ON MULTIPLIER AND REVERSAL MAPS ON $BCK/BCI$-ALGEBRAS

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Abstract

In this paper we develop two interesting mappings on $BCK/BCI$-algebras which are known as multiplier map and dual multiplier map. The reversal map on $BCK/BCI$-algebras, which is another typical map is also discussed. Different properties of these mappings are introduced.

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1 Introduction

Imai and Iseki [5] introduced two new types of abstract algebras in 1966 known as $BCK$-algebra $[8, 9, 10, 11]$ and $BCI$-algebra $[3, 4, 11, 18]$. In the last few decades mathematicians have made further studies in the field and developed concepts of similar type like dual $BCK$-algebras $[13]$ and many more $[6, 17]$. The scholars Kim and Yon $[13]$ presented their findings regarding dual $BCK$ algebra and M. V. algebra in 2007 which led to further research in the direction and findings were had in respect of a different algebra known as BE-algebra $[1, 12, 19]$. In this paper, we want to introduce multiplier map, dual multiplier map and reversal map on $BCK/BCI$-algebras.

2 Preliminaries

Definition 2.1. A system $(W; \oplus, 0)$ consisting of a set $W \neq \emptyset$, with $\oplus$ as binary Operation and 0 as fixed element is called $BCK$-algebra $[8, 9, 10]$ if

1. $((w_p \oplus w_r) \oplus (w_p \oplus w_t)) \oplus (w_t \oplus w_r) = 0$,  
2. $(w_p \oplus (w_p \oplus W_r)) \oplus w_r = 0$,  
3. $w_p \oplus w_p = 0$,  
4. $0 \oplus w_p = 0$,  
5. $w_p \oplus w_r \oplus 0$ and $w_r \oplus w_p = 0$ imply $w_p = w_r$ for all $w_p, w_r \in W$.

Definition 2.2. A system $(W; \oplus, 0)$ consisting of a set $W \neq \emptyset$, with $\oplus$ and 0 as defined above is called called a $BCI$-algebra $[3, 4, 11]$ if

1. $((w_p \oplus w_r) \oplus (w_p \oplus w_t)) \oplus (w_t \oplus w_r) = 0$,  
2. $(w_p \oplus (w_p \oplus W_r)) \oplus w_r = 0$,  
3. $w_p \oplus w_p = 0$,  
4. $w_p \oplus w_r \oplus 0$ and $w_r \oplus w_p = 0$ imply $w_p = w_r$,  
5. $w_p \oplus 0 = 0$ for all $w_p, w_r, w_t \in W$.

Example 2.1. Let $W = 0, 1, 2, 3$ and the binary operation $\oplus$ is given by the Table 2.1. Then $(W; \oplus, 0)$ is a $BCI$-algebra which is not a $BCK$ algebra.
Definition 2.3. A set $B \subseteq BCK/BCI$-algebra $(W; \oplus, 0)$ is a sub algebra [9,11] of $F$ if $w_p \oplus w_r \in B$ for all $w_p, w_r \in B$. It can be denoted as $B \leq W$.

Definition 2.4. A BCK-algebra $W$ is bounded [9] if $\exists$ an element $1 \in W$ such that $w_p \leq 1$ for each element $w_p \in W$. The element $1$ is called unit element of $W$.

Definition 2.5. If $(W; \oplus, 0)$ is a BCK-algebra and $w_p, w_r \in W$, then $w_p$ is called the right disjoint [7] from $w_r$ if $w_r \oplus w_p = w_r$.

3 Multiplier and dual multiplier maps

Definition 3.1. If $(W; \oplus, 0)$ is a BCK/BCI-algebra then a mapping $\zeta : W \to W$ is said to be

(i) multiplier, if $\zeta(w_p \oplus w_r) = \zeta(w_p) \oplus w_r$ and

(ii) dual multiplier, if $\zeta(w_p \oplus w_r) = w_p \oplus \zeta(w_r)$ for all $w_p, w_r \in W$.

Note 3.1. The identity map $e(w) = w$ is both multiplier and dual multiplier.

Example 3.1. Let us assume a BCK/BCI-algebra $(W; \oplus, 0)$.

For some fixed $q \in W$, $\zeta_q : W \to W$ is defined as $\zeta_q(w_p) = w_p \oplus q, \forall w_p \in W$.

Then $\zeta_q(w_p \oplus w_r) = (w_p \oplus w_r) \oplus q$

$= (w_p \oplus q) \oplus w_r$

$= \zeta_q(w_p) \oplus w_r$.

So, $\zeta_q$ is a multiplier.

In this case $\zeta_q(0) = 0 \oplus q = 0$ if $(W; \oplus, 0)$ is a BCK-algebra. But $\zeta_q(0)$ may not be 0 if $(W; \oplus, 0)$ is a BCI-algebra.

Notation 3.1. For a mapping $\zeta : W \to W$, denote

$L_\zeta = \{w_p \in W : \zeta(w_p) = 0\}$

and $M_\zeta = \{w_p \in W : \zeta(w_p) = w_p\}$.

Lemma 3.1. If $(W; \oplus, 0)$ is a BCK-algebra and $\zeta : W \to W$ is a dual multiplier, then

(i) $\zeta(0) = 0$, 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\oplus$ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 2 & 2 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 2 & 0 & 0 \\
3 & 3 & 2 & 1 & 0 \\
\hline
\end{tabular}
\end{table}
\[(ii)\] $W_p \leq \zeta(w_p), \forall w_p \in W,$
\[(iii)\] $w_p \leq w_r \implies w_p \leq \zeta(w_r), w_p, w_r \in W,$

and \[(iv)\] $M_\zeta \leq W.$

**Proof.** (i) We have, $0 = 0 \oplus 0.$ So $\zeta(0) = \zeta(0 \oplus 0) = 0 \oplus \zeta(0) = 0.$

(ii) We have, $0 = w_p \oplus w_p \implies \zeta(0) = \zeta(w_p \oplus w_p) = w_p \oplus \zeta(w_p).$

Thus $w_p \oplus \zeta(w_p) = 0 \implies w_p \leq \zeta(w_p) \forall w_p \in W.$

(iii) Again $w_p \leq w_r \implies w_p \oplus w_r = 0 \implies \zeta(w_p \oplus w_r) = \zeta(0) = \zeta(w_p) \oplus w_r, w_p, w_r \in W.$

Let $w_p, w_r \in M_\zeta.$ Then $\zeta(w_p) = w_p, \zeta(w_r) = w_r.$

Now, $\zeta(w_p \oplus w_r) = w_p \oplus \zeta(w_r) = w_p w_r.$

Therefore $w_p \oplus w_r \in M_\zeta.$

Hence $M_\zeta \leq W.$

**Lemma 3.2.** Let us assume a BCK-algebra $(W; \oplus, 0)$ and let $\zeta : W \to W$ be a multiplier. Then $L_\zeta \leq W.$

**Proof.** Let $w_p, w_r \in L_\zeta.$ Then $\zeta(w_p) = 0, \zeta(w_r) = 0.$

Now $\zeta(w_p \oplus w_r) = \zeta(w_p) \oplus w_r = 0 \oplus w_r = 0$ and so $w_p \oplus w_r \in L_\zeta.$

So $L_\zeta \leq W.$

**Lemma 3.3.** Let us assume a BCK/BCI-algebra $(W; \oplus, 0)$ and let $\zeta : W \to W$ be a multiplier such that $\zeta(0) = 0.$ Then

\[(i)\] $\zeta(w_p) \leq w_p, \forall w_p W,$
\[(ii)\] $w_p \leq w_r \implies \zeta(w_p) \leq w_r,$
\[(iii)\] $M_\zeta \leq W.$

**Proof.** (i) We have, $w_p \oplus w_p = 0, \forall w_p \in W$

\[\implies \zeta(w_p \oplus w_p) = \zeta(0)\]
\[\implies \zeta(w_p) \oplus w_p = 0\]
\[\implies \zeta(w_p) \leq w_p.\]

(ii) Let $w_p \leq w_r.$ Then $w_p \oplus w_r = 0$
\[
\Rightarrow \zeta(w_p \oplus w_r) = \zeta(0)
\]
\[
\Rightarrow \zeta(w_p) \oplus w_r = 0
\]
\[
\Rightarrow \zeta(w_p) \leq w_r.
\]

(iii) Let \(w_p, w_r \in M_\zeta\). \(\zeta(w_p) = w_p, \zeta(w_r) = w_r\).

Now \(\zeta(w_p \oplus w_r) = \zeta(w_p) \oplus w_r = w_p \oplus w_r\) and so \(w_p \oplus w_r \in M_\zeta\).

So \(M_\zeta \leq W\).

\begin{proof}

Now \(\zeta\) is a multiplier and, \(w_p \in W\) is a right disjoint from \(w_r\), then \(w_p\) is a right disjoint from \(\zeta(w_r)\).

This proves that \(w_p\) is right disjoint from \(\zeta(w_r)\).
\end{proof}

\textbf{Theorem 3.1.} Let us assume a BCK algebra \((W; \oplus, 0)\). If \(\zeta : W \to W\) is a multiplier and, \(w_p \in W\) is a right disjoint from \(w_r\), then \(w_p\) is a right disjoint from \(\zeta(w_r)\).

\begin{proof}

Since \(w_p\) is a right disjoint from \(w_r\), we have \(w_r \oplus w_p = w_r\).

\[
\Rightarrow \zeta(w_r \oplus w_p) = \zeta(w_r)
\]
\[
\Rightarrow \zeta(w_r) \oplus w_p = \zeta(w_r).
\]

This proves that \(w_p\) is right disjoint from \(\zeta(w_r)\).
\end{proof}

\textbf{Theorem 3.2.} Let us assume a BCK algebra \((W; \oplus, 0)\) and let \(\zeta : W \to W\) is a multiplier. Then either \(\zeta(0) = 0\) or every \(w_p \in W\) is a right disjoint from \(\zeta(0)\).

\begin{proof}

If \(\zeta(0) = 0\), then we have nothing to prove.

Suppose, \(\zeta(0) = k(0)\).

Then for every \(w_p \in W\), we have \(0 \oplus w_p = 0\).

\[
\Rightarrow \zeta(0 \oplus w_p) = \zeta(0)
\]
\[
\Rightarrow \zeta(0) \oplus w_p = \zeta(0)
\]
\[
\Rightarrow k \oplus w_p = k.
\]

Therefore \(w_p \in W\) is a right disjoint from \(k\) i.e \(\zeta(0)\).
\end{proof}

\textbf{Definition 3.2.} Let us assume a BCK/BCI-algebra \((W; \oplus, 0)\). A mapping \(\zeta : W \to W\) is called a reversal map if \(\zeta(w_p) \oplus w_r = \zeta(w_r) \oplus w_p\), \(\forall w_p, w_r \in W\).

\textbf{Remark 3.1.} The zero map is a reversal map.

\textbf{Example 3.2.} If \((W; \oplus, 0)\) is a BCK/BCI-algebra and \(\zeta : W \to W\) be defined as \(\zeta(w_p) = 1 \oplus w_p\) for some fixed \(1 \in W\). Then \(\zeta(w_p) \oplus w_r = (1 \oplus w_p) \oplus w_r\)

\[
= (1 \oplus w_r) \oplus w_p
\]
\[
= \zeta(w_r) \oplus w_p, \text{ for all } w_p, w_r \in W.
\]

Thus \(\zeta\) is a reversal map.
Theorem 3.3. Let us assume a bounded BCK algebra \((W; \oplus, 0)\) with unit element 1. If \(\zeta : W \to W\) is a non zero reversal map then \(\zeta(1) = 0\) and \(\zeta(0) \neq 0\).

Proof. Since \(\zeta\) is a reversal map, we have

\[
\zeta(1) \oplus w_p = \zeta(w_p) \oplus 1 = 0 \text{ for all } w_p \in W.
\]

This implies that \(\zeta(1) = 0\).

Next suppose that \(\zeta(0) = 0\). Then \(\forall w_p \in W\), we have

\[
\zeta(0) \oplus w_p = \zeta(w_p) \oplus 0
\]

\[
\implies 0 \oplus w_p = \zeta(w_p)
\]

\[
\implies 0 = \zeta(w_p) \text{ for all } w_p \in W.
\]

\[
\implies \zeta \text{ is the zero map.}
\]

Thus we get a contradiction and so \(\zeta(0) \neq 0\).

4 Conclusion

Here we want to mention the summary of the results included in the paper. Two interesting mappings are developed through this paper. They are known as multiplier map and dual multiplier map. Examples of both the maps are illustrated in Note 3.1. and Example 3.1.. Some characteristic properties are discussed in Lemma 3.1., 3.2., 3.3., Theorem 3.1., and 3.2.. Another typical map called as reversal map is also discussed. Examples are given in Remark 3.1. and Example 3.2.. A result is proved in Theorem 3.3.

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References


