CLOSURE OPERATOR AND $\alpha$-IDEALS IN 0-DISTRIBUTIVE LATTICES

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Abstract

A closure operator on the lattice of all ideals of a bounded 0-distributive lattice is introduced. It is observed that the ideals which are closed with respect to this closure operator are $\alpha$-ideals in it and conversely.

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1 Introduction

As a generalization of the concept of distributive lattices on one hand and pseudocomplemented lattices on the other, 0-distributive lattices are introduced by Varlet [14]. Jayaram [6] defined and studied $\alpha$-ideals in 0-distributive lattices. Additional properties of $\alpha$-ideals in a 0-distributive lattice are obtained by Pawar et al. [8, 9]. Separation theorem for $\alpha$-ideals in a 0-distributive lattice is proved in [5]. In [8], the authors have obtained a characterisation of an $\alpha$-ideal using a closure operator on the lattice of all ideals of a 0-distributive lattice. The notion of closed filter in CI-algebra with some characteristic properties, is studied by Sabhapandit et al. [11]. Subbarayan [12] has made contributions in different aspects of 0-distributive lattices.

In this paper we introduce a new closure operator on the lattice of all ideals of a 0-distributive lattice and characterise $\alpha$-ideals in terms of the ideals which are closed with respect to this closure operator. Further it is observed that in a given 0-distributive lattice the ideals which are closed under this closure operator are the $\alpha$-ideals in it and conversely.

2 Preliminaries

Following are some basic concepts and results needed in the sequel from references. For other non-explicitly stated elementary notions please refer to [3]. A lattice $L$ with 0 is said to be 0-distributive if $a \land b = 0$ and $a \land c = 0$ imply $a \land (b \lor c) = 0$ for any $a, b, c$ in $L$. Throughout this paper $L$ will denote a bounded 0-distributive lattice unless otherwise specified. For a lattice $L$, $\mathcal{I}(L)$ denotes the set of all ideals of $L$. Then $(\mathcal{I}(L), \land, \lor)$ is a lattice where $I \land J = I \cap J$ and $I \lor J = (I \cup J)$, for any two ideals $I$ and $J$ of $L$. For any non-empty subset $A$ of $L$, define $A^* = \{x \in L : x \land a = 0, \text{ for each } a \in A\}$. By $A^{**}$ we mean $(A^*)^*$. Note that when $A = \{a\}$ then $A^* = (a)^*$ and also denoted by $(a)^*$. An ideal $I$ in $L$ is called an annihilator ideal if $I = A^*$, for a non-empty subset $A$ of $L$. Let $L$ and $L'$ denote bounded 0-distributive lattices and $f : L \to L'$ be a homomorphism. $f$ is called an annihilator preserving homomorphism if $f(A^*) = \{f(A)\}^*$ for any non-empty subset $A$ of $L$. An ideal $I$ of $L$ is called an $\alpha$-ideal if $\{x\}^{**} \subseteq I$ for each $x \in L$. Closure operator on $L$ is a mapping $f : L \to L$ satisfying the following conditions: (i) $x \leq f(x)$, (ii) $x \leq y \Rightarrow f(x) \leq f(y)$ and (iii) $f(f(x)) = f(x)$.

Result 2.1. (Varlet [14]). A lattice $L$ with 0 is 0-distributive if and only if $A^*$ is an ideal for any non-empty subset $A$ of $L$.

Following result can be proved easily.

Result 2.2. In a 0-distributive lattice $L$, for all $a, b, c \in L$ we have
By Result 2.2 (ii) we get 

\( i \) \( \{a\}^* \cap \{b\}^* = \{a \land b\}^* \)

\( ii \) \( \{a\}^* \cap \{b\}^* = \{a \lor b\}^* \)

\( iii \) \( \{a\}^* = \{b\}^* \Rightarrow \{a \land c\}^* = \{b \land c\}^* \)

Result 2.3. (Pawar and Mane [8]). In a bounded 0-distributive lattice \( L \) following statements are equivalent.

\( i \) \( \text{For } x, y \in L, \{x\}^* = \{y\}^*, x \in I \Rightarrow y \in I \)

\( ii \) \( I = U \{\{x\}^*: x \in I\} \)

\( iii \) \( \text{For } x, y \in L, h(x) = h(y), x \in I \Rightarrow y \in I, \text{ where } h(x) = \{M : M \text{ is a minimal prime ideal containing } x\} \)

\( iv \) \( I \text{ is an } \alpha \text{-ideal.} \)

Result 2.4. (Jayaram [5]). Let \( L \) be a 0-distributive lattice. Let \( I \) be an \( \alpha \)-ideal and \( S \) be a meet sub semi lattice of \( L \) such that \( I \cap S = \emptyset \). Then there exists a prime \( \alpha \)-ideal \( P \) in \( L \) containing \( I \) and disjoint with \( S \).

Result 2.5. (Pawar and Mane [8]). Every annihilator ideal in a 0-distributive lattice \( L \) is an \( \alpha \)-ideal.

Result 2.6. (Pawar and Khopade [9]). Let \( L \) and \( L' \) be any two bounded 0-distributive lattices and let \( f : L \rightarrow L' \) be an annihilator preserving onto homomorphism, Then

\( i \) \( \text{If } I \text{ is an } \alpha \text{-ideal of } L, \text{ then } f(I) \text{ is an } \alpha \text{-ideal of } L' \)

\( ii \) \( \text{If } L' \text{ is an } \alpha \text{-ideal of } L, \text{ then } f^{-1}(I') \text{ is an } \alpha \text{-ideal of } L \)

3 Closure operator

In this section we introduce a closure operator on \( \mathcal{I}(L) \).

Define \( B(L) = \{\{a\}^*: a \in L\} \). \( L \) being 0-distributive lattice, \( B(L) \subseteq \mathcal{I}(L) \) (by Result 2.1) but \( B(L) \) is not necessarily a sub lattice of the lattice \( \mathcal{I}(L) \). For this consider the following example.

\begin{figure}[h]
  \centering
  \begin{tikzpicture}
    \node (0) at (0,0) {0};
    \node (a) at (1,1) {a};
    \node (b) at (2,1) {b};
    \node (c) at (2.5,0) {c};
    \node (d) at (2.8,1) {d};
    \node (e) at (3.5,1) {e};
    \draw (0) -- (a);
    \draw (0) -- (b);
    \draw (a) -- (b);
    \draw (a) -- (c);
    \draw (b) -- (c);
    \draw (b) -- (d);
    \draw (c) -- (d);
    \draw (c) -- (e);
    \draw (d) -- (e);
  \end{tikzpicture}
  \caption{Figure 3.1}
\end{figure}

Example 3.1. Consider the bounded 0 - distributive lattice \( L = \{0, a, b, c, d, e, 1\} \) as shown by the Hasse Diagramme in Figure 3.1. Here \( \{a\}^* = \{0, a, b\} \) and \( \{c\}^* = \{0, c\} \). Hence \( \{a\}^* \lor \{c\}^* = \{0, a, b, c, d\} \notin B(L) \). Hence the set \( B(L) \) is a poset under set inclusion but need not be a sub lattice of the lattice \( \mathcal{I}(L) \).

For \( \{a\}^*, \{b\}^* \in B(L) \). Define \( \{a\}^* \cap \{b\}^* = \{a \land b\}^* \) and \( \{a\}^* \lor \{b\}^* = \{a \lor b\}^* \). Then we have

Theorem 3.1. \( (B(L), \lor, \land) \) is a bounded lattice.

\begin{proof}
  \begin{itemize}
    \item \( \{a \land b\}^* \) is the infimum of \( \{a\}^* \) and \( \{b\}^* \) in \( (B(L), \subseteq) \). To prove \( \{a \lor b\}^* \) is the supremum of \( \{a\}^* \) and \( \{b\}^* \) in \( (B(L), \subseteq). \{a \lor b\}^* \) is an upper bound of \( \{a\}^* \) and \( \{b\}^* \) in \( (B(L), \subseteq). \) Let \( \{c\}^* \) be any other upper bound of \( \{a\}^* \) and \( \{b\}^* \) in \( (B(L), \subseteq). \) Let \( t \in \{a \lor b\}^* \). By Result 2.2 (ii) we get \( i \cap \{a\}^* \subseteq \{b\}^* \subseteq \{b\}^* \). But as \( \{b\}^* \subseteq \{c\}^* \) we get \( \{t\} \cap \{a\}^* \subseteq \{c\}^* \). Thus \( \{t\} \cap \{a\}^* \subseteq \{c\}^* \subseteq \{0\} \). Hence \( \{t\} \cap \{c\}^* \subseteq \{0\} \). Again, as \( \{t\} \subseteq \{c\}^* \), we get \( \{t\} \subseteq \{c\}^* \). Therefore \( \{t\} \subseteq \{c\}^* \) which yields \( t \in \{c\}^* \). This shows that \( \{a \lor b\}^* \subseteq \{c\}^* \) and hence \( \{a \lor b\}^* \) is the supremum of \( \{a\}^* \) and \( \{b\}^* \) in \( (B(L), \subseteq). \)
    \end{itemize}
  \end{proof}

Corollary 3.1. \( \text{The lattice } (B(L), \lor, \land) \text{ is a homomorphic image of the lattice } L. \)
Proof. Define \( \theta : L \to \mathcal{B}(L) \) by \( \theta(a) = \{a\}^{**} \) for each \( a \in L \). Then \( \theta(a \wedge b) = \{a \wedge b\}^{**} = \{a\}^{**} \cap \{b\}^{**} = \theta(a) \cap \theta(b) \) and \( \theta(a \vee b) = \{a \vee b\}^{**} = \{a\}^{**} \cup \{b\}^{**} = \theta(a) \cup \theta(b) \) hold for all \( a, b \in L \). Hence \( \theta \) is a homomorphism. As \( \theta \) is onto, the result follows.

Remark 3.1. Note that the homomorphism \( \theta \) is not necessarily one-one. For this consider the \( 0 \)-distributive lattice in Example 3.1. Here for \( a \neq b \in L \) we have \( \{a\}^{**} = \{b\}^{**} \).

For any ideal \( I \) of \( L \), define \( \delta(I) = \{\{a\}^{**} : a \in I\} \) and for any ideal \( \mathcal{T} \) of \( \mathcal{B}(L) \), define \( \overrightarrow{\delta}(\mathcal{T}) = \{a \in L : \{a\}^{**} \in \mathcal{T}\} \). With these notations we prove

Theorem 3.2.

(i) \( \mathcal{T} \) is an ideal of \( \mathcal{B}(L) \), for any ideal \( \mathcal{T} \) of \( L \).
(ii) \( \delta(I) \) is an ideal of \( L \), for any ideal \( I \) of \( \mathcal{B}(L) \).
(iii) For any two ideals \( I \) and \( J \) of \( L \), \( I \subseteq J \Rightarrow \delta(I) \subseteq \delta(J) \).
(iv) For any two ideals \( I \) and \( J \) of \( \mathcal{B}(L) \), \( \delta(I) \subseteq \delta(J) \). Hence \( \delta(I) \) is an ideal of \( \mathcal{B}(L) \).

Proof. (i) Let \( I \) be any ideal of \( L \). As \( 0 \in I \), \( \{0\}^{**} = \{0\} \in \delta(I) \). Hence \( \delta(I) \) is non-empty. Let \( \{a\}^{**}, \{b\}^{**} \in \delta(I) \) such that \( \{a\}^{**} \subseteq \{b\}^{**} \) and \( \{b\}^{**} \in \delta(I) \). Then \( \{b\}^{**} = \{x\}^{**} \) for some \( x \in I \). Thus \( \{a\}^{**} = \{a\}^{**} \cap \{b\}^{**} = \{a\}^{**} \cap \{x\}^{**} = \{a \wedge x\}^{**} \). As \( a \wedge x \in I \), we get \( \{a\}^{**} \in \delta(I) \). Let \( \{a\}^{**}, \{b\}^{**} \in \delta(I) \). Therefore \( \{a\}^{**} = \{x\}^{**} \) and \( \{b\}^{**} = \{y\}^{**} \) for some \( x, y \in I \). Hence \( \{a\}^{**} \cup \{b\}^{**} = \{x\}^{**} \cup \{y\}^{**} = \{x \vee y\}^{**} \). As \( x \vee y \in I \), we get \( \{x \vee y\}^{**} \in \delta(I) \). Hence \( \{a\}^{**} \cup \{b\}^{**} \in \delta(I) \). Therefore \( \delta(I) \) is an ideal of \( \mathcal{B}(L) \).

(ii) Let \( I \) be any ideal of \( \mathcal{B}(L) \). \( \{0\}^{**} = \{0\} \in I \) implies \( 0 \in \overrightarrow{\delta}(I) \). Hence \( \overrightarrow{\delta}(I) \) is non-empty. Let \( a, b \in L \) such that \( a \leq b \) and \( b \in \overrightarrow{\delta}(I) \). Then \( \{a\}^{**} \subseteq \{b\}^{**} \) and \( \{b\}^{**} \in I \). \( I \) being an ideal we get \( \{a\}^{**} \in \overrightarrow{\delta}(I) \). But then \( a \in \overrightarrow{\delta}(I) \). Let \( a, b \in \overrightarrow{\delta}(I) \). Then \( \{a\}^{**}, \{b\}^{**} \in I \) implies \( \{a\}^{**} \cup \{b\}^{**} = \{a \vee b\}^{**} \in \overrightarrow{\delta}(I) \). Therefore \( \overrightarrow{\delta}(I) \) is an ideal of \( L \).

(iii) Let \( I \) and \( J \) be two ideals of \( L \) such that \( I \subseteq J \). Let \( \{a\}^{**} \in \delta(I) \). Then \( \{a\}^{**} = \{x\}^{**} \) for some \( x \in I \). But then, since \( I \subseteq J \), we get \( x \in J \). This in turn gives \( \{a\}^{**} = \{x\}^{**} \in \delta(J) \). Hence \( \delta(I) \subseteq \delta(J) \).

(iv) Let \( I \) and \( J \) be any two ideals of \( \mathcal{B}(L) \) such that \( I \subseteq J \). Let \( x \in \overrightarrow{\delta}(I) \). Then \( \{x\}^{**} \in I \) implies \( \{x\}^{**} \in \overrightarrow{\delta}(J) \). Hence \( \delta(I) \subseteq \delta(J) \) and the result follows.

As \( \delta(I) \) is an ideal of \( \mathcal{B}(L) \), for any ideal \( I \) of \( L \), we have the mapping \( \delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L)) \) is well defined, where \( \mathcal{I}(\mathcal{B}(L)) \) denotes the lattice of all ideals of the lattice \( \mathcal{B}(L) \). Further we have

Theorem 3.3. \( \delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L)) \) is a \( \{0, 1\} \) homomorphism.

Proof. Let \( I \) and \( J \) be any ideals in \( \mathcal{I}(L) \). \( \delta(I \cap J) \subseteq \delta(I) \cap \delta(J) \) (by Theorem 3.2 - (iii)). Let \( \{a\}^{**} \in \delta(I) \cap \delta(J) \). Then \( \{a\}^{**} \in \delta(I) \) implies \( \{a\}^{**} = \{i\}^{**} \) for some \( i \in I \) and \( \{a\}^{**} \in \delta(J) \) gives \( \{a\}^{**} = \{j\}^{**} \) for some \( j \in J \). Thus \( \{a\}^{**} = \{i\}^{**} \cap \{j\}^{**} = \{i \wedge j\}^{**} \). As \( i \wedge j \in I \cap J \), we get \( \{a\}^{**} \in \delta(I \cap J) \). This shows that \( \delta(I \cap J) \subseteq \delta(I) \cap \delta(J) \). Combining both the inclusions we get \( \delta(I \cap J) = \delta(I) \cap \delta(J) \).

Now, again by Theorem 3.2 - (iii), \( \delta(I \cup J) \subseteq \delta(I \cup J) \). Let \( \{a\}^{**} \in \delta(I \cup J) \). Hence \( \{a\}^{**} = \{y\}^{**} \) for some \( y \in I \cup J \). Therefore \( y \leq \bot \) for some \( i \in I \) and \( \{i \wedge j\}^{**} = \{i \wedge j\}^{**} \). This yields \( \{y\}^{**} \subseteq \{i \wedge j\}^{**} = \{i \wedge j\}^{**} \). Therefore \( \{a\}^{**} = \{y\}^{**} \in \delta(I \cup J) \). Hence \( \delta(I \cup J) \subseteq \delta(I \cup J) \). Combining both the inclusions we get \( \delta(I \cup J) = \delta(I \cup J) \).

This proves that \( \delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L)) \) is a homomorphism. Again \( \delta(0) = \{0\}^{**} = \{0\} \) and \( \delta(1) = \{1\}^{**} = \{L\} \), shows \( \delta \) is a \( \{0, 1\} \) homomorphism.

By Theorem 3.2, we get two mappings \( \delta : \mathcal{I}(L) \to \mathcal{I}(\mathcal{B}(L)) \) and \( \overrightarrow{\delta} : \mathcal{B}(L) \to \mathcal{I}(L) \). Hence \( \overrightarrow{\delta} \circ \delta : \mathcal{B}(L) \to \mathcal{I}(L) \) and \( \overrightarrow{\delta} \circ \delta : \mathcal{I}(L) \to \mathcal{I}(L) \). About these two mappings we have

Theorem 3.4.

(i) \( \overrightarrow{\delta} \circ \delta \) is an identity mapping on \( \mathcal{I}(\mathcal{B}(L)) \).
(ii) \( \overrightarrow{\delta} \circ \delta \) is a closure operator on \( \mathcal{I}(L) \).

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In this section we show that the ideals in any ideal of \( \alpha \) are defined mapping. Proof. (i) Let \( I \) be any ideal of \( B(L) \). Let \( \{x\}^* \in \delta \circ \delta (I) = \delta(\delta(I)) \). Hence \( \{x\}^* = \{y\}^* \) for some \( y \in \delta(I) \). But then \( \{y\}^* \in I \), which implies \( \{x\}^* \in I \). This gives \( \delta \circ \delta(I) \subseteq I \). Conversely, let \( \{x\}^* \in I \). Then \( x \in \delta(I) \) and consequently \( \{x\}^* \in \delta(\delta(I)) \) (since \( \delta(I) \) is an ideal of \( L \)). Hence \( I \subseteq \delta \circ \delta(I) \). From both the inclusions we get \( \delta \circ \delta(I) = I \). Hence \( \delta \circ \delta \) is an identity mapping on \( I(\delta(I)) \).

(ii) Let \( I \in I(L) \) and \( x \in I \). Then \( \{x\}^* \in \delta(I) \) and by Theorem 3.2 - (i), \( \delta(I) \) is an ideal of \( B(L) \), which yields \( x \in \delta(I) \). Hence \( I \subseteq \delta \circ \delta(I) \). (3.1)

Let \( I, J \in I(L) \) and \( I \subseteq J \). As \( \delta \) and \( \psi \) are isotope mappings (by Theorem 3.2), we get \( \psi_I \subseteq \delta \circ \delta(J) \).

Finally, let \( I \in I(L) \). As \( I \subseteq \delta \circ \delta(I) \), applying (3.2) we get \( \delta \circ \delta(I) \subseteq \delta \circ \delta(\delta \circ \delta(I)) \). Conversely, let \( x \in \delta \circ \delta \left( \delta \circ \delta(I) \right) \). Then \( \{x\}^* \in \delta \left( \delta \circ \delta(I) \right) \) implies \( \{x\}^* = \{y\}^* \) for some \( y \in \delta \circ \delta(I) \). But then \( \{y\}^* \in \delta(I) \), which implies \( \{x\}^* \in \delta(I) \). This gives \( x \in \delta \circ \delta(I) \). This proves \( \delta \circ \delta \left( \delta \circ \delta(I) \right) \subseteq \delta \circ \delta(I) \). Combining both the inclusions we get \( \delta \circ \delta \left( \delta \circ \delta(I) \right) = \delta \circ \delta(I) \). (3.3)

From (3.1), (3.2) and (3.3) we get \( \delta \circ \delta \) is a closure operator on \( I(L) \).

\[ \Box \]

Remark 3.2. The mapping \( \delta: I(L) \rightarrow I(B(L)) \) is a homomorphism follows from Theorem 3.3. Let \( I \) be any ideal of \( B(L) \). As \( \delta(I) \) is an ideal of \( L \) and \( \delta \circ \delta(I) = I \), we get the mapping \( \delta: I(L) \rightarrow I(B(L)) \) is onto. Hence the lattice \( I(B(L)) \) is a homomorphic image of the lattice \( I(L) \).

4 \( \alpha \) - ideals

In this section we show that the ideals in \( L \) which are closed with respect to the closure operator \( \delta \circ \delta \) defined on \( I(L) \) are \( \alpha \)-ideals in \( L \) and conversely. Let \( C(L) \) denote the set of all ideals in \( L \) which are closed with respect to the closure operator \( \delta \circ \delta \) defined on \( I(L) \). Thus \( C(L) = \{I \in I(L) : \delta \circ \delta(I) = I \} \). Obviously, \( \{0 \} \) and \( \{1 \} \) belong to \( C(L) \). Hence \( C(L) \) is a non-empty subset of \( I(L) \) but not necessarily a sublattice of the lattice \( I(L) \). This follows by the \( 0 \)-distributive lattice given in Example 3.1. Here \( C(L) = \{(0), \{b\}, \{c\}\} \) and \( \{b\} \lor \{c\} = \{d\} \). As \( \{d\} \notin C(L) \), the subset \( C(L) \) is not a sublattice of the lattice \( I(L) \). Though \( C(L) \) does not form a sublattice of the lattice \( I(L) \), it forms a lattice on its own. This we prove in the following theorem.

Theorem 4.1. \( (C(L), \wedge, \vee) \) is a bounded lattice where \( \wedge \) and \( \vee \) are defined by \( I \wedge J = I \cap J \) and \( I \vee J = \delta \circ \delta(I \lor J) \) for \( I, J \in C(L) \)

Proof. (i) First we prove that for \( I, J \in C(L) \), \( I \wedge J \in C(L) \). As \( \delta \) and \( \delta \) are isotope mappings, we get \( \delta \circ \delta \) is also isotope. Hence \( \delta \circ \delta(I \wedge J) = \delta \circ \delta(I \cap J) \). Let \( x \in \delta \circ \delta(I \wedge J) \). Then \( \{x\}^* \in \delta(I \wedge J) \). This gives \( x \in \delta \circ \delta(I \wedge J) \). Hence \( \delta \circ \delta(I \wedge J) \subseteq \delta \circ \delta(I \wedge J) \). Combining both the inclusions we get \( \delta \circ \delta(I \wedge J) = \delta \circ \delta(I \wedge J) \). For \( I, J \in C(L) \), the supremum of \( I \) and \( J \) in \( C(L) \) is \( I \lor J \). Hence \( I \wedge J = I \cap J \).

(ii) First note that, by Theorem 3.4 - (ii), \( \delta \circ \delta(I) \in C(L) \), for any ideal \( I \) of \( L \). Let \( I, J \in C(L) \). Then \( I = \delta \circ \delta(I) \subseteq \delta \circ \delta(I \lor J) \) and \( J = \delta \circ \delta(J) \subseteq \delta \circ \delta(I \lor J) \) (since \( \delta \circ \delta \) is isotope). Thus \( \delta \circ \delta(I \lor J) \) is an upper bound of \( I \) and \( J \) in \( C(L) \). Let \( K \in C(L) \), such that \( I \subseteq K \) and \( J \subseteq K \). Then \( I \lor J \subseteq K \) implies \( \delta \circ \delta(I \lor J) \subseteq \delta \circ \delta(K) = K \) (since \( K \in C(L) \)). This shows that \( \delta \circ \delta(I \lor J) \) is the supremum of \( I \) and \( J \) in \( C(L) \). Thus \( I \lor J = \delta \circ \delta(I \lor J) \). As \( \{0\} \in C(L) \) and \( \{L\} \in C(L) \), \( (C(L), \wedge, \vee) \) is a bounded lattice.

We know that the lattice \( I(B(L)) \) is a homomorphic image of the lattice \( I(L) \) (see Remark 3.2). But interestingly we have

Theorem 4.2. The lattice \( C(L) \) is isomorphic with the lattice \( I(B(L)) \).

Proof. Define the mapping \( \psi: C(L) \rightarrow I(B(L)) \) by \( \psi(I) = \delta(I) \) for each \( I \in C(L) \), which is clearly a well defined mapping.

(i) Let \( \psi(I) = \psi(J) \) for \( I, J \in C(L) \). Then we have \( \delta(I) = \delta(J) \). Therefore \( \delta \circ \delta(I) = \delta \circ \delta(J) \) which implies \( I = J \) (since \( I, J \in C(L) \)). This shows that \( \psi \) is one-one.
(ii) Let $I$ be any ideal of $B(L)$. Then $\overrightarrow{\delta}(I)$ is an ideal of $L$ (by Theorem 3.2 - (ii)) and $\delta \circ \overrightarrow{\delta}(I) = I$ (by Theorem 3.4 - (i)). Then $\overrightarrow{\delta} \circ \delta(\overrightarrow{\delta}(I)) = \overrightarrow{\delta}(\delta(\overrightarrow{\delta}(I))) = \overrightarrow{\delta}(\delta \circ \overrightarrow{\delta}(I)) = \overrightarrow{\delta}(I)$. This shows that $\overrightarrow{\delta}(I) \subseteq C(L)$. As $\psi(\overrightarrow{\delta}(I)) = \delta(\overrightarrow{\delta}(I)) = \delta \circ \overrightarrow{\delta}(I) = I$, we get $\psi$ is onto.

(iii) Let $I, J \in C(L)$. Then by definition of $\psi$ and by Theorem 3.3 we get $\psi(I \cap J) = \psi(I) \cap \psi(J) = \delta(I) \cap \delta(J) = \psi(I \cap J)$. And by definition of $\psi$ in $C(L)$ we get $\psi(I \cup J) = \delta(I \cup J) = \delta(\overrightarrow{\delta}(I \cup J)) = \delta(I) \cup \delta(J) = \psi(I) \cup \psi(J)$. This proves that $\psi$ is a homomorphism. From (i) - (iii) we get $\psi$ is an isomorphism.

Following theorem gives a necessary and sufficient conditions for an ideal $I$ of $L$ to be a member of $C(L)$.

**Theorem 4.3.** For any ideal $I$ of $L$, following statements are equivalent.

(i) $I \in C(L)$.

(ii) For $x, y \in L$, $\{x\}^{**} \subseteq \{y\}^{**}$, $x \in I \Rightarrow y \in I$.

(iii) For $x, y \in L$, $\{x\}^{*} \subseteq \{y\}^{*}$, $x \in I \Rightarrow y \in I$.

(iv) $I = \bigcup \{\{x\}^{**} : x \in I\}$.

(v) For $x, y \in L$, $h(x) = h(y)$, $x \in I \Rightarrow y \in I$.

where $h(x) = \{M : M$ is a minimal prime ideal containing $x\}$.

(vi) $I$ is an $\alpha$-ideal.

**Proof.** The equivalence of the statements (iii) to (vi) follows by Result 2.3.

(ii) $\Leftrightarrow$ (iii): As $\{x\}^{**} \subseteq \{y\}^{**}$ implies $\{x\}^{*} = \{y\}^{*}$ for any $x, y \in L$, the equivalence follows.

(i) $\Rightarrow$ (ii): Let $I \in C(L)$. Let $x, y \in L$ such that $\{x\}^{**} = \{y\}^{**}$ and $x \in I$. As $x \in I$, we have $\{x\}^{**} \subseteq \delta(I)$. But then, by assumption, we get $\{y\}^{**} \subseteq \delta(I)$. This gives $y \in \overrightarrow{\delta}\delta(I)$. Again by assumption that $I \in C(L)$, we get $y \in I$. Thus the implication follows.

(ii) $\Rightarrow$ (i): Let $I \in \mathcal{I}(L)$ satisfying condition in (ii). By Theorem 3.4, we have $I \subseteq \overrightarrow{\delta} \circ \delta(I)$. To prove $\overrightarrow{\delta} \circ \delta(I) \subseteq I$. On contrary assume that $\overrightarrow{\delta} \circ \delta(I) \not\subseteq I$. Then there exists $x \in \overrightarrow{\delta} \circ \delta(I)$ such that $x \notin I$. Then $\{x\}^{**} \subseteq \delta(I)$ which implies $\{x\}^{**} = \{y\}^{**}$ for some $y \in I$. But then, by assumption, $x \in I$: a contradiction. Hence $\overrightarrow{\delta} \circ \delta(I) \subseteq I$. Combining both the inclusions, we get $\overrightarrow{\delta} \circ \delta(I) = I$. Hence $I \in C(L)$ and the implication follows. Hence all the statements are equivalent.

Using the property that $I \in C(L)$ if and only if $I$ is an $\alpha$-ideal, proved in above theorem, we get

**Corollary 4.1.** $[a] \in C(L)$ if and only if $[a] = \{a\}^{**}$ for any $a \in L$.

**Proof.** Let $[a] \in C(L)$. Then by Theorem 4.3, $[a]$ is an $\alpha$-ideal of $L$. This gives $\{a\}^{**} \subseteq [a]$ (by definition of $\alpha$-ideal). As we obviously have $[a] \subseteq \{a\}^{**}$, the proof of if part follows. Conversely, suppose $[a] = \{a\}^{**}$. We know that every annihilator ideal is an $\alpha$-ideal, therefore $\{a\}^{**} = [a]$ is an $\alpha$-ideal. Thus again by Theorem 4.3, we get $[a] \in C(L)$.

$I^{*} \in C(L)$ for any ideal $I$ in $L$, because $I^{*}$ is an $\alpha$-ideal of $L$ (see Result 2.5). Hence we have

**Corollary 4.2.** The lattice $(C(L), \cap)$ is a pseudo complemented lattice.

Define $A_{0}(L) = \{\{x\}^{*} : x \in L\}$. Then $(A_{0}(L), \wedge, \vee)$ is a lattice, where $\{x\}^{*} \wedge \{y\}^{*} = \{x \wedge y\}^{*}$ and $\{x\}^{*} \vee \{y\}^{*} = \{x \vee y\}^{*}$. This lattice is called as a lattice of all annules of $L$. For any ideal $I$ in $L$, the set $\{\{x\}^{*} : x \in I\}$ is a filter in $A_{0}(L)$ and for any filter $F$ in $A_{0}(L)$, the set $\{x \in L : \{x\}^{*} \in F\}$ is an ideal of $L$. Let $F(A_{0}(L))$ denote the lattice of all filters in $A_{0}(L)$. Then the maps $\alpha : \mathcal{I}(L) \to \mathcal{F}(A_{0}(L))$ defined by $\alpha(I) = \{\{x\}^{*} : x \in I\}$ and $\beta : \mathcal{F}(A_{0}(L)) \to \mathcal{I}(L)$ defined by $\beta(F) = \{x \in L : \{x\}^{*} \in F\}$ are well defined isomote maps.

We need the following results from [8]:

**Lemma 4.1.** ([8], Theorem 9). The map $\beta \circ \alpha : \mathcal{I}(L) \to \mathcal{I}(L)$ is a closure operator on $\mathcal{I}(L)$.

**Lemma 4.2.** ([8] Theorem 10).

For any ideal $I$ in $L$, following statements are equivalent.

(i) $I$ is an $\alpha$-ideal.
(ii) \( \beta \circ \alpha (I) = I \).

Using above two lemmas and Theorem 4.3 we get
\[
\mathcal{C}(L) = \{ I \in \mathcal{I}(L) : \delta \circ \delta (I) = I \} = \{ I \in \mathcal{I}(L) : \beta \circ \alpha (I) = I \}.
\]
Hence an ideal \( I \) in \( L \) is closed with respect to the closure operator \( \delta \circ \delta \) if and only if it is closed with respect to the closure operator \( \beta \circ \alpha \) defined on \( \mathcal{I}(L) \). Thus we have

**Corollary 4.3.** For any ideal \( I \) of \( L \), \( \delta \circ \delta (I) = I \) if and only if \( \beta \circ \alpha (I) = I \).

Let \( I \) be an ideal of \( L \). If there exists a prime ideal \( P \) of \( L \) such that \( I \subseteq P \) and \( P \) is minimal in the class of all prime ideals containing \( I \), then \( P \) is called a prime ideal belonging to \( I \). We know that any prime ideal of \( L \) need not be an \( \alpha \)-ideal. For this consider the lattice \( L = \{0, a, b, c, d, e, f, g, h, i\} \) whose Hasse diagram is as in Figure 3.1. The ideal \( (e) \) is a prime ideal but not an \( \alpha \)-ideal. For, \( d \in (e) \) but \( (d)^{**} = L \not\subseteq (e) \).

In the following theorem we show that a prime ideal belonging to an \( \alpha \)-ideal is an \( \alpha \)-ideal.

**Theorem 4.4.** Let \( I \) be an \( \alpha \)-ideal of \( L \). Let \( P \) be a prime ideal belonging to \( I \), then \( P \) is an \( \alpha \) ideal.

**Proof.** Suppose \( P \) is not an \( \alpha \)-ideal. Hence there exist \( x, y \) in \( L \) such that \( \{x\}^{**} = \{y\}^{**}, x \in P \) but \( y \notin P \) (see Theorem 4.3). Consider the filter \( F = (L \setminus P) \cup \{x \land y\} \). Claim that \( F \cap I = \emptyset \). Let \( F \cap I \neq \emptyset \). Select \( a \in F \cap I \). Then \( a \in F \) implies \( a \geq r \land s \) for some \( r \in (L \setminus P) \) and \( s \geq x \land y \). But then \( a \geq r \land x \land y \) and therefore \( r \land x \land y \in I \) (as \( a \in I \)). Since \( \{x\}^{**} = \{y\}^{**} \), using the Result 2.2, we get \( \{r \land x\}^{**} = \{r \land y\}^{**} \)

and hence \( \{r \land x \land y\}^{**} = \{r \land y\}^{**} \). Since \( r \land x \land y \in I \) and \( I \) is an \( \alpha \)-ideal, by Theorem 4.3, we get \( r \land y \in I \). Hence \( r \land y \in P \) (since \( I \subseteq P \)). Now \( r \land y \in P, P \) is a prime ideal and \( r \notin P \) imply \( y \in P \); which contradicts our assumption. Hence we must have \( F \cap I = \emptyset \). Therefore, by Result 2.4, there exists a prime ideal \( Q \) containing \( I \) and disjoint with \( F \). Thus \( Q \subseteq P \). Moreover \( F \cap Q = \emptyset \) and \( x \land y \notin F \) implies \( x \land y \notin Q \). Hence \( Q \neq P \) (since \( x \in P \Rightarrow x \land y \in P \)). But this contradicts to the fact that \( P \) is minimal in the class of all prime ideals containing \( I \). Hence we must have \( P \) is an \( \alpha \)-ideal.

Making an appeal to Theorem 4.1, Theorem 4.3 and Result 2.6, we establish

**Corollary 4.4.** Let \( L \) and \( L' \) be bounded 0-distributive lattices and let \( f : L \rightarrow L' \) be an annihilator preserving onto homomorphism. Then we have

(i) If \( I \in \mathcal{C}(L) \), then \( f(I) \in \mathcal{C}(L') \).

(ii) If \( I' \in \mathcal{C}(L') \), then \( f^{-1}(I') \in \mathcal{C}(L) \).

5 Conclusion

The present investigation provides a new way to define closure operator on the lattice of all ideals of a bounded 0-distributive lattice. Moreover the ideals closed with respect to this closure operator are \( \alpha \)-ideals. Therefore this work will motivate and useful to study more properties of \( \alpha \)-ideals.

**References**


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