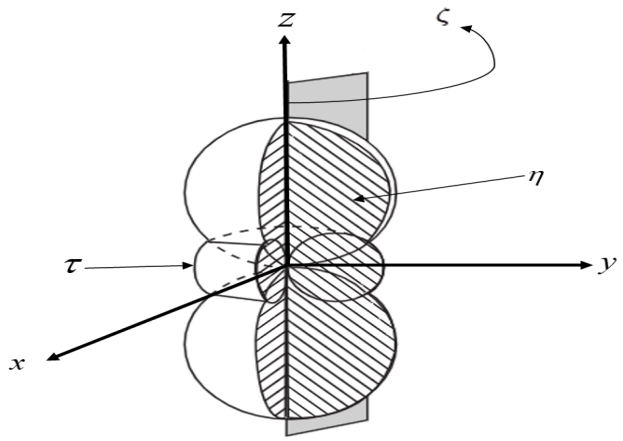

***Figure* 2.1:** Geometry of Tangent Sphere Coordinates

The metric coefficients $h_1,\ h_2$ and h_3 along coordinate curves $\tau,\ \eta$ and ζ are given by

$$h_1 = h_2 = (\tau^2 + \eta^2) \text{ and } h_3 = \frac{\tau}{\tau^2 + \eta^2}, \tag{2.2}$$

while Stokes operator E^2 in tangent sphere coordinate system (τ, η, ζ) assumes the form

$$E^2 = (\tau^2 + \eta^2)^2 \Big[\frac{\partial^2}{\partial \tau^2} + \frac{\tau^2 - \eta^2}{\tau(\tau^2 + \eta^2)} \frac{\partial}{\partial \tau} + \frac{2\eta}{\tau^2 + \eta^2} \frac{\partial}{\partial \eta} + \frac{\partial^2}{\partial \eta^2} \Big] \equiv \nabla^2. \tag{2.3}$$

2.2 Cardioid Coordinates

As it rolls around another fixed circle with the same radius, the cardioid is traced by a point on the circle's perimeter. A set of points of reflection of a fixed point on a circle through all of the circle's tangents is an additional description of a cardioid. In the study of curves and coordinate systems, the cardioid is a lovely topic because of its graceful shape and fascinating features that have interested mathematicians for a long time. The position of a moving fluid element P in the Cartesian coordinate system (x, y, z) can be tranformed in the cardioid coordinate system $P(\lambda, \sigma, \zeta)$, employing the following relations [15]:

$$x = \frac{\lambda\sigma\cos\zeta}{(\lambda^2 + \sigma^2)^2},\ y = \frac{\lambda\sigma\sin\zeta}{(\lambda^2 + \sigma^2)^2} \text{ and } z = \frac{\lambda^2 - \sigma^2}{2(\lambda^2 + \sigma^2)^2}, \tag{2.4}$$

where $\lambda \geq 0,\ \sigma \geq 0$ and $\zeta \in [0, 2\pi)$.

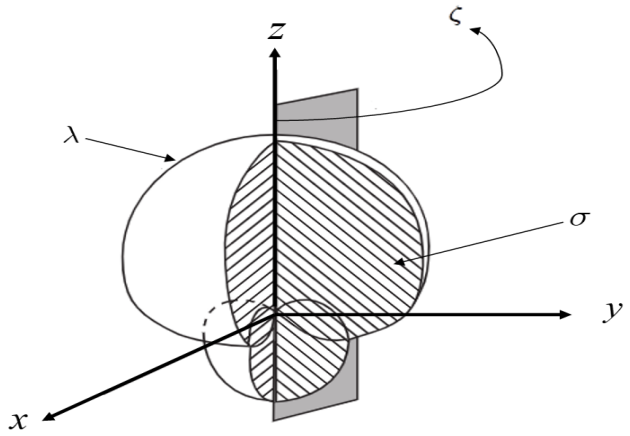

***Figure* 2.2:** *Geometry of Cardioid Coordinates*

The metric coefficients will be of the form

$$h_1 = h_2 = \frac{(\lambda^2 + \sigma^2)^3}{\sqrt{\lambda^2(\lambda^2 - 3\sigma^2)^2 + \sigma^2(\sigma^2 - 3\lambda^2)^2}} \text{ and } h_3 = \frac{\lambda\sigma}{(\lambda^2 + \sigma^2)^2}. \tag{2.5}$$

The operator E^2 assumes the form

$$E^2 = (\lambda^2+\sigma^2)^3\Big[\frac{\partial^2}{\partial\lambda^2} + \frac{3\lambda^2-\sigma^2}{\lambda(\lambda^2+\sigma^2)}\frac{\partial}{\partial\lambda} + \frac{3\sigma^2-\lambda^2}{\sigma(\lambda^2+\sigma^2)}\frac{\partial}{\partial\sigma} + \frac{\partial^2}{\partial\sigma^2}\Big] \equiv \nabla^2. \tag{2.6}$$

3 Mathematical Formulation and Solution

A form of fluid motion where the velocity at each point is parallel to a fixed plane is known as two-dimensional flow. To be more specific, the velocity to that fixed plane remains constant at every point along a given normal. Let us consider two dimensional flow of an incompressible viscous fluid in special cases of orthogonal curvilinear coordinates system such as tangent sphere and cardioid, respectively. Let $\vec{v}$ be velocity of moving fluid in two-dimensional motion having velocity components $(v_1(u_1,u_2), v_2(u_1,u_2), 0)$ in orthogonal curvilinear coordinates (u_1,u_2,u_3). The relationship between fluid velocity, cross-sectional area, and density can be described by the equation of continuity. The concept of conservation of mass in fluid dynamics is embodied in the continuity equation. In simple terms, it guarantees the conservation of mass throughout fluid flow within a system. It is essential to understand the behavior of fluid in conduits such as rivers, hoses, and pipes. The concept of conservation of mass in fluid dynamics is embodied in the continuity equation, which enables us to examine and estimates fluid flow patterns. For incompressible viscous fluid, the equation of continuity in mathematical form will imply

$$\nabla.\vec{v} = 0, \quad i.e. \quad \frac{\partial(h_2h_3v_1)}{\partial u_1} + \frac{\partial(h_3h_1v_2)}{\partial u_2} = 0. \tag{3.1}$$

Introducing a scalar-valued function ψ so that the equation of continuity is consequently satisfied. The aforementioned kind of function is known as a stream function. This function will represent all flow patterns that occur in fluid flows, which are the pathways followed by fluid particles as they move through the flow. Stream function can be employed for illustrating and clarifying fluid flow fields which enables us to examine and estimate fluid flow patterns. In order to this assumption, we may express the velocity components in terms of stream function by following formulae:

$$v_1 = \frac{1}{h_2h_3}\frac{\partial\psi}{\partial u_2} \qquad \text{and} \qquad v_2 = -\frac{1}{h_3h_1}\frac{\partial\psi}{\partial u_1}. \tag{3.2}$$

Let $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$ be the unit vectors along coordinate curves (u_1,u_2,u_3), then

$$\nabla\times\vec{v} = \frac{\hat{e}_3}{h_3}\,E^2\psi, \qquad \text{and} \qquad \nabla\times\nabla\times\nabla\times\vec{v} = \frac{\hat{e}_3}{h_3}\,E^2(E^2\psi), \tag{3.3}$$

where $E^2 = \dfrac{h_3}{h_1h_2}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2}{h_3h_1}\dfrac{\partial}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1}{h_2h_3}\dfrac{\partial}{\partial u_2}\right)\right]$.

Operating the curl operator on the Brinkman equation (1.3) in orthogonal curvilinear coordinates and eliminating the pressure, we will obtain

$$E^2(E^2-\alpha^2)\psi = 0, \quad \text{where} \quad \alpha = \sqrt{\frac{\mu}{k\mu_e}}. \tag{3.4}$$

3.1 In Tangent Sphere coordinates

To obtain the general stream function solution of the Brinkman equation (3.4) in tangent sphere coordinate system, assuming that $\psi(\tau,\eta) = \psi_1(\tau,\eta) + \psi_2(\tau,\eta)$ such that

$$E^2\psi_1(\tau,\eta) = 0, \tag{3.5}$$

and

$$(E^2-\alpha^2)\psi_2(\tau,\eta) = 0. \tag{3.6}$$

3.1.1 Solution for $\psi_1(\tau,\eta)$:

To solve the equation (3.5), assuming that

$$\psi_1(\tau,\eta) = \frac{1}{\sqrt{\tau^2+\eta^2}}M(\tau)N(\eta). \tag{3.7}$$

Applying the method of separation of variables and using equations (2.3) and (3.5), we will get that

$$\frac{M^{''}(\tau)}{M(\tau)} - \frac{1}{\tau}\frac{M^{'}(\tau)}{M(\tau)} = -\frac{N^{''}(\eta)}{N(\eta)} = m^2 > 0. \tag{3.8}$$

The above equation (3.8) exhibits

$$\tau^2M^{''}(\tau) - \tau M^{'}(\tau) - \tau^2m^2M(\tau) = 0, \tag{3.9}$$

$$\text{and} \qquad N^{''}(\eta) + m^2 N(\eta) = 0. \tag{3.10}$$

By substituting $M(\tau) = \tau U(z)$ and $z = m\tau$, in the above equation (3.9), we can transform it as follows:

$$z^2 U''(z) + zU'(z) - (z^2 + 1)U(z) = 0. \tag{3.11}$$

The differential equation mentioned above is a modified Bessel differential equation of first order whose two linearly independent solutions are $I_1(\cdot)$ and $K_1(\cdot)$. Thus, the solution of differental equations (3.9) and (3.10) will be of the form

$$M(\tau) = A_m \tau I_1(m\tau) + B_m \tau K_1(m\tau), \tag{3.12}$$

$$\text{and} \quad N(\eta) = C_m \cos(m\eta) + D_m \sin(m\eta). \tag{3.13}$$

Therefore, the solution of differential equation (3.5) can be expressed in the following form

$$\psi_1(\tau, \eta) = \frac{\tau}{\sqrt{\tau^2 + \eta^2}} \sum_{m=1}^{\infty} \Big[A_m I_1(m\tau) + B_m K_1(m\tau) \Big]^{\cos(m\eta)}_{\sin(m\eta)}, \tag{3.14}$$

where A_m, B_m, C_m and D_m are arbitrary constants.

3.1.2 Solution for $\psi_2(\tau, \eta)$:

Now, we are planning to solve $(E^2 - \alpha^2)\psi_2 = 0$. For this, we will choose

$$\psi_2(\tau, \eta) = \frac{1}{\sqrt{\tau^2 + \eta^2}} R(\tau) S(\eta). \tag{3.15}$$

Applying the method of separation of variables by keeping the equations (2.3) and (3.6), we will obtain the following expressions:

$$\frac{R^{''}(\tau)}{R(\tau)} - \frac{1}{\tau} \frac{R^{'}(\tau)}{R(\tau)} - \alpha^2 = -\frac{S^{''}(\eta)}{S(\eta)} = m^2 > 0. \tag{3.16}$$

The mathematical equation (3.16) implies that

$$\tau^2 R^{''}(\tau) - \tau R^{'}(\tau) - \tau^2 (m^2 + \alpha^2) R(\tau) = 0, \tag{3.17}$$

$$\text{and} \qquad S^{''}(\eta) + m^2 S(\eta) = 0. \tag{3.18}$$

Therefore, solution of equations (3.17) and (3.18) will be

$$R(\tau) = E_m \tau I_1(\tau \sqrt{m^2 + \alpha^2}) + F_m \tau K_1(\tau \sqrt{m^2 + \alpha^2}), \tag{3.19}$$

$$\text{and} \qquad S(\eta) = G_m \cos(m\eta) + H_m \sin(m\eta). \tag{3.20}$$

Hence, analytical expression of ψ_2 in concise form will be

$$\psi_2(\tau, \eta) = \frac{\tau}{\sqrt{\tau^2 + \eta^2}} \sum_{m=1}^{\infty} \Big[E_m I_1(\tau \sqrt{m^2 + \alpha^2}) + F_m K_1(\tau \sqrt{m^2 + \alpha^2}) \Big]^{cos(m\eta)}_{sin(m\eta)}, \tag{3.21}$$

where E_m, F_m, G_m and H_m are arbitrary constants.

Hence, the general stream function solution of Brinkman equation (3.4) in the tangent sphere coordinates can be expressed in the following form

$$\psi(\tau, \eta) = \frac{\tau}{\sqrt{\tau^2 + \eta^2}} \sum_{m=1}^{\infty} \left[A_m I_1(m\tau) + B_m K_1(m\tau) + E_m I_1(\tau \sqrt{m^2 + \alpha^2}) + F_m K_1(\tau \sqrt{m^2 + \alpha^2}) \right]^{\cos(m\eta)}_{\sin(m\eta)}. \tag{3.22}$$

3.2 In Cardioid Coordinates:

In queue of determination of the general stream function solution of Brinkman equation (3.4) in cardioid coordinates, we may assume that $\psi(\lambda, \sigma) = \psi_1(\lambda, \sigma) + \psi_2(\lambda, \sigma)$ with hypothesis $E^2 \psi_1(\lambda, \sigma) = 0$ and $(E^2 - \alpha^2)\psi_2(\lambda, \sigma) = 0$.

3.2.1 Determination of $\psi_1(\lambda,\sigma)$:

Assuming that

$$\psi_1(\lambda,\sigma)=\frac{M(\lambda)N(\sigma)}{(\lambda^2+\sigma^2)\sqrt{2}}. \tag{3.23}$$

Utilizing the analytical expression of differential operator E^2 from equation (2.6) and applying the separation of variables techniques, we obtain

$$\frac{M^{''}(\lambda)}{M(\lambda)}-\frac{1}{\lambda}\frac{M^{'}(\lambda)}{M(\lambda)}=-\frac{N^{''}(\sigma)}{N(\sigma)}+\frac{1}{\sigma}\frac{N^{'}(\sigma)}{N(\sigma)}=m^2>0. \tag{3.24}$$

From equation (3.24), we have

$$\lambda^2M^{''}(\lambda)-\lambda M^{'}(\lambda)-m^2\lambda^2M(\lambda)=0, \tag{3.25}$$

$$\text{and}\quad \sigma^2N^{''}(\sigma)-\sigma N^{'}(\sigma)+m^2\sigma^2N(\sigma)=0. \tag{3.26}$$

Substituting $M(\lambda)\to\lambda M(m\lambda)$, in the equation (3.25), then it can be transformed into a modified Bessel differential equation of first order whose two linearly independent solutions are $I_1(\cdot)$ and $K_1(\cdot)$. On the other hand, employing similar substitution, equation (3.26) can be reduced into Bessel differential equation of first order whose two linearly independent solutions are $J_1(\cdot)$ and $Y_1(\cdot)$. Hence the solution of differential equations (3.25) and (3.26) will be

$$M(\lambda)=\Big[A_m\lambda I_1(m\lambda)+B_m\lambda K_1(m\lambda)\Big], \tag{3.27}$$

$$\text{and}\qquad N(\sigma)=\Big[C_m\sigma J_1(m\sigma)+D_m\sigma Y_1(m\sigma)\Big], \tag{3.28}$$

where A_m, B_m, C_m and D_m are arbitrary constants.

Now putting these values of $M(\lambda)$ and $N(\sigma)$ from equations (3.27) and (3.28) in equation (3.23), $\psi_1(\lambda,\sigma)$ will be of the form

$$\psi_1(\lambda,\sigma)=\frac{\lambda\sigma}{\sqrt{2}(\lambda^2+\sigma^2)}\sum_{m=1}^{\infty}\Big[A_mI_1(m\lambda)+B_mK_1(m\lambda)\Big]_{Y_1(m\sigma)}^{J_1(m\sigma)}. \tag{3.29}$$

Solution of $\psi_2(\lambda,\sigma)$:

Since, scalar valued function $\psi_2(\lambda,\sigma)$ is satisfying the equation $(\nabla^2-\alpha^2)\psi_2(\lambda,\sigma)=0$, therefore, we may assume $\psi_2(\lambda,\sigma)$ in the following form:

$$\psi_2(\lambda,\sigma)=\frac{R(\lambda)S(\sigma)}{(\lambda^2+\sigma^2)\sqrt{2}}. \tag{3.30}$$

Applying separation of variables, we will obtain a pair of ordinary differential equations

$$\lambda^2R^{''}(\lambda)-\lambda R^{'}(\lambda)-\lambda^2(m^2+\alpha^2)R(\lambda)=0, \tag{3.31}$$

$$\text{and}\qquad \sigma^2S^{''}(\sigma)-\sigma S^{'}(\sigma)+\sigma^2m^2S(\sigma)=0. \tag{3.32}$$

Solving these differential equations as usual, we will get

$$R(\lambda)=\Big[E_m\lambda I_1(\lambda\sqrt{\alpha^2+m^2})+F_m\lambda K_1(\lambda\sqrt{\alpha^2+m^2})\Big], \tag{3.33}$$

$$\text{and}\qquad S(\sigma)=\Big[G_m\sigma J_1(m\sigma)+H_m\sigma Y_1(m\sigma)\Big], \tag{3.34}$$

where E_m, F_m, G_m and H_m are arbitrary constants .

Substituting the values of $R(\lambda)$ and $S(\sigma)$ from equations (3.33) and (3.34) in equation (3.30), we obtain the explicit expression as follows:

$$\psi_2(\lambda,\sigma)=\frac{\lambda\sigma}{\sqrt{2}(\lambda^2+\sigma^2)}\sum_{m=1}^{\infty}\left[E_mI_1(\lambda\sqrt{\alpha^2+m^2})+F_mK_1(\lambda\sqrt{\alpha^2+m^2})\right]_{Y_1(m\sigma)}^{J_1(m\sigma)}. \tag{3.35}$$

Therefore, the stream function solution of Brinkman equation (3.4) in the cardioid coordinates in the precise analytical form will be

$$\psi(\lambda,\sigma)=\frac{\lambda\sigma}{\sqrt{2}(\lambda^2+\sigma^2)}\sum_{m=1}^{\infty}\Bigg[A_mI_1(m\lambda)+B_mK_1(m\lambda)+E_mI_1(\lambda\sqrt{\alpha^2+m^2}) + F_mK_1(\lambda\sqrt{\alpha^2+m^2})\Bigg]_{Y_1(m\sigma)}^{J_1(m\sigma)}. \tag{3.36}$$

4 Determination of Velocity Components

4.1 In Tangent Sphere Coordinates

The velocity components v_τ and v_η in the tangent sphere coordinate system can be determined using the equations (3.2) and (3.22). These velocity components are

$$v_\tau = -\frac{1}{(\eta^2+\tau^2)^{3/2}}\Bigg[\sin(\eta m)\left(\eta C_m\left(\eta^2+\tau^2\right)+\eta D_m\right)+\cos(\eta m)\left(\eta C_m - mD_m\left(\eta^2+\tau^2\right)\right)\Bigg] \Bigg[A_m I_m(\tau)+B_m K_m(\tau)+F_m K_{\sqrt{m^2+\alpha^2}}(\tau)+e_m I_{\sqrt{m^2+\alpha^2}}(\tau)\Bigg], \quad (4.1)$$

and

$$\begin{aligned} v_\eta = \frac{1}{(\eta^2+\tau^2)^{3/2}}\Bigg[& -\frac{\left(\eta^2+\tau^2\right)}{\tau}\Big[A_m I_m(\tau)+B_m K_m(\tau)+F_m K_{\sqrt{n^2+\alpha^2}}+E_m I_{\sqrt{n^2+\alpha^2}}\Big] \\ & -\frac{1}{2}\left(\eta^2+\tau^2\right)\Big(A_m(I_{m-1}(\tau)+I_{m+1}(\tau))-B_m(K_{m-1}(\tau)+K_{m+1}(\tau)) \\ & -F_m\big(K_{\sqrt{n^2+\alpha^2}-1}(\tau)+K_{\sqrt{n^2+\alpha^2}+1}(\tau)\big)+E_m\big(I_{\sqrt{n^2+\alpha^2}-1}(\tau)+I_{\sqrt{n^2+\alpha^2}+1}(\tau)\big)\Big) \\ & +\tau\Big(A_m I_m(\tau)+B_m K_m(\tau)+F_m K_{\sqrt{n^2+\alpha^2}}(\tau)+E_m I_{\sqrt{n^2+\alpha^2}}(\tau)\Big)\Bigg]_{\sin n\eta}^{\cos n\eta}. \end{aligned} \quad (4.2)$$

4.2 In Cardioid Coordinates

The velocity components v_λ and v_σ in cardioid coordinates can be obtained from the equation (3.2), will be of the form

$$v_\lambda = \frac{1}{2}\Bigg[C_m\left(\frac{2\lambda^2 J_m(\sigma)}{\sigma^3+\sigma\lambda^2}+J_{m-1}(\sigma)-J_{m+1}(\sigma)\right)+D_m\Bigg(\frac{2\lambda^2 Y_m(\sigma)}{\sigma^3+\sigma\lambda^2}+Y_{m-1}(\sigma) - Y_{m+1}(\sigma)\Bigg)\Bigg]\left(A_m I_m(\lambda)+B_m K_m(\lambda)+F_m K_{\sqrt{m^2+\alpha^2}}(\lambda)+E_m I_{\sqrt{m^2+\alpha^2}}(\lambda)\right), \quad (4.3)$$

and

$$\begin{aligned} v_\sigma = \frac{1}{2}\left(C_m J_m(\sigma)+D_m Y_m(\sigma)\right)\Bigg[& \left(A_m I_m(\lambda)+B_m K_m(\lambda)+F_m K_{\sqrt{m^2+\alpha^2}}(\lambda)\right. \\ & \left.+E_m I_{\sqrt{m^2+\alpha^2}}(\lambda)\right)\times\frac{2\lambda}{\sigma^2+\lambda^2}-\frac{2}{\lambda}\Big(A_m I_m(\lambda)+B_m K_m(\lambda)+F_m K_{\sqrt{m^2+\alpha^2}}(\lambda) \\ & +E_m I_{\sqrt{m^2+\alpha^2}}(\lambda)\Big)-A_m\left(I_{m-1}(\lambda)+I_{m+1}(\lambda)\right)+B_m(K_{m-1}(\lambda)+K_{m+1}(\lambda)) \\ & +F_m\left(K_{\sqrt{m^2+\alpha^2}-1}(\lambda)+K_{\sqrt{m^2+\alpha^2}+1}(\lambda)\right)-E_m\left(I_{\sqrt{m^2+\alpha^2}-1}(\lambda)+I_{\sqrt{m^2+\alpha^2}+1}(\lambda)\right)\Bigg]. \end{aligned} \quad (4.4)$$

5 Deduction from Obtained Solution

In this section, authors have tried to reduce the obtained solution into stream function solution of Stokes equation in tangent sphere and cardioid coordinate systems for the special case of permeability. We are familiar that Brinkman equation reduces to Stokes equation as permeability becomes infinite.

5.1 When $\alpha \to 0$ in Tangent Sphere Coordinate System

Substituting the value of permeability parameter $k \to \infty$, *i.e.* $\alpha \to 0$ in the stream function solution reported in equation (3.22), we will obtain the following expression:

$$\psi(\tau,\eta) = \frac{\tau}{\sqrt{\tau^2+\eta^2}}\sum_{m=1}^{\infty}\Big[A_m I_1(m\tau)+B_m K_1(m\tau)\Big]_{\sin(m\eta)}^{\cos(m\eta)}. \quad (5.1)$$

The deduced result matches with the stream function solution of Stokes equations of motion in tangent sphere coordinate system, reported earlier by Protopapas [17].

5.2 When $\alpha \to 0$ in Cardioid Coordinate System

Substituting $\alpha \to 0$ in the stream function solution given in equation (3.36), we will obtain

$$\psi(\lambda, \sigma) = \frac{\lambda\sigma}{\sqrt{2}(\lambda^2 + \sigma^2)} \sum_{m=1}^{\infty} \Big[A_m I_1(m\lambda) + B_m K_1(m\lambda) \Big]_{Y_1(m\sigma)}^{J_1(m\sigma)}. \tag{5.2}$$

The above result matches and validates with the solution of Stokes equation of motion in cardioid coordinate system, published earlier by Protopapas [17].

6 Result and Discussion

Authors have determined the stream function solution of the Brinkman equation in two coordinate systems namely cardioid (τ, η, ζ) and tangent sphere (λ, σ, ζ). Method of separation of variables is used to transform the non-linear partial differential equations into linear homogeneous differential equations. Obtained analytical expression of the stream function and velocity components is representing the algebraic functions, trigonometric functions $(\sin(\cdot), \cos(\cdot))$, Bessel functions $(J_1(\cdot), K_1(\cdot))$ of first and second kinds in first order and modified Bessel functions $(I_1(\cdot), K_1(\cdot))$ of first and second kinds.

This analytical solution can be implemented for both two-dimensional vortex flow and creeping motion. This stream function solution for the Brinkman equation can be used to a variety of fluid flow problems that include passing through a heart-shaped or spherically-shaped obstruction. Analysis of the motion of a porous sphere rolling on a porous surface can be performed with the obtained solution. The Brinkman equation offers analytical solutions to a number of physical problems involving cardioid systems and tangent spheres. In fluid dynamics, the Brinkman equation constitutes a connection between microscopic and macroscopic scales. The Brinkman equation is used in petroleum engineering to simulate steady-state viscous flow through porous media. The fluid flow of biofilms, which are communities of microorganisms confined to surfaces, can be modeledusing the help of the Brinkman equation. Blood flow in porous medium, such clots, must be understood by taking the Brinkman equation into account. This equation can also be used to explain the motion of flagella, which are structures resembling whips, in gels.

7 Conclusion

The stream function solution of the Brinkman equation in the tangent sphere and cardioid coordinate systems is presented in this research work. Transcendental functions including trigonometric functions, modified Bessel functions, and Bessel functions are employed in the analytical expressions of the stream function and velocity components. Authors have discussed the dependence of the permeability parameter on the obtained solution for the Brinkman equation in the tangent sphere and cardioid coordinate systems. Authors have deduced their solutions for particular case of permeability which matches with available results [17]. Fluid flow in intricate domains inside porous medium is modeled using the Brinkman equation. In addition to having practical applications in a variety of domains, including biological flows, complex domain modeling, and petroleum engineering, the Brinkman equation serves as vital role fortheunderstanding offluid flow through porous media.

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