The quartic model of an elliptic curve represented by the system of Diophantine equations \( x^2 - 7y^2 = 1 \) and \( x = az^2 - b \), where \( a \) and \( b \) are integers, will be examined for integral solutions. In this work, the system of equations is solved using the algebraic number theory method. Solving a system of equations using the parameters \( 1 \leq a \leq 10 \) and \( 1 < b < 11 \) is an application.

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1 Introduction

The study of polynomial equation solutions or systems of integer equations is known as the study of the Diophantine equation. It is one of the earliest subfields of number theory and indeed of mathematics. To address number theoretic issues, mathematics frequently invents or extensively develops brand-new tools. Recently the authors Das, Somanath and Bindu [5] discussed integer solution analysis for Diophantine equation. It is one of the earliest subfields of number theory and indeed of mathematics. To

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Here we consider the system of Diophantine equation

\[
 x^2 - 7y^2 = 1 \quad \text{and} \quad x = az^2 - b,
\]

where \( a \) and \( b \) are integers.

A quartic model of an elliptic curve is represented by our system of equations. We resolve the problems in this paper. Many authors [1,2,3,7] have solved type \( x^2 - Dy^2 = 1 \) and \( x = az^2 - b \), where \( D \) is a square free integer and \( x, y, z, a \) and \( b \) are integers. Here we consider the system of Diophantine equation

\[
 x^2 - 7y^2 = 1 \quad \text{and} \quad x = az^2 - b,
\]

where \( a \) and \( b \) are integers.

We resolve the problems in this paper using algebraic number theory to solve (1.1). Following are definitions for the notations used in this paper.

Here, we suppose that \( u = 8 + 3\sqrt{7} \) is the fundamental unit in the field \( \mathbb{Q}\sqrt{7} \) and \( \alpha = \frac{um + \overline{m}}{\omega + \overline{\omega}} \) where \( \omega^2 = u.v \) or \( \omega^2 = \pm v \) and \( v = 8 + 3\sqrt{7} \). Here \( \omega \) denotes the conjugate of \( \omega \) in \( \mathbb{Q}\sqrt{7} \).

2 Preliminaries

Lemma 2.1. If \( m, u, \omega, \overline{\omega} \) are defined as before, then \( \alpha = \frac{um + \overline{m}}{\omega + \overline{\omega}} \) is an algebraic number in the field of \( \mathbb{Q} \).

Proof. Consider \( \alpha = \frac{um + \overline{m}}{\omega + \overline{\omega}} = \frac{um + \overline{m}}{(\omega + \overline{\omega})/2} (\omega + \overline{\omega}) = \frac{\omega^2 m + \overline{m} \omega + \overline{m} (\omega + \overline{\omega})}{\omega + \overline{\omega}} \).

Here we see \( \omega^2 + \overline{\omega}^2, \omega^2 u^m \) and \( u^m + \overline{\omega}^m \) are all rational integers and therefore it is clear that \( \alpha \) is an algebraic number.
3 Main Results

**Theorem 3.1.** Let \((x,y,z)\) be an integral solution of (1.1). Then \(z\) exists only if it satisfies any of the following two equations.
- \(2ax^2 - \theta^2 \alpha^2 = 2(b - 1), \) where \(\theta^2 \alpha^2 = 2(x + 1),\)
- \(2ax^2 + \theta^2 \alpha^2 = 2(b + 1), \) where \(\theta^2 \alpha^2 = -2(x + 1).\)

*Proof.* The theorem is proved considering four different cases. Factorizing equation (1.1) in the field \(\mathbb{Q}\sqrt{7}\) gives

\[(x + \sqrt{7}y)(x - \sqrt{7}y) = (8 + 3\sqrt{7})(8 - 3\sqrt{7}).\]  

Therefore, we have,

\[(x + \sqrt{7}y) = \pm vu^n \text{ and } (x - \sqrt{7}y) = \pm \overline{vu}^n, \text{ where } n \in \mathbb{Z},\]  

from which we arrive at the relation

\[2x = \pm (vu^n + \overline{vu}^n).\]  

The solution of (3.2) is split into four cases.

**Case 1:** Assume \(n = 2m + 1\) which is an odd number.
Equation (3.3) gives

\[
\begin{cases}
2x = vu^n + \overline{vu}^n \\
= \infty^{2m+1} + \overline{\square}^{2m+1} \\
= (\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 - 2\omega\overline{\omega}, \text{ with } \omega^2 = u.v \\
= \theta^2 \alpha^2 - 2\omega\overline{\omega},
\end{cases}
\]

where

\[
\theta^2 \alpha^2 = (\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 = 256(\omega^{m} + \overline{\omega}^{m})^2 \text{ and } \omega\overline{\omega} = 1.
\]

Substituting (3.5) and second equation of (1.1) in (3.4), we get \(2ax^2 - \theta^2 \alpha^2 = 2(b - 1)\) where \(\theta^2 \alpha^2 = 2(x + 1).\)

**Case 2:** Assume \(n = 2m + 1\) which is an odd number.
Equation (3.3) gives

\[
\begin{cases}
2x = -(vu^n + \overline{vu}^n) \\
= -(\infty^{2m+1} + \overline{\square}^{2m+1}) \\
= -(\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 - 2\omega\overline{\omega}, \text{ with } \omega^2 = -u.v \\
= -\theta^2 \alpha^2 + 2\omega\overline{\omega},
\end{cases}
\]

where

\[
\theta^2 \alpha^2 = (\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 = -252(\omega^{m} + \overline{\omega}^{m})^2 \text{ and } \omega\overline{\omega} = 1.
\]

Substituting (3.7) and second equation of (1.1) in (3.6), we get \(2ax^2 + \theta^2 \alpha^2 = 2(b - 1)\) where \(\theta^2 \alpha^2 = -2(x - 1).\)

**Case 3:** Assume \(n = 2m\) which is an even number.

\[
\begin{cases}
2x = \pm (vu^n + \overline{vu}^n) \\
= \pm (\infty^{2m} + \overline{\square}^{2m}) \\
= (\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 - 2\omega\overline{\omega}, \text{ with } \omega^2 = v \\
= \theta^2 \alpha^2 - 2\omega\overline{\omega},
\end{cases}
\]

where

\[
\theta^2 \alpha^2 = (\omega + \overline{\omega})^2(\omega^{m} + \overline{\omega}^{m})^2 = 18(\omega^{m} + \overline{\omega}^{m})^2 \text{ and } \omega\overline{\omega} = 1.
\]
Substituting (3.9) and second equation of (1.1) in (3.7), we get $2az^2 - \theta^2\alpha^2 = 2(b-1)$ where $\theta^2\alpha^2 = 2(x+1)$.

**Case 4:** Assume $n = 2m$ which is an even number.

$$
\begin{align*}
2x &= -(vu^n + v\overline{u}^n) \\
   &= -(vu^{2m} + v\overline{u}^{2m}) \\
   &= -(\omega + \overline{\omega})^2 (\frac{\omega u^n + \overline{u}^n}{\omega + \overline{\omega}})^2 - 2\omega\overline{\omega}, \quad \text{with } \omega^2 = v \\
   &= -\theta^2\alpha^2 - 2\omega\overline{\omega},
\end{align*}
$$

(3.10)

where

$$
\theta^2\alpha^2 = (\omega + \overline{\omega})^2 (\frac{\omega u^n + \overline{u}^n}{\omega + \overline{\omega}})^2 = -14(\frac{\omega u^n + \overline{u}^n}{\omega + \overline{\omega}})^2 \text{ and } \omega\overline{\omega} = 1.
$$

(3.11)

Substituting (3.11) and second equation of (1.1) in (3.10), we get $2az^2 + \theta^2\alpha^2 = 2(b+1)$ where $\theta^2\alpha^2 = -2(x-1)$.

4 Analysis

From (1.1), consider the Pellian equation $x^2 - 7y^2 = 1$ with $(x_0, y_0) = (8, 3)$ as the initial solution.

Then its general solution is given by

$$
x_n = \frac{(8+3\sqrt{7})^n + (8-3\sqrt{7})^n}{2\sqrt{7}},
$$

$$
y_n = \frac{(8+3\sqrt{7})^n - (8-3\sqrt{7})^n}{2\sqrt{7}},
$$

where $n \geq 0$, $n \in \mathbb{Z}$.

Applying recurrence formula to the solution, it generates infinite sequence of solutions for $x$ and $y$.

To apply this solution to the second part of (1.1), we have to choose $x$ in such a way that $z = \sqrt{\frac{x+b}{a}}$ represents an integer. This solution depends on the values of $a$ and $b$.

For instance, consider the system (1.1) for $a = 3$ and $b = 1$. This has exactly one integral solution $(x, y, z) = (-1, 0, 0)$.

With the initial solution $(8, 3)$, we can view that several triplets $(a, b, z)$ can be constituted.

When $a = 1, b = 1, z$ possess the values 0, 3, 45, 717, 11427, 182115, 2902413, 46256493, 11748967107, 737201475, 5598155946723, 298419138685, 18724627237.

When $a = 2, b = 1, z$ obtains the values 0, 1, 8, 127, 32257, 514088, 8193151, 130576328, 208102809, 33165873224, 528572943487, 842400122568, 134255446617601.

For $a = 3, 4, 5, 7, 10, b = 1, z$ gets the solution as 0.

When $a = 8, b = 1, z = 0, 4, 1012, 257044, 16582936612, 4212000611284$.

For $a = 9, b = 1, z = 0, 1, 15, 239, 3809, 60705, 967471, 15418831, 245733825, 241542079, 994730462895, 15853271982241$.

As an application, in this paper we have found solutions to the system of equations with bounds defined for the parameters $1 \leq a < 11$ and $1 < b < 11$, and the results are reported in the Table 4.1 below. We have only listed the positive values of $z$ for the sake of simplicity.

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where NS represents no solution.
5 Conclusion
We can use a variety of techniques to solve the family of systems of Diophantine equations, including the one in (1.1). This study makes an effort to identify the issues and use software to discover solutions.

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References
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