FIXED POINT THEOREM FOR F-KANNAN MAPPING

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Abstract
The aim of this paper is to prove a common fixed point theorem for F-Kannan mapping in metric spaces. Also, we extend the concept of F-contraction into F-Kannan mappings motivated by Batra et al. [4].

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1 Introduction
In 1922, Banach established the most famous fundamental fixed point theorem known as Banach contraction principle [5] which states that if X is a complete metric space and T : X → X is a contraction map, that is, d(Tφ, Tξ) ≤ kd(φ, ξ) for all φ, ξ ∈ X and k ∈ [0, 1), then T has a unique fixed point or Tφ = φ has a unique solution. Later in 1968, Kannan [18] gave an alternative contraction condition which was different from the Banach contraction condition and generalization of the Banach contraction principle.

Theorem 1.1 ([18]). Let (X, d) be a complete metric space and a self-mapping T : X → X such that
d(Tφ, Tξ) ≤ k{d(φ, Tφ) + d(ξ, Tξ)},
for all φ, ξ ∈ X and k ∈ [0, ½]. Then, T has a unique fixed point ψ ∈ X and for any φ ∈ X the sequence of iterate \{Tnφ\} converges to ψ.

In 1969, Nadler [22] started the investigations concerning fixed point theory for set-valued contractions see e.g. [6, 8, 17, 20, 24, 25, 28, 29, 30, 32, 33, 35]. Rhoades [23] in 1977 explored 250 different types of contractive conditions and obtained relationship amongst them. Since then, many generalizations of Kannan mapping have been introduced in literature by many researchers in different directions [6, 11, 12, 13, 14, 19, 32, 33, 34]. In 2008, Azam and Arshad [1] extended Kannan’s fixed point result to a generalized metric space where triangle inequality was replaced by inequalities involving four or more points as introduced earlier by Branciari [2].

We will use the following definitions and preliminary results to prove our result.

Definition 1.1 ([1]). Let X be a non-empty set. Suppose that the mapping d : X × X → R satisfies
(i) d(φ, ξ) ≥ 0, for all φ, ξ ∈ X and d(φ, ξ) = 0 if and only if φ = ξ.
(ii) d(φ, ξ) = d(ξ, φ), for all φ, ξ ∈ X.
(iii) d(φ, ξ) ≤ d(φ, α) + d(α, ψ) + d(ψ, ξ), for all φ, ξ ∈ X and for all distinct points α, ψ ∈ X – {φ, ξ} (rectangular property).
Then d is called a generalized metric and (X, d) is a generalized metric space.

Definition 1.2 ([2]). Let (X, d) be a metric space. A mapping T : X → X is said to be sequentially convergent if we have, for every sequence \{ξn\}, if \{Tξn\} is convergent then \{ξn\} is also convergent. T is said to be consequentially convergent if we have, for every sequence \{ξn\}, if \{Tξn\} is convergence then \{ξn\} has a convergent subsequence.
Definition 1.3 ([19]). Let $X$ be a topological space. If $\{\phi_n\}$ is a sequence of points of $X$, and if $n_1 < n_2 < n_3 < \ldots < n_i < \ldots$ is an increasing sequence of positive integers, then the sequence $\{\xi_n\}$ defined by setting $\xi_n = \phi_{n_i}$ is called a subsequence of the sequence $\{\phi_n\}$. The space $X$ is said to be sequentially compact if every sequence of points of $X$ has a convergent subsequence.

For detailed study sequentially convergent property, one can see [3, 4].

In 2011, Moradi and Alimohammadi [21] extended Kannan’s mapping [18] by using the concept of Branciari [2]. They proved results on two self-mappings as follows:

Theorem 1.2 ([21]). Let $(X, d)$ be a complete metric space and $T, S : X \to X$ be mappings such that $T$ is continuous, one-to-one, and subsequentially convergent. If $\lambda = [0, \frac{1}{2})$ and

$$d(TS\phi, TS\xi) \leq \lambda [d(T\phi, TS\phi) + d(T\xi, TS\xi)], \text{ for all } \phi, \xi \in X$$

(1.2) then $S$ has a unique fixed point. Also, if $T$ is sequentially convergent then for every $\phi_0 \in X$, the sequence of iterates $\{S^n\phi_0\}$ converges to this fixed point.

In 2012, Wardowski [31] introduced a special class of contraction mapping called $F$ contraction involving a real valued function $F$ defined on $R^+$ and satisfying some properties.

Wardowski [31] gave the following definitions.

Let $F$ be function defined as $F : R^+ \to R$, which satisfies the following conditions:

(F1) $F$ is strictly increasing, that is, for all $\alpha, \beta \in R^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(F2) For each sequence $\{\alpha_n\}_{n \in N}$ of positive numbers $\lim_{n \to \infty} \alpha_n = 0$ if and only if $F(\alpha_n) = -\infty$.

(F3) There exists $k \in (0, 1)$ such that

$$\lim_{n \to \infty} (\alpha_n)^k F(\alpha_n) = 0.$$  

(1.3)

Definition 1.4 ([31]). A mapping $T : X \to X$ is said to be a $F$-contraction if there exists $\tau > 0$ such that for all $\phi, \xi \in X$.

$$d(T\phi, T\xi) > 0 \Rightarrow \tau + F(d(T\phi, T\xi)) \leq F(d(\phi, \xi)).$$  

(1.4)

Wardowski [31] introduced a generalization of Banach contraction principle, which is as follows:

Theorem 1.3 ([31]). Let $(X, d)$ be a complete metric space and $T : X \to X$ be a $F$-contraction. Then, $T$ has a unique fixed point $\phi^* \in X$ and for every $\phi_0 \in X$ a sequence $\{T^n\phi\}_{n \in N}$ is convergent to $\phi^*$.

In 2019, Goswami et al. [13] introduced the concept of Kannan-$F$-contractive type mapping in $b$-metric spaces. Batra et al. [4] noticed that the definition introduced by Goswami et al. [13] is not meaningful in general. They support that argument by providing suitable example on the definition of Goswami et al. [13]. Therefore, Batra et al. [4] gave the notions of $F$-contraction and Kannan mapping to define a new class of contractions called $F$-Kannan mappings which is in true sense a generalization of Kannan mappings.

Batra et al. [4] gave a new generalization family of contraction called $F$-Kannan mapping and introduced the forthcoming definition.

In this paper, we use the following notations: Let $X$ be a nonempty set and $(X, d)$ denotes the metric space. Define the cardinality of a set $A$ by $\text{card}(A)$ and $\text{Fix} T$ denotes the set of all fixed points of a mapping $T$.

Definition 1.5 ([4]). Let $F$ be a mapping satisfying (F1)-(F3). A mapping $T : X \to X$ is said to be an $F$-Kannan mapping if the following holds:

(K1) $T\phi \neq T\xi \Rightarrow T\phi \neq \phi$ or $T\xi \neq \xi$.

(1.5)

(K2) There exists $\tau > 0$ such that

$$\tau + F(d(T\phi, T\xi)) \leq F \left[ \frac{d(\phi, T\phi) + d(\xi, T\xi)}{2} \right], \text{ for all } \phi, \xi \in X, \text{ with } T\phi = T\xi.$$  

(1.6)

Remark 1.1 ([4]). By above properties, it follows that every $F$-Kannan mapping $T$ on a metric space $(X, d)$ satisfies the following condition

$$d(T\phi, T\xi) \leq \frac{d(\phi, T\phi) + d(\xi, T\xi)}{2}, \text{ for every } \phi, \xi \in X.$$  

(1.7)

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Further, it is concluded that \( \text{card}\{\text{Fix} \, T\} \leq 1 \). Let \( T \) be a self-map of a metric space \((X,d)\) we say that \( T \) is a Picard operator if \( T \) has a unique fixed point \( \phi^* \) and \( \lim_{n \to \infty} T^n \phi = \phi^* \) for all \( \phi \in X \).

**Lemma 1.1** ([4]). Let \((X,d)\) be a metric space and \( T : X \to X \) be an F-Kannan mapping. Then, \( d(T^n \phi, T^{n+1} \phi) \to 0 \) as \( n \to \infty \) for all \( \phi \in X \).

Batra et al. [4] defined the following result of an F-Kannan mapping using the properties of Subrahmanyam [27] which is an extension of Goswami et al. [13], and Wardowski [31] results.

**Theorem 1.4** ([4]). Let \((X,d)\) be a complete metric space and suppose \( T : X \to X \) is an F-Kannan mapping, then \( T \) is a Picard operator.

In 2017, Gopal et al. [9] discussed the fundamental properties for a fixed point theorem which ensures the existence of a common fixed point for suitable assumptions. Those assumptions are sufficient and include conditions of commutativity, containment of ranges of mappings, continuity of at least one mapping, contractive, and all necessary common fixed point theorems attempts to obtain or soften required values of one or more such conditions.

**Definition 1.6** ([15]). Let \((S,T)\) be a pair of self-mappings on a metric space \((X,d)\). Then the coincidence point of the pair \((S,T)\) is a point \( \phi \in X \) such that \((S\phi) = (T\phi) = \phi^*\), then \( \phi^* \) is called coincidence point of the pair \((S,T)\). If \( \phi^* = \phi \), then \( \phi \) is said to be a common fixed point.

**Definition 1.7** ([16]). Let \( S, T \) be self-mappings of a nonempty set \( X \). A point \( \phi \in X \) is the coincidence point of \( S \) and \( T \) if \( \alpha = S\phi = T\phi \). The set of coincidence point of \( S \) and \( T \) is denoted by \( C(S,T) \).

**Definition 1.8** ([16, 26]). Let \((S,T)\) be a pair of self-mappings on a metric space \((X,d)\). Then the pair \((S,T)\) is said to be:

(i) Commuting if, for all \( \phi \in X \), \( S(T\phi) = T(S\phi) \).

(ii) Weakly commuting if, for all \( d(S(T\phi), T(S\phi)) \leq d(S\phi, T\phi) \).

(iii) Compatible if \( \lim_{n \to \infty} d(ST\phi_n, TS\phi_n) = 0 \), whenever \( \phi_n \) is a sequence in \( X \) such that \( \lim_{n \to \infty} ST\phi_n = \lim_{n \to \infty} S\phi_n = \alpha \).

(iv) Weakly compatible if, for all \( S(T\phi) = T(S\phi) \), for every coincidence point \( \phi \in X \).

In 2021, Wangwe and Kumar [34] proved the following lemma for a pair of self-mappings in metric spaces.

**Lemma 1.2** ([34]). Let \((X,d)\) be a metric space and \( T, S : X \to X \) be an F-Kannan mapping. Then,
\[
d(TS^i\phi_0, TS^{i+1}\phi_0) \to 0 \quad \text{as} \quad i \to \infty, \quad \text{for all} \quad \phi \in X. \tag{1.8}
\]

2 Main Result

We prove the main result by assuming a map to be sequentially convergent with a pair of two self-mappings in F-Kannan mappings. We shall start with the extension of Definition 1.5 using a pair of two self-mappings in the F-Kannan mapping setting.

**Definition 2.1.** Let \( F \) be a mapping satisfying \((F1)-(F3)\). A pair of two self-mapping \( T, S : X \to X \) is said to be an F-Kannan mapping if the following holds:

FK1 \[
TS\phi \neq TS\psi \Rightarrow TS\phi \neq \phi \quad \text{or} \quad TS\psi \neq \psi. \tag{2.1}
\]

FK2 \[
\text{There exists } \tau > 0 \text{ such that for all } \phi, \psi \in X,
\]
\[
d(T\phi, T\psi) > 0 \Rightarrow \tau + F(d(TS\phi, TS\psi)) \leq F(M(T\phi, T\psi)). \tag{2.2}
\]

where
\[
M(T\phi, T\psi) = \max \left\{ \frac{d(T\phi, T\psi)}{d(T\phi, TS\phi)), \frac{d(T\phi, TS\phi) + d(T\psi, TS\phi)}{d(T\phi, TS\phi) + d(T\psi, TS\phi) + 1}, d(T\phi, T\psi) \right\}.
\]

**Theorem 2.1.** Let \((X,d)\) be a complete metric space and \( T, S : X \to X \) be an F-Kannan mapping such that \( T \) is continuous, injection, and subsequentially convergent. If \( \lambda = [0, \frac{1}{2}] \), \( \tau > 0 \) and
\[
d(T\phi, T\psi) > 0 \Rightarrow \tau + F(d(TS\phi, TS\psi)) \leq F(M(T\phi, T\psi)),
\]

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where

\[ M(T\phi, T\psi) = \max \left\{ \frac{d(T\phi, T\psi) + d(T\psi, T\phi)}{d(T\phi, T\phi) + d(T\psi, T\psi) + 1} \right\}. \]

Then \( S \) has a unique fixed point. Also, if \( T \) is subsequentially convergent then for every \( \phi_0 \in X \) the sequence of iterates \( \{S^i\phi_0\} \) converges to this fixed point.

**Proof.** Let \( \phi_0 \in X \) be an arbitrary point in \( X \). Let the sequence \( \{\phi_i\}_{i \geq 1} \) be defined by \( \phi_{i+1} = S\phi_i \) and \( \phi_i = S^i\phi_0 \), for \( i = 1, 2, 3, \ldots \).

Using the inequality (2.2), we have

\[ d(T\phi_i, T\phi_{i+1}) = d(TS\phi_{i-1}, TS\phi_i), \]

\[ \tau + F(d(T\phi_i, T\phi_{i+1})) \leq F(M(T\phi_{i-1}, T\phi_i)), \]

\[ F(d(T\phi_i, T\phi_{i+1})) \leq F(M(T\phi_{i-1}, T\phi_i)) - \tau. \]

(2.3)

Now for any \( i \in N \), we have

\[ \tau + F(d(TS\phi_{i-1}, TS\phi_i)) \leq F(M(T\phi_{i-1}, T\phi_i)). \]

(2.4)

Therefore,

\[ F(d(T\phi_i, T\phi_{i+1})) \leq F(d(TS\phi_{i-1}, TS\phi_i)) \leq F(M(T\phi_{i-1}, T\phi_i)) - \tau. \]

(2.5)

Now,

\[ M(T\phi_{i-1}, T\phi_i) = \max \left\{ \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1}, \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1} \right\} \]

\[ = \max \left\{ \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1}, \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1} \right\} \]

\[ \leq \max \left\{ \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1}, \frac{d(T\phi_{i-1}, T\phi_i)}{d(T\phi_{i-1}, T\phi_{i-1}) + d(T\phi_i, T\phi_i) + 1} \right\} \]

\[ = \max\{d(T\phi_{i-1}, T\phi_i), d(T\phi_i, T\phi_{i+1})\}. \]

So, we have

\[ F(d(T\phi_i, T\phi_{i+1})) = F(d(TS\phi_{i-1}, TS\phi_i)) \leq F(\max\{d(T\phi_{i-1}, T\phi_i), d(T\phi_i, T\phi_{i-1})\}) - \tau. \]

The case \( M(T\phi_{i-1}, T\phi_i) = d(T\phi_i, T\phi_{i+1}) \) is impossible. Indeed,

\[ F(d(T\phi_i, T\phi_{i+1})) = F(d(TS\phi_{i-1}, TS\phi_i)) \]

\[ \leq F(d(T\phi_i, T\phi_{i+1})) - \tau \]

\[ < F(d(T\phi_i, T\phi_{i+1})), \]

which is a contradiction.

Thus from (2.5), we have

\[ F(d(T\phi_i, T\phi_{i+1})) \leq F(d(T\phi_{i-1}, T\phi_i)) - \tau. \]

Continuing this process, we have

\[ F(d(T\phi_i, T\phi_{i+1})) \leq F(d(T\phi_{i-1}, T\phi_i)) - \tau \]

\[ = F(d(TS\phi_{i-2}, TS\phi_{i-1})) - \tau \]

\[ \leq F(d(T\phi_{i-2}, T\phi_{i-1})) - 2\tau \]

\[ = F(d(TS\phi_{i-3}, TS\phi_{i-2})) - 2\tau \]

\[ \leq F(d(T\phi_{i-3}, T\phi_{i-2})) - 3\tau \]

\[ : \]

\[ \leq F(d(T\phi_0, T\phi_1)) - i\tau. \]
This implies that
\[ F(d(T\phi_i, T\phi_{i+1})) \leq F(d(T\phi_0, T\phi_1)) - \tau. \] (2.6)

Letting \( i \to \infty \) in (2.6) and using property (F2), we have
\[ \lim_{i \to \infty} d(T\phi_i, T\phi_{i+1}) = 0. \] (2.7)

By Lemma 1.2, we have \( d(T\phi_i, T\phi_{i+1}) \to 0 \) as \( i \to \infty \).

Denote \( d(T\phi_i, T\phi_{i+1}) = v_i \), for all \( i = 1, 2, 3, \ldots \) and \( i \in N \), for \( F \)-Kannan mappings. Using condition (F3) of the function \( F \) there exists \( k \in (0, 1) \) such that
\[ \lim_{i \to \infty} (v_i)^k F(v_i) = 0. \] (2.8)

From (2.6), for every \( i \in N \), we have
\[ (u_i)^k F(v_i) \leq (v_i)^k F(v_{i-1}) - \tau(v_i)^k, \] (2.9)
\[ (u_i)^k F(v_i) - (v_i)^k F(v_{i-1}) \leq -\tau(v_i)^k, \] (2.10)
\[ (u_i)^k [F(v_i) - F(v_{i-1})] \leq -\tau(v_i)^k \leq 0. \] (2.11)

On taking limit as \( i \to \infty \) in (2.9), we have
\[ \lim_{i \to \infty} (\tau v_i)^k = 0. \] (2.12)

From (2.12), there exists \( i_1 \in N \) such that \( \tau(v_i)^k \leq 1 \), for all \( i \geq i_1 \), which follows that
\[ v_i \leq \frac{1}{\tau^k}, \quad \text{for all} \quad i \geq i_1. \] (2.13)

Therefore \( \sum_{i=0}^{\infty} d(T\phi_i, T\phi_{i+1}) \) converges.

By (2.13), we prove that \( \{T\phi_i\} \) is a Cauchy sequence since \((X, d)\) is complete. Consider \( i, j \in N \) such that \( j \geq i \),
\[ d(T\phi_i, T\phi_j) \leq d(T\phi_i, T\phi_{i+1}) + d(T\phi_{i+1}, T\phi_{i+2}) + d(T\phi_{i+2}, T\phi_{i+3}) + \ldots + d(T\phi_{j-1}, T\phi_j) \]
\[ \leq v_i + v_{i+1} + v_{i+2} + \ldots + v_{j-1} \]
\[ = \sum_{i=0}^{j-1} d(T\phi_i, T\phi_{i+1}) \leq \sum_{i=1}^{\infty} v_i \leq \sum_{i=1}^{\infty} \frac{1}{\tau^i}. \] (2.14)

This shows that the series \( \sum_{i=1}^{\infty} \frac{1}{\tau^i} \) converges, which implies that
\[ \lim_{i \to \infty} d(T\phi_i, T\phi_j) = 0. \] (2.15)

So, \( T\phi_i = T\phi_j \) for every \( j \geq i \) in \( X \). Hence, \( \{T\phi_i\} \) is a Cauchy sequence in \( X \). The completeness of \( X \) ensures the existence of \( \phi^* \in X \) such that
\[ d(T\phi^*, T\phi^*) = \begin{cases} \lim_{i,j \to \infty} d(T\phi_i, T\phi_j) = 0, \\ \lim_{i \to \infty} d(T\phi_i, \phi^*) = 0. \end{cases} \] (2.16)

By (2.16), it follows that \( \phi_{i+1} \to \phi^* \) as \( i \to \infty \). By continuity of \( S \) and \( T \), we have
\[ \phi^* = \lim_{i \to \infty} \phi_i = \lim_{i \to \infty} S\phi_i = S\phi^*, \]
\[ \phi^* = \lim_{i \to \infty} \phi_i = \lim_{i \to \infty} T\phi_i = T\phi^*. \] (2.17)

Since \( X \) is a complete metric space, there exists \( \phi^* \in X \) such that
\[ \lim_{i \to \infty} T\phi_i = \phi^*. \] (2.18)

Now, we prove that \( \phi^* \) is a fixed point of \( T \).

Now, suppose \( F \) is continuous. In this case, we claim that \( \phi^* = T\phi^* \). Assume the contrary, that is, \( \phi^* \neq T\phi^* \). In this case, there exists an \( n_0 \in N \) and a subsequence \( \{\phi_{i_k}\} \) of \( \{\phi_i\} \) such that \( d(T\phi_{i_k}, T\phi^*) > 0 \) for all \( n_k \geq n_0 \). (Otherwise, there exists \( i_1 \in N \) such that \( \phi_i = T\phi^* \) for all \( i \geq i_1 \), which implies that \( \phi_i \to T\phi^* \). This is a contradiction. Since \( d(T\phi_{i_k}, T\phi^*) > 0 \) for all \( i_k \geq i_0 \), then from (2.3), we have
\[ \tau + F(d(\phi_{i_k+1}, T\phi^*)) = \tau + F(d(T\phi_{i_k}, T\phi^*)) \]
Taking the limit $k \to \infty$ and using the continuity of $F$, we have $\tau + F(d(\phi^*, T\phi^*)) \leq F(d(\phi^*, T\phi^*))$, which is a contradiction. Therefore, our claim is true, that is $\phi^* = T\phi^*$.

Next, we prove that $\phi^*$ is a unique fixed point of $T$. Assume the contrary, that is, there exists $\alpha^* \in \text{card}\{\text{Fix} T\}$ such that $\phi^* \neq \alpha^*$. Let $T\phi_i \to \alpha^*$ and $\alpha^*$ is a fixed point of $T$. Using Lemma 1.2, it follows that $\phi^* \neq \alpha^*$. Which is a contradiction. Hence $\phi^*$ is a unique fixed point of $T$.

Moreover, $T$ is subsequentially convergent, $\{\phi_i\}$ has a convergent subsequence and there exists $\alpha^* \in X$ and $\{\phi_{i_k}\}_{k=1}^\infty$ so that $\lim_{k \to \infty} \phi_{i_k} = \alpha^*$. Since $T$ is continuous and $\lim_{k \to \infty} \phi_{i_k} = \alpha^*$.

Due to continuity of $T$, it implies that

$$\lim_{k \to \infty} T\phi_{i_k} = T\alpha^*.$$  

From (2.18), we conclude that

$$T\alpha^* = \phi^*.$$  

Using Remark 1.1 and (ii) of Definition 1.1, we have

$$d(T\phi_i, T\phi_{i+1}) = d(TS\phi_{i-1}, TS\phi_i) \leq \lambda d(T\phi_{i-1}, TS\phi_{i-1}) + d(T\phi_i, TS\phi_i) = \lambda d(T\phi_{i-1}, T\phi_i) + d(T\phi_i, T\phi_{i+1}) \leq \frac{\lambda}{1-\lambda} d(T\phi_{i-1}, T\phi_i).$$

Thus, using the (2.2) and (iii) of Definition 1.1, we get

$$F(d(TS\alpha^*, T\alpha^*)) \leq F(d(TS\alpha^*, T\alpha^*)) \leq F(d(TS\alpha^*, T\alpha^*)) \leq \frac{\lambda}{1-\lambda} d(T\phi_{i-1}, T\phi_i).$$

As $F$ is sequentially increasing, this implies that

$$d(TS\alpha^*, T\alpha^*) \leq d(TS\alpha^*, T\alpha^*) \leq d(TS\alpha^*, T\alpha^*) \leq \frac{\lambda}{1-\lambda} d(T\phi_{i-1}, T\phi_i).$$

By Lemmas 1.2 when $TS\phi_{i_k} \neq TS^{i_k+1}\phi_0$ for any $i \in N$ and (2.2), we obtain

$$d(TS\alpha^*, TS^{i_k}\phi_0) \leq \lambda d(T\alpha^*, T\alpha^*) + d(TS^{i_k}\phi_0, TS^{i_k+1}\phi_0) \leq \lambda d(T\alpha^*, T\alpha^*) + d(T\phi_0, T\phi_{i_k}) \leq \lambda d(T\alpha^*, T\alpha^*) \leq \left(\frac{\lambda}{1-\lambda}\right)^{i_k} d(T\phi_0, T\phi_{i_k}).$$

Using (2.21) and (2.22) in (2.20), we establish

$$d(TS\alpha^*, T\alpha^*) \leq \lambda d(T\alpha^*, T\alpha^*) + \lambda \left(\frac{\lambda}{1-\lambda}\right)^{i_k} d(T\phi_0, T\phi_{i_k}) + \frac{\lambda}{1-\lambda} d(T\phi_{i_k}, T\alpha^*),$$

which follows

$$d(TS\alpha^*, T\alpha^*) \leq \left(\frac{\lambda}{1-\lambda}\right)^{i_k} d(T\phi_0, T\phi_{i_k}) + \left(\frac{\lambda}{1-\lambda}\right)^{i_k+1} d(T\phi_{i_k}, T\alpha^*).$$

Letting $k \to \infty$ in (2.24), we get

$$d(TS\alpha^*, T\alpha^*) = 0.$$  

Since $T$ is injection, $S\alpha^* = \alpha^*$. So, $S$ has a fixed point. As $T$ is subsequentially convergent, we conclude that $\{\phi_i\}$ converges to the fixed point of $S$. Implying that $S\phi \in X$ and $T\phi \in X$, then, there exists a point $\alpha^* \in X$ such that $\alpha^* \in S\alpha^* \cap T\alpha^*$, that is $\alpha^*$ is a common fixed point of $S$ and $T$. Which satisfies the fundamental property of Definition 1.7.
Example 2.1. Let $F_1 : R^+ \rightarrow R$ be defined as $F_1(\psi) = \ln(\psi)$. Then, clearly (F1)-(F3) are satisfied by $F_1(\psi)$. In fact, (F3) satisfies for every $k \in (0, 1)$. Moreover, equation (2.2) takes the form:

$$d(TS\phi, TS\xi) \leq e^{-\tau}[F(M(T\phi, T\xi))], \quad \text{for all } \phi, \xi \in X \text{ with } TS\phi \neq TS\xi. \quad (2.26)$$

Thus, if $T, S : X \rightarrow X$ is a Kannan mapping with constant $k \in (0, 1)$ satisfying

$$d(TS\phi, TS\xi) \leq k[F(M(T\phi, T\xi))], \quad \text{for every } \phi, \xi \in X. \quad (2.27)$$

Then, it also satisfies (2.25) and (2.2) with $\tau = \ln(\frac{1}{k})$.

3 Conclusion

If two self-mappings satisfy $(FK_1) - (FK_2)$, that is, $F$-Kannan mapping then both have unique common fixed point.

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References


