FRACTIONAL TAYLOR EXPANSION OF CLASSICAL AND LAGUERRE-TYPE FUNCTIONS

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Abstract

Beginning with the fractional exponential, which is frequently used in applications, fractional Taylor expansions of several elementary functions in both the ordinary and Laguerrian cases are introduced, showing that some basic formulas continue to apply to new expansions. Some graphs obtained using the computer algebra system Mathematica® are given for illustration purposes.

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1 Introduction

Fractional calculus is becoming increasingly important both in the field of applied sciences (see e.g. [16]–[19]) and in that of special functions, including multivariate ones (see e.g. [13]).

In a recent article [3], we have used a fractional version of the exponential function, defined as

\[ \text{Exp}_\alpha(t) = 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots + \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} + \cdots, \]  

(1.1)

to derive a generalized form of the Bernoulli and Euler numbers and polynomials. Using the computer algebra system Mathematica, we have derived several tables featuring these new special numbers and fractional polynomials for certain values of the parameter \( \alpha \) between 0 and 1, showing that these entities converge to the classical numbers as \( \alpha \) approaches 1.

It is worth to recall that the fractional exponential function (1.1) is a special case of the Mittag-Leffler function \( E_\alpha(x) \) [10], since it results

\[ \text{Exp}_\alpha(x) = E_\alpha(x^\alpha). \]

Note that the fractional exponential function satisfies the eigenvalue property

\[ D_x^\alpha \text{Exp}_\alpha(xt) = t^\alpha \text{Exp}_\alpha(xt), \]

with respect the fractional derivative \( D_x^\alpha \), defined by the Euler equation

\[ D_x^\alpha x^n = \begin{cases} \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \alpha)} x^{n-\alpha}, & \text{if } n > [\alpha] - 1, \\ 0, & \text{if } n = 0, 1, \ldots, [\alpha] - 1, \end{cases} \]  

(1.2)

where \( n \geq 0 \) and \( [\alpha] \) denotes the ceiling function (the smallest integer greater than or equal to \( \alpha \)).

If \( c \) is a constant then \( D_x^\alpha c = 0 \).

This definition falls as a special case, of the fractional derivative introduced by Caputo [1], defined as follows

\[ D_{a+}^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m - \alpha)} \int_a^x \frac{f^{(m)}(\tau)}{(x - \tau)^{\alpha-m+1}} d\tau, & \text{where } m = [\alpha], \text{ if } \alpha \notin \mathbb{N} \\ f^{(\alpha)}(x), & \text{if } \alpha \in \mathbb{N}, \end{cases} \]

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and reduces to the above equation (1.2) when \( a = 0 \) and \( f(x) = x^n \).

As the fractional exponential function (1.1) has been used in the article [3], as well as in the Laguerre-type fractional version of the Bernoulli and Euler numbers and polynomials [14], and in several other different applications [2, 4, 5], we are motivated to study the fractional versions of other elementary function and their fractional Laguerre-type analogues.

In Section 2 we derive the fractional Taylor expansion of the circular, hyperbolic and logarithm functions, while in Section 3 we examine the case of the fractional Laguerre-type version of some classical functions.

Several graphs of the considered functions are depicted with the aid of the computer algebra system Mathematica®.

2 Fractional Taylor expansions

Consider the Taylor (McLaurin) expansion

\[
f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad a_n := D_x^n f(x)_{|x=0}
\]

and suppose that this expansion is convergent for \( |x| < R \), \( R \) real positive number or \(+\infty\).

According to the result in Groza-Jianu [11], putting \( D^n f(x) = D_x^n D_x^a \cdots D_x^2 f(x) \), \( n \) fractional derivatives), the expansion

\[
F_\alpha(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{\Gamma(n\alpha+1)}, \quad a_n := D^{\alpha n} f(x)_{|x=0},
\]

is convergent for \( |x| < R^2 \) or \( |x| < +\infty \).

This means that the Taylor fractional version \( F_\alpha(x) \) of a given function \( f(x) \) can be written with the same coefficients and replacing the power \( x^n \) with \( x^{\alpha n} \) and writing the factorial in terms of the corresponding Gamma function.

The main fractional Taylor expansions are reported in the following sections.

2.1 Circular functions

\[
\sin_{\alpha}(x) = \frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{x^{5\alpha}}{\Gamma(5\alpha+1)} + \cdots + (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} + o(x^{(2n+2)\alpha}),
\]

\[
\cos_{\alpha}(x) = 1 - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots + (-1)^n \frac{x^{2\alpha n}}{\Gamma(2n\alpha+1)} + o(x^{(2n+1)\alpha}).
\]

We prove the following result

**Theorem 2.1.** The fractional \( \sin_{\alpha}(x) \) and \( \cos_{\alpha}(x) \) functions satisfy the same differentiation property, with respect to the Fractional derivative \( D_x^\alpha \), as the classical ones with respect to the ordinary derivative, that is

\[
D_x^\alpha \sin_{\alpha}(x) = \cos_{\alpha}(x), \tag{2.1}
\]

and

\[
D_x^\alpha \cos_{\alpha}(x) = -\sin_{\alpha}(x). \tag{2.2}
\]

**Proof.** In fact, for the \( \sin_{\alpha}(x) \) function, it results

\[
D_x^\alpha (-1)^n \frac{x^{2\alpha n(2n+1)}}{\Gamma((2n+1)\alpha+1)} = (-1)^n \frac{1}{\Gamma((2n+1)\alpha+1)} \Gamma((2n+1)\alpha+1) x^{2\alpha n} = (-1)^n \frac{x^{2\alpha n}}{\Gamma(2n\alpha+1)},
\]

so that equation (2.1) follows.

For the \( \cos_{\alpha}(x) \) function, we have

\[
\cos_{\alpha}(x) = 1 - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots + (-1)^n \frac{x^{2\alpha n}}{\Gamma(2n\alpha+1)} + o(x^{(2n+1)\alpha}) =
\]

\[
= 1 - \left( \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots + (-1)^{n+1} \frac{x^{2\alpha n}}{\Gamma(2n\alpha+1)} \right) + o(x^{(2n+1)\alpha})
\]

Recalling that the fractional derivative of the constant vanishes, it results

\[
D_x^\alpha (-1)^n+1 \frac{x^{2\alpha n}}{\Gamma(2n\alpha+1)} = (-1)^n \frac{1}{\Gamma((2n\alpha+1-\alpha))} \Gamma((2n\alpha+1-\alpha)) x^{2\alpha n - \alpha} =
\]

\[
= (-1)^n \frac{x^{2\alpha n}}{\Gamma((2n-1)\alpha+1)} \Gamma((2n-1)\alpha).\]

Posing \( n+1 \) instead of \( n \) we find the result in equation (2.2).

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For the classical tangent function we have the Taylor expansion using Bernoulli numbers:
\[
\tan(x) = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \frac{17}{315} x^7 + \frac{62}{2835} x^9 + \cdots + \frac{2^{2n}(2^{2n} - 1) B_{2n}}{(2n)!} x^{2n - 1} + o(x^{2n}).
\]
We pose by definition:
\[
\tan_{\alpha}(x) = \frac{\sin_{\alpha}(x)}{\cos_{\alpha}(x)} = c_1 x^\alpha + c_3 x^{3\alpha} + c_5 x^{5\alpha} + c_7 x^{7\alpha} + \cdots + c_{2n+1} x^{(2n+1)\alpha} + \cdots,
\]
Since
\[
\tan_{\alpha}(x) = \frac{x^{\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{5\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{2n\alpha}}{\Gamma((\alpha+1)/2)} + \cdots = \\
\left(1 + \frac{x^{2\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{4\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{2n\alpha}}{\Gamma((\alpha+1)/2)} + \cdots \right),
\]
we find, for the first few values
\[
c_1 = \frac{1}{\Gamma(\alpha+1)},
\]
\[
c_3 = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)},
\]
\[
c_5 = -\frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} + \frac{1}{\Gamma(5\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)},
\]
\[
c_7 = -\frac{1}{\Gamma(2\alpha+1)^2} + \frac{1}{\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} + \frac{1}{\Gamma(5\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} - \frac{1}{\Gamma(\alpha+1)},
\]
\[
\cdots.
\]

2.2 Hyperbolic functions
We pose, by definition:
\[
\sinh_{\alpha}(x) = \frac{x^{\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{3\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{5\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{(2n+1)\alpha}}{\Gamma((\alpha+1)/2)} + o(x^{(2n+1)\alpha})
\]
\[
\cosh_{\alpha}(x) = 1 + \frac{x^{2\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{4\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{2n\alpha}}{\Gamma((\alpha+1)/2)} + o(x^{(2n+1)\alpha}).
\]
For the classical hyperbolic tangent function we have the Taylor expansion:
\[
\tanh(x) = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \frac{62}{2835} x^9 - \cdots + (-1)^n \frac{x^{2n-1}}{(2n)!} x^{2n-1} + o(x^{2n}).
\]
We pose, by definition:
\[
\tanh_{\alpha}(x) = \frac{\sinh_{\alpha}(x)}{\cosh_{\alpha}(x)} = c_1 x^\alpha + c_3 x^{3\alpha} + c_5 x^{5\alpha} + c_7 x^{7\alpha} + \cdots + c_{2n+1} x^{(2n+1)\alpha} + \cdots,
\]
Since
\[
\tanh_{\alpha}(x) = \frac{x^{\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{3\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{5\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{(2n+1)\alpha}}{\Gamma((\alpha+1)/2)} + \cdots = \\
\left(1 + \frac{x^{2\alpha}}{\Gamma((\alpha+1)/2)} + \frac{x^{4\alpha}}{\Gamma((\alpha+1)/2)} + \cdots + \frac{x^{2n\alpha}}{\Gamma((\alpha+1)/2)} + \cdots \right),
\]
we find, for the first few values
\[
c_1 = \frac{1}{\Gamma(\alpha+1)},
\]
\[
c_3 = -\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)},
\]
\[
c_5 = -\frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} + \frac{1}{\Gamma(5\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} + \frac{1}{\Gamma(5\alpha+1)} - \frac{1}{\Gamma(\alpha+1)},
\]
\[
c_7 = -\frac{1}{\Gamma(2\alpha+1)^2} + \frac{1}{\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} + \frac{1}{\Gamma(5\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} - \frac{1}{\Gamma(\alpha+1)},
\]
\[
\cdots.
\]
2.3 Fractional logarithm
We pose, by definition:
\[ \alpha \log(1 + x) = x^\alpha - \frac{\Gamma(\alpha)}{\Gamma(1+\alpha)} x^{2\alpha} + \frac{\Gamma(2\alpha)}{\Gamma(3\alpha)} x^{3\alpha} + \cdots + (-1)^n \frac{\Gamma((n-1)\alpha)}{n! \Gamma(n\alpha)} x^{n\alpha} + o(x^{n\alpha}). \]

3 The Laguerre-type case
As the exponential function \( e^{ax} \) is an eigenfunction of the derivative operator \( D := d/dx \), since \( De^{ax} = ae^{ax} \), likewise the Laguerre-type exponential
\[ e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}. \]
is an eigenfunction of the Laguerre derivative
\[ \hat{D}_L := DXD = D + xD^2. \]
In fact it results
\[ \hat{D}_L e_1(ax) = ae_1(ax). \]

The above result has been extendes as follows [6, 15].
Considering the differential operator, containind \( n + 1 \) derivatives
\[ \hat{D}^{(n+1)}_L := DX \cdots DXDXD = \]
\[ = D(xD + x^2D^2 + \cdots + x^{n-1}D^{n-1}) = \]
\[ = S(n,1)D + S(n,2)xD^2 + \cdots + S(n,n)x^{n-1}D^n, \]
(where \( S(n,k), k = 1,2, \ldots, n \), denote Stirling numbers of the second kind), and the function:
\[ e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \]
we have proven in [6] that the function \( e_n(ax) \) is an eigenfunction of the operator \( \hat{D}_{nL} \), that is
\[ \hat{D}_{nL} e_n(ax) = ae_n(ax). \]

Obviously, \( \hat{D}_{1L} := \hat{D}_L \).

For this reason we will refer to such functions as L-exponential functions, or shortly L-exponentials.

Remark 3.1. For completeness, we recall that the operators \( D_L = DXD \) and its itertes as \( D_{nL} = DXDxDX \cdots DXD \) can be considered as particular cases of the hyper-Bessel differential operators when \( \alpha_0 = \alpha_1 = \cdots = \alpha_n = \alpha \) (the special case considered in operational calculus by Ditkin and Prudnikov [9]).
In general, the Bessel-type differential operators of arbitrary order \( n \) were introduced by Dimovski, in 1966 [8] and later called by Kiryakova hyper-Bessel operators, because are closely related to their eigenfunctions, called hyper-Bessel by Delerue [7], in 1953. These operators were studied in 1994 by Kiryakova in her book [12], Ch. 3.

4 Fractional Laguerre-type expansions
We limit ourselves to consider only the 1st order Laguerre fractional derivative, as the general case is similar.

4.1 1st order Laguerre-type fractional circular functions
We have the definitions
\[ \text{L} \sin_\alpha(x) = \frac{x^\alpha}{\Gamma((\alpha+1)\alpha)} - \frac{x^{3\alpha}}{\Gamma((3\alpha+1)\alpha)} + \cdots + (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)(\alpha+1)\alpha)} + o(x^{(2n+2)\alpha}) \]
\[ \text{L} \cos_\alpha(x) = 1 - \frac{x^{2\alpha}}{\Gamma((2\alpha+1)\alpha)} + \frac{x^{4\alpha}}{\Gamma((4\alpha+1)\alpha)} + \cdots + (-1)^n \frac{x^{2n\alpha}}{\Gamma((2n\alpha+1)\alpha)} + o(x^{(2n+1)\alpha}). \]
We prove the following result

Theorem 4.1. The fractional \( \text{L} \sin_\alpha(x) \) and \( \text{L} \cos_\alpha(x) \) functions satisfy the differentiation properties
\[ D_x^\alpha \text{L} \sin_\alpha(x) = \text{L} \cos_\alpha(x), \quad (4.1) \]
and
\[ D_x^\alpha \text{L} \cos_\alpha(x) = -\text{L} \sin_\alpha(x). \quad (4.2) \]
Proof. In fact, for the \( L\text{Sin}_a(x) \) function, it results

\[
D^a_x x^a D^a_x (-1)^n \frac{x^{(2n+1)a}}{[\Gamma(2(n+1)a+1)]^2} =
\]

\[
= (-1)^n \frac{1}{[\Gamma(2n+1)\alpha+1)]^2} \frac{\Gamma(2n+1)^2}{[\Gamma(2n+1)]^2} x^{2\alpha} = (-1)^n \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2},
\]

so that equation (4.1) follows.

For the \( L\text{Cos}_a(x) \) function, we have

\[
L\text{Cos}_a(x) = 1 - \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + \frac{x^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} + o(x^{2n+1}) =
\]

\[
= 1 - \left( \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + \frac{x^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} \right) + o(x^{2n+1}).
\]

Recalling that the fractional derivative of the constant vanishes, it results

\[
D^a_x x^a D^a_x (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} = (-1)^n \frac{1}{[\Gamma(2(n+1)\alpha+1)]^2} \frac{\Gamma(2n+1)^2}{[\Gamma(2(n+1)\alpha+1)]^2} x^{2\alpha} =
\]

\[
= (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} [\Gamma(2(n+1)\alpha+1)]\alpha x^{(2n+1)\alpha} =
\]

\[
= (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} x^{(2n+1)\alpha}.
\]

Posing \( n+1 \) instead of \( n \) we find the result in equation (4.2).

For the Laguerre-type fractional tangent function we pose, by definition

\[
L\text{Tan}_a(x) = \frac{L\text{Sin}_a(x)}{L\text{Cos}_a(x)} = d_1 x^\alpha + d_3 x^{3\alpha} + d_5 x^{5\alpha} + d_7 x^{7\alpha} + \cdots + d_{2n+1} x^{(2n+1)\alpha} + \cdots,
\]

Since

\[
\frac{x^\alpha}{[\Gamma(\alpha+1)]^2} - \frac{x^{3\alpha}}{[\Gamma(3\alpha+1)]^2} + \frac{x^{5\alpha}}{[\Gamma(5\alpha+1)]^2} + \cdots + (-1)^n \frac{x^{(2n+1)\alpha}}{[\Gamma(2n\alpha+1)]^2} + \cdots =
\]

\[
\left( 1 - \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + \frac{x^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + (-1)^n \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} + \cdots \right) \cdot \left( d_1 x^\alpha + d_3 x^{3\alpha} + d_5 x^{5\alpha} + d_7 x^{7\alpha} + \cdots + d_{2n+1} x^{(2n+1)\alpha} + \cdots \right),
\]

we find, for the first few values

\[
d_1 = \frac{1}{[\Gamma(\alpha+1)]^2},
\]

\[
d_3 = \frac{1}{[\Gamma(\alpha+1)]^2} \frac{1}{[\Gamma(3\alpha+1)]^2} - \frac{1}{[\Gamma(3\alpha+1)]^2},
\]

\[
d_5 = -\frac{1}{[\Gamma(2\alpha+1)]^2} \frac{1}{[\Gamma(3\alpha+1)]^2} + \frac{1}{[\Gamma(4\alpha+1)]^2} \frac{1}{[\Gamma(5\alpha+1)]^2} + \frac{1}{[\Gamma(5\alpha+1)]^2} - \frac{1}{[\Gamma(6\alpha+1)]^2} - \frac{1}{[\Gamma(7\alpha+1)]^2},
\]

\[
\cdots
\]

4.2 Hyperbolic functions

In a similar way we find, for the Laguerre-type fractional hyperbolic functions the definitions

\[
L\text{Sin}_a(x) = \left( \frac{x^\alpha}{[\Gamma(\alpha+1)]^2} + \frac{x^{3\alpha}}{[\Gamma(3\alpha+1)]^2} + \cdots + \frac{x^{(2n+1)\alpha}}{[\Gamma(2n\alpha+1)]^2} \right) + o(x^{2n+2})
\]

\[
L\text{Cos}_a(x) = 1 + \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + \frac{x^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} + o(x^{2n+1}).
\]

For the Laguerre-type fractional hyperbolic tangent function we pose, by definition

\[
L\text{Tan}_a(x) = \frac{L\text{Sin}_a(x)}{L\text{Cos}_a(x)} = d_1 x^\alpha + d_3 x^{3\alpha} + d_5 x^{5\alpha} + d_7 x^{7\alpha} + \cdots + d_{2n+1} x^{(2n+1)\alpha} + \cdots,
\]

Since

\[
\frac{x^\alpha}{[\Gamma(\alpha+1)]^2} + \frac{x^{3\alpha}}{[\Gamma(3\alpha+1)]^2} + \frac{x^{5\alpha}}{[\Gamma(5\alpha+1)]^2} + \frac{x^{(2n+1)\alpha}}{[\Gamma(2n\alpha+1)]^2} + \cdots =
\]

\[
\left( 1 + \frac{x^{2\alpha}}{[\Gamma(2\alpha+1)]^2} + \frac{x^{4\alpha}}{[\Gamma(4\alpha+1)]^2} + \cdots + \frac{x^{2\alpha}}{[\Gamma(2n\alpha+1)]^2} + \cdots \right) \cdot \left( d_1 x^\alpha + d_3 x^{3\alpha} + d_5 x^{5\alpha} + d_7 x^{7\alpha} + \cdots + d_{2n+1} x^{(2n+1)\alpha} + \cdots \right),
\]

\[
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\]
we find, for the first few values

\[ d_1 = \frac{1}{\Gamma(\alpha+1)} \],

\[ d_3 = -\frac{1}{\Gamma(\alpha+1)^2} \left[ \Gamma(2\alpha+1) \right]^2 + \frac{1}{\Gamma(3\alpha+1)^2} \],

\[ d_5 = -\frac{1}{\Gamma(2\alpha+1)^2} \left[ \Gamma(3\alpha+1) \right]^2 + \frac{1}{\Gamma(5\alpha+1)^2} \],

\[ d_7 = \frac{1}{\Gamma(2\alpha+1)^2} - \frac{1}{\Gamma(4\alpha+1)^2} - \frac{1}{\Gamma(2\alpha+1)^2} \left[ \Gamma(5\alpha+1) \right]^2 + \frac{1}{\Gamma(6\alpha+1)^2} + \frac{1}{\Gamma(7\alpha+1)^2} \].

5 Conclusion

In the framework of the fractional calculus and its applications it seems that the fractional exponential function occupies a prominent place, as it is used in the solution of differential problems such as the Bagley-Torvik problem and others related to it [2], as well as in the extension to the fractional case of polynomial and number functions of Number Theory [3, 14]. In this area we thought it appropriate to extend other classical elementary functions to the fractional case. The topic was analyzed in this paper by showing that formulas analogous to the traditional ones continue to hold in the fractional domain, even in the case of extensions to the Laguerrian case.

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