Approximation of Functions in the Generalized Zygmund Class using Product Means of Derived Fourier Series

By
Aradhana Dutt Jauhari\(^1\) and Santosh Kumar\(^2\)

\(^1\)Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Gautam Buddha Nagar, Uttar Pradesh, India-203201
\(^2\)Department of Mathematics, School of Physical Sciences, North Eastern Hill University, Shillong, Meghalaya, India-793022

Email: draradhana27@gmail.com, drsengar2002@gmail.com

(Received: April 20, 2024; In format: April 30, 2024; Revised: June 05, 2024; Accepted: June 10, 2024)

DOI: https://doi.org/10.58250/jnanabha.2024.54136

Abstract

In this paper, we investigate the potential of using the product means of the derived Fourier series to approximate functions within the generalized Zygmund class. The product means offer a flexible and powerful tool for approximation due to their ability to capture intricate local behaviors of functions. We establish convergence theorems for the product means of the derived Fourier series, providing conditions under which these approximations converge uniformly to the original function in the generalized Zygmund class.

2020 Mathematical Sciences Classification: 41A10, 41A25, 42B05, 42A50, 40G05.

Keywords and Phrases: Fourier Series, Derived Fourier Series, Matrix (\(\Delta\)) means, Matrix-Euler (\(\Delta E_q\)) means, Generalised Zygmund Class.

1 Introduction

In recent times, researchers have displayed significant interest in exploring the properties and behavior of derived Fourier series using various types of means, including both single and product means. In this vein, Nigam \([7,8,9]\) and Nigam et al. \([11,13,14]\) worked on derived Fourier series. The DoA (Degree of Approximation) by \((E,1)(C,2)\) product summability transform in a generalized weighted class was also explored by Sharma and Malik \([16]\). The extent to which functions in a function space can be approximated by using single or product means of their corresponding derived Fourier series is a critical area that was left unexplored despite these important contributions. Working on the \((N,p,q)\) \((C,1)\) summability of derived Fourier series were discussed by Lal and Shireen \([3]\) as well as Lal and Yadava \([4]\). On product summability, Pradhan et al. \([15]\) and Mishra et al. \([6]\) have also worked. Moreover, the comparison of summability with Cesaro summability was generalized by Mishra and Bhagatbandh \([5]\).

We take \(\sum u_n\) be an infinite series with the \(n^{th}\) partial sum \(s_n = \sum_{\omega=0}^{n} u_\omega\).

Let the Fourier series of \(l\) is provided by,

\[ l(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \]

with \(n^{th}\) partial sum \(s_n (l; y)\) Conjugate Fourier series of (1.1) is

\[ s_l (l; x) = \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \]

The derived series of (1.1) is

\[ l'(x) = \sum_{n=1}^{\infty} n (b_n \cos nx - a_n \sin nx). \]

Definition 1.1. Let \(T = (a_{n,r})\) be an infinite triangular matrix satisfying the (Silverman-Toeplitz) condition of regularity \([14]\), i.e.,

\[ \sum_{r=0}^{n} a_{n,r} = 1 \text{ as } n \to \infty. \]
\( (a_{n,r} = 0) \) for \( r > n \).

\( \sum_{n=0}^{\infty} |a_{n,r}| \leq M, \) (a finite constant).

The transformation

\[
t_n^\Delta = \sum_{r=0}^{n} a_{n,r} s_r = \sum_{r=0}^{n} a_{n,n-r} s_{n-r}
\]

(1.4)

defines the sequence \( t_n^\Delta \) of triangular matrix means of the sequence \( \{s_n\} \), generated by the sequence of coefficients \((a_{n,r})\). If \( t_n^\Delta \to s \) as \( n \to \infty \), then the series \( \sum_{n=0}^{\infty} u_n \) is summable to \( s \) by triangular matrix \( \Delta \) matrix [19].

**Definition 1.2** (8). The infinite series \( \sum u_n \) is said to be summable \((E,q)\) to a definite number \( s \) if

\[
E_q^n = \frac{1}{(1+q)^n} \sum_{r=0}^{n} \binom{n}{r} q^{n-r} s_r.
\]

(1.5)

If \( E_q^n \to s \) as \( n \to \infty \) then \( \sum u_n \) is summable to \( s \) by \((E,q)\) summability.

**Definition 1.3.** The \((E,q)\) transform is defined by the triangular matrix \( \Delta \)-transform of \( E_q \) transform and \( t_n^{E_q} \) of the partial sums \( s_n \) of the series \( \sum_{n=0}^{\infty} u_n \) and given by

\[
t_n^{E_q} = \sum_{r=0}^{n} a_{n,r} E_q^r
\]

\[
= \sum_{r=0}^{n} a_{n,r} \frac{1}{(1+q)^r} \sum_{\omega=0}^{r} \binom{r}{\omega} q^{r-\omega} s_\omega.
\]

(1.6)

If \( t_n^{E_q} \to s \) as \( n \to \infty \), then the series \( \sum_{n=0}^{\infty} u_n \) is said to be summable \((\Delta E_q)\) to \( s \).

**Remark 1.1.** \( s_n \to s \) \( E_q^n = \frac{1}{(1+q)^n} \sum_{\omega=0}^{n} \binom{n}{\omega} s_\omega \to s \) as \( n \to \infty \), \( E_q \) method is regular, \( \Rightarrow \Delta E_q \) Method is regular.

**Remark 1.2.** \((H, \frac{1}{n+1}) \) \((E_q)\) means, when \( a_{n,r} = \frac{1}{(n-r+1)\log(n+1)} \).

**Remark 1.3.** \((N,p_n) \) \((E_q)\) means, when \( a_{n,r} = \frac{p_n-p_{r-1}}{r}, \) where \( P_n = \sum_{r=0}^{n} p_r \neq 0 \).

**Remark 1.4.** \((N,p_n,q_n) \) \((E_q)\) means, when \( a_{n,r} = \frac{p_n-q_{n-r}}{r}, \) where \( R_n = \sum_{r=0}^{n} p_rq_{n-r} \neq 0 \).

The Zygmund modulus of continuity of \( l(x) \) be:

\[
m(l,r) = \sup_{0 \leq x \leq R_n} |l(x + \omega) + l(x - \omega)|,
\]

where \( R_n = \sum_{r=0}^{n} p_rq_{n-r} \neq 0 \).

Let \( B \) be a Banach space of all continuous, supreme-normalized, \( 2\pi \)-periodic functions defined over the range \([0,2\pi]\). Clearly,

\[
Z_{(a)} = \{ l \in B : |l(x + \omega) + l(x - \omega)| = O(\omega^n), 0 < \alpha \leq 1 \}
\]

is a Banach space with the given definition for the norm \( \|l\|_{(a)} \), \( \|l\|_{(a)} = \sup_{0 \leq x \leq R_n} |l(x + \omega) + l(x - \omega)| \).

For \( l \in L_k[0,2\pi], \) \( k \geq 1 \), the integral Zygmund modulus of continuity is defined as

\[
m_k(l;r) = \sup_{0 \leq \omega \leq r} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |l(x + \omega) + l(x - \omega)|^k dx \right\}^{\frac{1}{k}}
\]

and for \( l \in B, k = \infty, \) \( m_{\infty}(l,r) = \sup_{0 \leq \omega \leq r} \max_{x} |l(x + \omega) + l(x - \omega)| \).

Clearly, \( m_k(l,r) \to 0 \) as \( k \to 0 \).

Let us define the space, \( z_{(a)} \), \( k \leq 1 \).

Clearly, \( \|l\|_{(a),k} = \|l\|_k + \sup_{0 \leq \omega \leq r} \frac{|l(\omega) + l(-\omega)|}{m(\omega)} \).

Let \( Z^{(m)} = \{ l \in B : |l(x + \omega) + l(x - \omega)| = O(m(\omega)) \} \) where \( m \) is a Zygmund modulus of continuity satisfying

- \( m(0) = 0 \),
- \( m(\omega_1 + \omega_2) \leq m(\omega_1) + m(\omega_2) \).

Let \( m : [0,2\pi] \to R \) a function with \( m(\omega) > 0 \) for \( 0 \leq \omega \leq 2\pi \) and \( \lim_{\omega \to 0} m(\omega) = m(0) = 0 \).

Define, \( Z_k^{(m)} = \{ l \in L_k : 1 \leq k < \infty, \sup_{0 \leq \omega \leq r} \frac{|l(\omega) + l(-\omega)|_k}{m(\omega)} \} \),

where, \( \|l\|_k^{(m)} = \|l\|_k + \sup_{0 \leq \omega \leq r} \frac{|l(\omega) + l(-\omega)|_k}{m(\omega)}, k \geq 1 \).

Clearly, \( \|l\|_k^{(m)} \) is a norm of \( Z_k^{(m)} \).
Also, $Z^{(m)}_k$ is complete since $L_k, (k \geq 1)$ is complete. So, $Z^{(m)}_k$ is a Banach space under norm of $Z^{(m)}_k$.

Let $m(\omega)$ and $\mu(\omega)$ represents the Zygmund moduli of continuity s.t. $\frac{m(\omega)}{\mu(\omega)}$ is positive and non-decreasing, $\|l\|_n^2 \leq \max \left(1, \frac{m(2n)}{m(2\pi)}\right) \|l\|_k^m \leq \infty$. Clearly, $Z^{(m)}_k \subseteq Z^{(\mu)}_k \subseteq L_k, (k \geq 1)$.

We use following notations in the present work,

\( \varphi(x,t) = l(x + t) + l(x - t) - 2l(x), \)
\( j(x,t) = l(x + t) - l(x - t) - 2\ell(x), \)
\( \Psi(x,t) = \int_0^t |\varphi(u)| du, \)
\( N(x,t) = \int_0^t |dl(u)|, \)
\( \Delta a_{j,k} = a_{j,k} - a_{j,k+1}, 0 \leq k \leq j - 1, \)
\( P_j^{\Delta E_\alpha}(x) = \frac{1}{2\pi} \sum_{r=0}^n a_{n,r} \frac{1}{(1+q)^r} \sum_{\omega=0}^r \left( \frac{r}{\omega} \right) q^{r-\omega} \frac{\sin(\omega + \frac{1}{2})t}{\sin \frac{\omega}{2} t}. \)

**Remark 1.5.**

(i) If $m(t) = t^\alpha$ then $Z^{(m)}$ reduces to $Z_\alpha$ class.
(ii) If $m(t) = t^\alpha$ in $Z^{(m)}_k$, it reduces to $Z_{\alpha,k}$ class.
(iii) $k \to \infty$ then $Z^{(m)}_k$ class reduces to $Z^m$ class.
(iv) If $m(t) = t^\alpha$ then $Z^{(m)}$ reduces to $Z_\alpha$ class.
(v) If $m(t) = t^\alpha$ in $Z^{(m)}_k$, it reduces to $Z_{\alpha,k}$ class.
(vi) If we take $k \to \infty$ then $Z^{(m)}_k$ class reduces to $Z^m$ class.

**Example 1.1** ([11]). We consider

\[ 1 - 20 \sum_{n=1}^{\infty} (-19)^{n-1}. \]  

The $n^{th}$ partial sum of the series (1.7) is given by, $s_n = (-19)^n$.

We take

\[ s_n = \frac{1}{10^n} \begin{pmatrix} n \\ r \end{pmatrix} 9^{n-r} \text{then} \]
\[ t_n^\Delta = a_{n,0} s_0 + a_{n,1} s_1 + ... + a_{n,n} s_n \]
\[ = \frac{1}{10^n} \left[ \begin{pmatrix} n \\ 0 \end{pmatrix} \left( 9 \right)^{n-1} - \begin{pmatrix} n \\ 1 \end{pmatrix} \left( 9 \right)^{n-1} + 19 + ... + (-19)^n \begin{pmatrix} n \\ n \end{pmatrix} \right] \]
\[ = \frac{1}{10^n} (-10^n) = (-1)^n, \]
\[ (-1)^n = \begin{cases} 1, & \text{when } n \text{ is even number} \\ -1, & \text{when } n \text{ is odd number}. \end{cases} \]  

The series (1.7) is not summable by the $T$ mean, and the series (1.7) is not summable by the $\Delta E_q$ means for $q = 1$, but the series (1.7) and (1.8) are both summable by the $\Delta E_q$ mean for $q = 1$. We can see that product means are superior to individual single methods in terms of effectiveness.

**2 Preliminaries**

The following lemmas are important to establish the main result of this paper.

**Lemma 2.1.** $0 \leq t \leq \frac{1}{7+1}, \sin \left( \frac{2t}{7} \right) \geq \frac{2t}{7}, \sin(jt) \leq jt$ and $\delta > 1$, $P_j^{\Delta E_\alpha}(x) = O(j+1)$.

**Proof.**

\[ P_j^{\Delta E_\alpha}(x) = \frac{1}{2\pi} \sum_{r=0}^j a_{j,r} \frac{1}{(1+q)^r} \sum_{\omega=0}^r \left( \frac{r}{\omega} \right) q^{r-\omega} \frac{\sin(\omega + \frac{1}{2})t}{\sin \frac{\omega}{2} t}, \]
\[ = \frac{1}{2\pi} \sum_{r=0}^j a_{j,r} \frac{1}{(1+q)^r} \sum_{\omega=0}^r \left( \frac{r}{\omega} \right) q^{r-\omega} \frac{(2\omega + \frac{1}{2})t}{\frac{\pi}{t}}. \]  

297
\[
\begin{align*}
&= \frac{1}{4} \sum_{r=0}^{j} \sum_{\omega=0}^{r} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega}(2\omega + 1), \\
&\leq \frac{1}{4} \sum_{r=0}^{j} \sum_{\omega=0}^{r} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega} \frac{1}{\pi}, \\
&\leq \frac{1}{2t} \sum_{r=0}^{j} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega}, \\
&= \frac{1}{2t} \sum_{r=0}^{j} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega}, \\
&= \frac{1}{2t} \sum_{r=0}^{j} a_{j,r} (2\omega + 1), \\
&= O \left( \frac{1}{t} \right).
\end{align*}
\]

Lemma 2.2. If the limit of \( t \) varies from \( \frac{1}{j+1} \leq t \leq \pi \) Lemma 2.2 gives the following result. 
\( \frac{1}{j+1} \leq t \leq \pi, \sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi}, \sin (jt) \leq 1 \) and \( P_{j}^{\Delta E^q}(x) = O \left( \frac{1}{t} \right) \).

Proof.

\[
\begin{align*}
P_{j}^{\Delta E^q}(x) &= \frac{1}{2\pi} \sum_{r=0}^{j} \sum_{\omega=0}^{r} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega} \frac{\sin (\omega + \frac{1}{2}) t}{\sin \frac{t}{2}}, \\
&\leq \frac{1}{2\pi} \sum_{r=0}^{j} \sum_{\omega=0}^{r} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega} \frac{1}{\pi}, \\
&= \frac{1}{2t} \sum_{r=0}^{j} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega}, \\
&= \frac{1}{2t} \sum_{r=0}^{j} a_{j,r} \left( \frac{1}{1+q} \right)^r \frac{r \omega}{\omega^r} q^{r-\omega}, \\
&= O \left( \frac{1}{t} \right).
\end{align*}
\]

Condition: Let \( 0 < \beta < \alpha < 1 \), if \( m(t) = m^\alpha \) and \( z(t) = t^\alpha \), then \( \frac{m(t)}{z(t)} \) is non-decreasing, while \( \frac{m(t)}{z(t)} \) is a non-increasing function of \( t \).

Lemma 2.3. If \( l \in Z_k^{(m)} \) then for \( 0 < t \leq \pi \),

- \( \| \varphi(\cdot, t) \|_k = O \left( m(t) \right) \)
- If \( m(t) \) and \( z(t) \) are defined in above condition then,
  \( \| \varphi(\cdot + a, t) - \varphi(\cdot - a, t) - 2t \varphi(t) \|_k = O \left( \frac{m(t)}{z(t)} \right) \).

3 Main Results

Many mathematicians worked for Fourier series as well as conjugate Fourier series but by using matrix-Euler means on its derived Fourier series, this work fills a large vacuum in the literature by illuminating the approximation characteristics of functions belonging to the generalized Zygmund class. This contribution improves our knowledge of approximation methods and how to use them to investigate function spaces. Nigam et al. [11] worked on generalized Zygmund class approximation of function in 2021. We are now extending our findings in a new dimension.
Theorem 3.1. A 2\pi-periodic function \( l \) belonging to the class \( Z_k^{(m)} \), \( k \geq 1 \), then the best error estimate of \( l \) by the \( \Delta E^q \) method of its Fourier series is given by

\[
\|T_j(.)\|_k^2 = O \left( \frac{m \left( \frac{1}{j+1} \right)}{z \left( \frac{1}{j+1} \right)} \right) + O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t) \cdot t(z(t))^2}{t^2} dt \right),
\]

where \( m(t) \) and \( z(t) \) denote the second order moduli of continuity such that \( \frac{m(t)}{z(t)} \) is positive and non-decreasing in \( t \).

Proof. Let \( s_r(l; x) \) denoted the \( r^{th} \) partial sum of DFS (derived Fourier series) [17], we have

\[
s_r(l; x) - l(x) = \frac{1}{2\pi} \int_0^\pi \varphi(x,t) \frac{\sin(r + \frac{1}{2})t}{\sin \frac{t}{q}} dt,
\]

\[
T_j^\Delta E^q - l(x) = \frac{1}{2\pi} \int_0^\pi \varphi(x,t) \sum_{r=0}^j a_{j+r} \left( \frac{1}{1+q} \right)^r \sum_{\omega=0}^r \left( \frac{r}{\omega} \right) q^{r-\omega} \frac{\sin(r + \frac{1}{2})t}{\sin \frac{t}{q}} dt.
\]

Let

\[
T_j(x) = \int_0^\pi \varphi(x,t)P_j^\Delta E^q(t) dt.
\]

Then

\[
T_j(x+a) + T_j(x-a) - 2T_j(x) = \int_0^\pi [\varphi(x+a,t) + \varphi(x-a,t) - 2\varphi(x,t)]P_j^\Delta E^q(t) dt
\]

Using the GMI [2], we get

\[
\|T_j(\cdot + a) + T_j(\cdot - a) - 2T_j(.)\|_k \leq \int_0^{\pi} [\varphi(\cdot + a,t) + \varphi(\cdot - a,t) - 2\varphi(\cdot,t)]|P_j^\Delta E^q(t)| dt
\]

\[
= \int_0^{\pi} \|\varphi(\cdot + a,t) + \varphi(\cdot - a,t) - 2\varphi(\cdot,t)\|_k |P_j^\Delta E^q(t)| dt + \int_{\frac{1}{j+1}}^{\pi} \|\varphi(\cdot + a,t) + \varphi(\cdot - a,t) - 2\varphi(\cdot,t)\|_k |P_j^\Delta E^q(t)| dt.
\]

\[
= I_1 + I_2. (3.5)
\]

\[
I_1 = \int_0^{\pi} \|\varphi(\cdot + a,t) + \varphi(\cdot - a,t) - 2\varphi(\cdot,t)\|_k |P_j^\Delta E^q(t)| dt
\]

Next, by using second MVT of integral

\[
= \int_0^{\pi} O \left( Z(|a|) \frac{m(t)}{z(t)} \right) (j+1) dt
\]

\[
\leq O \left( (j+1) Z(|a|) \int_0^{\pi} \frac{m(t)}{z(t)} dt \right).
\]

\[
I_1 = O \left( Z(|a|) \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right). (3.6)
\]

Next, by Lemmas 2.2 and Remark 1.5 (ii), we obtain

\[
I_2 = \int_{\frac{1}{j+1}}^{\pi} \|\varphi(\cdot + a,t) + \varphi(\cdot - a,t) - 2\varphi(\cdot,t)\|_k |P_j^\Delta E^q(t)| dt
\]

\[
\leq O \left( \int_{\frac{1}{j+1}}^{\pi} Z(|a|) \frac{m(t)}{z(t)} \frac{1}{t} dt \right).
\]

\[
I_2 = O \left( Z(|a|) \int_{\frac{1}{j+1}}^{\pi} \frac{m(t)}{z(t)} \frac{1}{t} dt \right). (3.7)
\]

299
By (3.5), (3.6) and (3.7), we have
\[ \|T_j(\cdot + a) + T_j(\cdot - a) - 2T_j(t)\|_k = O \left( Z([a]) \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) + O \left( Z([a]) \int_{\frac{1}{j+1}}^{\pi} \frac{m(t)}{z(t) t} dt \right). \]

Therefore,
\[ \sup_{a \neq 0} \|T_j(\cdot + a) + T_j(\cdot - a) - 2T_j(t)\|_k = O \left( \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) + O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t)}{z(t) t} dt \right). \]

Using lemmas (2.1), (2.2) and Remark-1.5(i), we obtain
\[ \|T_j(\cdot)\|_k = \|T_j^\Delta E^\pi - t\|_k \leq \left( \int_0^{1} + \int_{\frac{1}{j+1}}^{\pi} \right) \|\varphi(\cdot, t)\|_k |P_j^\Delta E^\pi(t)| dt \]
\[ = \int_0^{1} \|\varphi(\cdot, t)\|_k |P_j^\Delta E^\pi(t)| dt + \int_{\frac{1}{j+1}}^{\pi} \|\varphi(\cdot, t)\|_k |P_j^\Delta E^\pi(t)| dt \]
\[ = O \left( (j+1) \int_0^{1} m(t) dt + \int_{\frac{1}{j+1}}^{\pi} m(t) \frac{1}{z(t) t} dt \right). \]

As we know that \( \|T_j(\cdot)\|_k = \|T_j(\cdot)\|_k + \sup_{a \neq 0} \|T_j(\cdot + a) + T_j(\cdot - a) - 2T_j(t)\|_k \).

By (3.6), (3.7), we get
\[ \|T_j(\cdot)\|_k = O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t)}{t} dt \right) + O \left( m(\frac{1}{j+1}) \right) + O \left( \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) + O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t)}{z(t) t} dt \right). \]

In view of monotonically of the function \( z(t) \), for \( 0 \leq t \leq \pi \), we have
\[ m(t) = \frac{m(t)}{z(t)} z(t) \leq z(\pi) \frac{m(t)}{z(t)} z(t) = O \left( \frac{m(t)}{z(\pi)} \right). \]

Hence, \( O \left( m(\frac{1}{j+1}) \right) = O \left( \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) \) for \( t = \frac{1}{j+1} \).

Again, due to monotonically of the function \( z(t) \),
\[ \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t) \frac{1}{t} z(t) dt}{z(t) t} \right) \leq z(\pi) \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t) \frac{1}{z(t) t}}{z(t) t} dt \right) = O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t) \frac{1}{z(t) t}}{z(t) t} dt \right). \]

Thus,
\[ \|T_j(\cdot)\|_k = O \left( \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) + O \left( \int_{\frac{1}{j+1}}^{\pi} \frac{m(t) \frac{1}{z(t) t}}{z(t) t} dt \right). \]

Hence, proof of the Theorem 3.1 is completed.

\[ \boxed{\text{Theorem 3.2.} \text{ A } 2\pi-	ext{periodic function } l \text{ belonging to the class } Z_k^{(m)}, k \geq 1, \text{ then the best error estimate of } l' \text{ by the } \Delta E^\pi \text{ method of its Derived Fourier series is given by}} \]
\[ \|l' - \Delta E^\pi \|_k = O \left( (j+1) \left( \frac{m(\frac{1}{j+1})}{z(\frac{1}{j+1})} \right) \int_0^{\pi} dh(t) \right), \]

where \( m(t) \) and \( z(t) \) denote the second order moduli of continuity such that \( \frac{m(t)}{z(t)} \) is positive and non-decreasing in \( t \).

\[ \text{Proof.} \text{ Let } S_r(l; x) \text{ denoted the } r^{th} \text{ partial sum of derived Fourier series, given by} \]
\[ s_r(l; x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]
The partial sum of (1.1) can be represented as a definite integral, we have
\[ s_r(l; x) = \frac{1}{2\pi} \int_0^{2\pi} l(v)dv + \frac{1}{\pi} \sum_{n=1}^{r} \{ \cos nx \int_0^{2\pi} l(v) \cos nv dv + \sin nx \int_0^{2\pi} l(v) \sin nv dv \} \]
Putting, \( v = x + t \), this becomes,
\[ s_r(l; x) = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} f(x+t)dt. \]

Since the integrated has the period \( 2\pi \), and so take the same values in \( (2\pi - y, 2\pi) \) as in \( (-y, 0) \)
\[ s_r(l; x) = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} f(x+t)dt. \]

We may also write-
\[ S_r(l; x) = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} [l(x+t) + l(x-t)]dt. \]

Denoting the sum of the first term of the DFS (derived Fourier series) (1.3) by \( s_r(x) \), we get
\[ S_r'(l; x) = \frac{1}{\pi} \int_0^{\pi} \{ l(x+t) + l(x-t) \} \left( \frac{d}{dt} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right) \]
\[ = \frac{1}{\pi} \int_0^{\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} d[l(x+t) + l(x-t)] \]
\[ = \frac{1}{\pi} \int_0^{\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dh(t) + l'(x). \]

Hence,
\[ S_r'(x) - l'(x) = \frac{1}{\pi} \int_0^{\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dh(t). \]

\[ P_j^{\Delta E_0}(x) - l'(x) = \frac{1}{\pi} \int_0^{\pi} h(x,t) \frac{1}{(1+q)^r} \sum_{n=0}^{r} \int_0^{\pi} \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dh(t), \]
\[ i_j^{\Delta E_0}(x) - l'(x) = \sum_{r=0}^{j} a_{j,r} \left( P_j^{\Delta E_0}(x) - l'(x) \right), \]
\[ = \frac{1}{\pi} \int_0^{\pi} h(x,t) \sum_{r=0}^{j} a_{j,r} \frac{1}{(1+q)^r} \sum_{n=0}^{r} \left( \frac{r}{\omega} \right) q^{-r-n} \sin \left( \omega + \frac{1}{2} \right) t \frac{\sin \left( r + \frac{1}{2} \right) t}{\sin \frac{t}{2}} dh(t), \]
\[ R_j'(x) = i_j^{\Delta E_0}(x) - l'(x). \]

Then,
\[ R_j'(x+v) + R_j'(x-v) - 2tR_j'(x) = \int_0^{\pi} \{ h(x+v,t) + h(x-v,t) - 2th'(x,t) \} P_j^{\Delta E_0}(t)dh(t). \]

Using the GMI [1],
\[ \| R_j(.+v) + R_j(.+v) - 2tR_j'(x) \|_k \leq \int_0^{\pi} \| h(.+v,t) + h(.+v,t) - 2th'(.,t) \|_k \| P_j^{\Delta E_0}(x) \| dh(t) \]
\[ \| R_j(.+v) + R_j(.+v) - 2tR_j'(x) \|_k \leq \left( \int_0^{\pi} \| h(.+v,t) + h(.+v,t) - 2th'(.,t) \|_k \| P_j^{\Delta E_0}(x) \| \right). \]
Thus, by (3.11), (3.12) and (3.13), we have

\[
[R_j' (. + v) + R_j' (. - v) - 2t R_j' (x)]_k = I_3 + I_4. \quad (3.13)
\]

Lemmas 2.2 and 2.3, we obtain

\[
I_3 = \int_0^{\pi} \frac{1}{z(t)} \int_0^{\pi} \frac{m(t)}{z(t)} \, dh \, dt
\]

Now, by Lemma 2.2 and 2.3, we also derive

\[
I_3 = O \left( \frac{m(1)}{z(t)} \int_0^{\pi} dh \right). \quad (3.14)
\]

Therefore

\[
\begin{align*}
I_3 &= O \left( \frac{m(1)}{z(t)} \int_0^{\pi} dh \right) \\
&= O \left( \frac{m(1)}{z(t)} \int_0^{\pi} dh \right).
\end{align*}
\]

Now, by Lemma 2.2 and 2.3, we also derive

\[
I_4 = \int_0^{\pi} \frac{1}{z(t)} \int_0^{\pi} \frac{m(t)}{z(t)} \, dh \, dt
\]

By (3.11), (3.12) and (3.13), we have

\[
\sup_{\nu \neq 0} \frac{[R_j' (. + v) + R_j' (. - v) - 2t R_j' (x)]_k}{|\nu|} = O \left( \frac{m(1)}{z(t)} \int_0^{\pi} dh \right) + O \left( \frac{m(1)}{z(t)} \int_0^{\pi} dh \right). \quad (3.16)
\]

Thus, using Lemmas 2.1, 2.2 and 2.3, we obtain

\[
\begin{align*}
\| R_j' \|_k &= \| t_j^{\Delta E^\nu} - g' \|_k \\
&\leq \left( \int_0^{\pi} + \int_0^{\pi} \right) \| h(., t) \|_k \| P_j^{\Delta E^\nu}(t) \| dh(t) \\
&= \int_0^{\pi} \| h(., t) \|_k \| P_j^{\Delta E^\nu}(t) \| dh(t) + \int_0^{\pi} \| h(., t) \|_k \| P_j^{\Delta E^\nu}(t) \| dh(t) \\
&= O \left( \frac{m(1)}{z(1)} \int_0^{\pi} dh \right) + O \left( \frac{m(1)}{z(1)} \int_0^{\pi} dh \right) \\
\| R_j' \|_k &= O \left( \frac{m(1)}{z(1)} \int_0^{\pi} dh \right) + O \left( \frac{m(1)}{z(1)} \int_0^{\pi} dh \right). \quad (3.17)
\end{align*}
\]
By (3.14) and (3.15), we have
\[ \| R_j' \|_k = \| R_j' \|_k = \sup_{v \in [0, 1]} \| R_j' (., v) + R_j' (., -v) - 2t R_j (x) \|_k \]
\[ = O \left( (j + 1) m \left( \frac{1}{j + 1} \right) \int_{0}^{1 \pi + 1} dh(t) \right) + O \left( \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt \right) \]
\[ + O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dh(t) \right) + O \left( \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt \right). \]

Due to the monotonicity of the function \( z(t) \),
\[ m(t) = \frac{m(t)}{z(t)} z(t) \leq z(\pi) \frac{m(t)}{z(t)} = O \left( \frac{m(t)}{z(t)} \right), 0 \leq t \leq \pi. \]

Hence for \( t = \frac{1}{j + 1} \).

Again, due to the monotonicity of the function \( z(t) \),
\[ \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt = \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt = O \left( \frac{m(t)}{z(t)} dt \right). \]

Thus,
\[ \| R_j' \|_k = O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dh(t) \right) + O \left( \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt \right). \]

Using Remark 1.3 and the second MVT,
\[ \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt \geq (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dt. \]

By (3.16) and (3.17), we drive
\[ \| t_j^{\Delta E^r} - t' \|_k = O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dh(t) \right) + O \left( \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt \right). \]

Hence, proof of the Theorem 3.2 is completed. \( \Box \)

4 Corollaries
The following corollaries are derived from Theorems 3.1 and 3.2.

**Corollary 4.1.** If we put \( m(t) = t^{\gamma_1} z(t) = t^{\gamma_2}, 0 \leq \gamma_2 < \gamma_1 \leq 1 \) in theorem 3.1, (since \( m(t) \) is non-increasing function) then
\[ \| T_j(\cdot) \|_k = O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dt \right). \]

**Proof.** For \( m(t) = t^{\gamma_1} z(t) = t^{\gamma_2}, 0 \leq \gamma_2 < \gamma_1 \leq 1 \) in Theorem 3.1, (since \( m(t) \) is non-increasing function) we derive
\[ \| T_j(\cdot) \|_k = O \left( \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \right) + \int_{0}^{1 \pi + 1} \frac{m(t)}{z(t)} dt = O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dt \right). \]

**Corollary 4.2.** If we put \( m(t) = t^{\gamma_1} z(t) = t^{\gamma_2}, 0 \leq \gamma_2 < \gamma_1 \leq 1 \) in Theorem 3.2, then
\[ \| t_j^{\Delta E^r} - t' \|_k = O \left( (j + 1) \frac{m \left( \frac{1}{j + 1} \right)}{z \left( \frac{1}{j + 1} \right)} \int_{0}^{1 \pi + 1} dh(t) \right). \]
Corollary 4.3. By putting $m(t) = t^{\gamma_1}, z(t) = t^{\gamma_2}$, $0 \leq \gamma_2 < \gamma_1 \leq 1$ and using Remark 1.2, then
\[
\|t_j^{H_{E^q}} - l\|_k^{(z)} = O \left( (j + 1)^{\frac{m(1/2)}{z(1/2)}} \int_{\pi/2}^{\pi} dh(t) \right).
\]

Corollary 4.4. By putting $m(t) = t^{\gamma_1}, z(t) = t^{\gamma_2}$, $0 \leq \gamma_2 < \gamma_1 \leq 1$ and using Remark 1.3, we drive
\[
\|t_j^{N_{p_n,q} E^q} - l\|_k^{(z)} = O \left( (j + 1)^{\frac{m(1/2)}{z(1/2)}} \int_{\pi/2}^{\pi} dh(t) \right).
\]

Corollary 4.5. By putting $m(t) = t^{\gamma_1}, z(t) = t^{\gamma_2}$, $0 \leq \gamma_2 < \gamma_1 \leq 1$ and using Remark 1.4, we establish
\[
\|t_j^{N_{p_n,q} E^q} - l\|_k^{(z)} = O \left( (j + 1)^{\frac{m(1/2)}{z(1/2)}} \int_{\pi/2}^{\pi} dh(t) \right).
\]

5 Conclusion
We establish connections between the convergence properties of the product means and the specific smoothness conditions inherent in the Generalized Zygmund class. By leveraging insights from harmonic analysis and function approximation theory, we derive conditions under which the convergence of the product means guarantees accurate approximations of functions in the Generalized Zygmund class. Our analysis encompasses both pointwise and uniform convergence aspects, shedding light on the rate of convergence and the interplay between approximation accuracy and the properties of the underlying functions. Numerical calculations and applications highlight the practical efficacy of the proposed approximation technique. We present instances where the product means of derived Fourier series efficiently capture intricate features of functions, offering a valuable alternative to existing approximation methods. These findings open new avenues for approximating functions in the Generalized Zygmund class across diverse domains, such as signal processing, image analysis, and scientific computing.

Acknowledgement. Authors are thankful for the valuable comment from the reviewer. Also, Dr. Ananya Manas and Mr. Pankaj Tiwari, Division of Mathematics, School of Basic Sciences, Galgotias University, for their valuable suggestion to complete this work.

References


