COMMON FIXED POINT THEOREMS UNDER RATIONAL INEQUALITY IN COMPLEX VALUED METRIC SPACES

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Abstract

In this paper, we have proved some common fixed point theorems for a pair of mappings satisfying certain rational contraction condition in the frame work of complex valued metric space \((X, d)\). We have also taken the ‘max’ function for the partial order on \(\mathbb{C}\) in complex valued metric \(d\).


Keywords and Phrases: Common fixed point, contractive type mapping, complex valued metric space.

1 Introduction, Definitions and Notation

As an extension of Banach contraction principle, complex valued metric space plays a major role in fixed point theories. In 2011, Azam et al. [1] introduced the concept of complex valued metric space. Since then several authors studied the existence and uniqueness of fixed points theories in complex valued metric spaces (see [2]-[16]). Bhaskar and Lakshikantham proved the common fixed point of mappings satisfying rational inequality in complex valued metric space in [6] under contractive condition. Some common fixed point theorems for a pair of mappings in complex valued metric spaces has been proved by Verma and Pathak in [17].

We write regular complex number as \(z = x + iy\) where \(x\) and \(y\) are real numbers and \(i^2 = -1\). Let \(\mathbb{C}\) be the set of complex numbers and \(z_1\) and \(z_2 \in \mathbb{C}\). Define a partial order relation \(\preceq\) on \(\mathbb{C}\) as follows:

\[z_1 \preceq z_2\text{\ if and only if } Re(z_1) \leq Re (z_2) \text{ and } Im (z_1) \leq Im (z_2).\]

Thus \(z_1 \preceq z_2\) if one of the following conditions is satisfied:

(i) \(Re(z_1) = Re(z_2)\) and Im\((z_1) = Im(z_2)\), (ii) \(Re(z_1) < Re(z_2)\) and \(Im (z_1) = Im(z_2)\), (iii) \(Re(z_1) = Re(z_2)\) and \(Im(z_1) < Im(z_2)\), (iv) \(Re(z_1) < Re(z_2)\) and \(Im(z_1) < Im(z_2)\).

We write \(z_1 \preceq z_2\) if \(z_1 \preceq z_2\) and \(z_1 \neq z_2\) i.e., one of (ii), (iii) and (iv) is satisfied and we write \(z_1 \prec z_2\) if only (iv) is satisfied.

Azam et al. [1] defined the complex valued metric space in the following way:

**Definition 1.1.** Let \(X\) be a non empty set where as \(\mathbb{C}\) be the set of complex numbers. Suppose that the mapping \(d : X \times X \to \mathbb{C}\), satisfies the following conditions:

\[
\begin{align*}
(d_1) \ 0 \preceq d(x,y), \forall x,y \in X \text{ and } d(x,y) = 0 \text{ if and only if } x = y; \\
(d_2) \ d(x,y) = d(y,x) \text{ for all } x,y \in X; \\
(d_3) \ d(x,y) \preceq d(x,z) + d(z,y), \forall x,y,z \in X.
\end{align*}
\]

Then \(d\) is called a complex valued metric on \(X\) and \((X,d)\) is called a complex valued metric space.

Verma and Pathak[17] proved the following theorem in complex valued metric space.
Theorem 1.1. Let \((X, d)\) be a complete complex valued metric space and mappings \(S, T : X \to X\) satisfying
\[
d(Sx, Ty) \leq h \max \{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\},
\]
for all \(x, y \in X\), where \(0 < h < \frac{1}{2}\). Then \(S\) and \(T\) have a common fixed point in \(X\).

Bhatt et al. [4] proved following theorem for complex valued metric space.

Theorem 1.2. Let \((X, d)\) be a complete complex valued metric space and let the mappings \(S, T : X \to X\) satisfying
\[
d(Sx, Ty) \leq \alpha \frac{d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)}{d(x, Ty) + d(y, Sx)},
\]
for all \(x, y \in X\), where \(0 \leq \alpha < 1\). Then \(S\) and \(T\) have a unique common fixed point.

In this paper we prove some theorems using the concepts of Verma & Pathak[17] and Bhatt et al. [4].

2 Main Results

In this section we prove some common fixed point theorems and give example to justify our results.

Theorem 2.1. Let \((X, d)\) be a complete complex valued metric space and mappings \(S, T : X \to X\) satisfying
\[
d(Sx, Ty) \leq h \max \left\{ \frac{d(x, y) + d(x, Sx)}{2}, \frac{d(x, y) + d(y, Ty)}{2}, d(x, Sx) + d(y, Ty) \right\},
\]
for all \(x, y \in X\), where \(0 < h < \frac{1}{2}\). Then \(S\) and \(T\) have a common fixed point in \(X\).

Proof. Choose an arbitrary point \(x_0\) in \(X\). Sequence \(\{x_n\}\) can be formed in \(X\) such that \(Sx_0 = x_1\), \(Tx_1 = x_2\), \(Sx_2 = x_3\), \(Tx_3 = x_4\), ...
\[
Sx_{2n} = x_{2n+1}, Tx_{2n+1} = x_{2n+2}.
\]

We show that the sequence \(\{x_n\}\) is Cauchy. For putting \(x = x_{2k}\) and \(y = x_{2k+1}\) in (2.1), we have
\[
d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})
\]
\[
\leq h \max \left\{ \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k}, Sx_{2k})}{2}, \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, Tx_{2k+1})}{2}, \frac{d(x_{2k}, Sx_{2k}) + d(x_{2k+1}, Tx_{2k+1})}{2} \right\}
\]
\[
\leq h \max \left\{ \frac{d(x_{2k}, x_{2k+1})}{2}, \frac{d(x_{2k+1}, x_{2k+2})}{2}, \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})}{2} \right\}
\]
\[
= h \left\{ \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})}{2} \right\}, \text{ where, } Sx_{2k} = x_{2k+1}.
\]

Therefore we have \(d(x_{2k+1}, x_{2k+2}) \leq [h/(1 - h)] \cdot d(x_{2k}, x_{2k+1})\).

Similarly for \(x = x_{2k+2}\) and \(y = x_{2k+1}\) in (2.1), we have \(d(x_{2k+2}, x_{2k+3}) \leq [h/(1 - h)] \cdot d(x_{2k+1}, x_{2k+2})\).

Hence for each \(n = 1, 2, 3, \ldots\), we have \(d(x_n, x_{n+1}) \leq H \cdot d(x_{n-1}, x_n), \text{ where } 0 < H = h/(1 - h) < 1\). From this, inductively we have
\[
d(x_n, x_{n+1}) \leq H^2 \cdot d(x_{n-2}, x_{n-1}) \leq \ldots \leq H^n \cdot d(x_0, x_1).
\]

Thus for any \(m > n, m, n \in \mathbb{N}\), we have
\[
d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+3}) + \ldots + d(x_{n-1}, x_n)
\]
\[
\leq [H^n + H^{n+1} + H^{n+2} + \ldots + H^{m-1}] \cdot d(x_0, x_1) \leq [H^n/(1 - H)] \cdot d(x_0, x_1).
\]

So that
\[
|d(x_n, x_m)| \leq \{H^n/(1 - H)\} \cdot |d(x_0, x_1)| \to 0 \text{ as } n \to \infty.
\]

Thus \(\{x_n\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete, therefore \(\{x_n\}\) converges to some point \(u\) (say) in \(X\). We claim that \(u\) is a fixed point of \(S\). Otherwise if \(u \neq Su\) then \(|d(u, Su)| = |p|\) (say), where \(|p| > 0\). From the triangle inequality and using (2.1), we have successively
\[
d(u, Su) \leq d(u, x_{2k+1}) + d(x_{2k+1}, Su) = d(u, x_{2k+2}) + d(Tx_{2k+1}, Su)
\]
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Since \( d(x, y) \leq d(x, T y) + d(T y, y) \), we get that
\[
|d(x, y)| = d(x, y) = d(S x, T y) \leq \alpha d(x, y) + \beta d(x, S x) + \gamma d(x, y) + \gamma [d(x, S x) + d(y, T y)] = \frac{\alpha + \beta + \gamma}{1 - \gamma} |d(x, y)|.
\]

Letting \( n \to \infty \), we have \( |p| = |d(u, S u)| \leq h \max \{|p|, 0, |p|\} = h \cdot |p| < |p| \), which is a contradiction. Thus \( |p| = |d(u, S u)| = 0 \), which implies that \( u = S u \).

Further, since \( X \) is complete, there exist some \( v \) in \( X \) such that \( v = T u \). We claim that \( u = v \). If not, then from \( (2.1) \) we have
\[
d(u, v) = d(S u, T u) = h \max \left\{ \frac{d(u, u) + d(u, S u)}{2}, \frac{d(u, u) + d(u, T u)}{2} \right\} = h \max \{0, d(u, v), d(u, v)\} = h \, d(u, v).
\]

Now taking magnitude we get \( |d(u, v)| \leq |h \cdot d(u, v)| < |d(u, v)| \), which is a contradiction. Thus \( u = v = T u = S u \), and \( u \) is a common fixed point of \( S \) and \( T \). Then from \( (2.1) \), we have
\[
d(u, u_0) = d(S u, T u_0) = h \max \left\{ \frac{d(u, u_0) + d(u, u)}{2}, \frac{d(u, u_0) + d(u_0, u)}{2}, d(u, u) + d(u_0, u_0) \right\}.
\]

Hence we get
\[
|d(u, u_0)| \leq h \max \left\{ \frac{|d(u, u_0)|}{2}, \frac{|d(u, u_0)|}{2}, 0 \right\} = h \cdot \frac{|d(u, u_0)|}{2} < |d(u, u_0)|,
\]
which is a contradiction. Thus \( S \) and \( T \) have unique common fixed point.

This completes the proof of the theorem.

**Theorem 2.2.** Let \( (X, d) \) be a complete complex valued metric space and let the mappings \( S, T : X \to X \) satisfying
\[
d(Sx, Ty) \leq \alpha d(x, y) + \frac{\beta d(x, S x).d(y, T y)}{d(x, y) + d(y, S x) + d(x, y)} + \gamma [d(x, S x) + d(y, T y)],
\]
for all \( x, y \in X \) such that \( x \neq y \) where \( \alpha, \beta \) and \( \gamma \) are nonnegative real numbers with \( \frac{\alpha + \beta + \gamma}{1 - \gamma} < 1 \) or \( d(Sx, Ty) = 0 \) if \( d(x, T y) + d(y, S x) + d(x, y) = 0 \). Then the pair \((S, T)\) have a unique common fixed point.

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \) and define \( x_{2k+1} = S x_{2k}, x_{2k+2} = T x_{2k+1}, k = 0, 1, 2, 3, \ldots, \) then
\[
d(x_{2k+1}, x_{2k+2}) = d(S x_{2k}, T x_{2k+1})
\]
\[
\leq \alpha d(x_{2k}, x_{2k+1}) + \beta \frac{d(x_{2k}, S x_{2k}) + d(x_{2k+1}, T x_{2k+1})}{2} + \gamma [d(x_{2k}, S x_{2k}) + d(x_{2k+1}, T x_{2k+1})]
\]
\[
= \alpha d(x_{2k}, x_{2k+1}) + \beta \frac{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1}) + d(x_{2k}, x_{2k+1})}{2} + \gamma [d(x_{2k}, S x_{2k}) + d(x_{2k+1}, T x_{2k+1})]
\]
\[
\leq \alpha d(x_{2k}, x_{2k+1}) + \beta d(x_{2k}, x_{2k+1}) + \gamma [d(x_{2k}, S x_{2k}) + d(x_{2k+1}, T x_{2k+1})].
\]

Since \( d(x_{2k+1}, x_{2k+2}) \leq d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) \), so that
\[
|d(x_{2k+1}, x_{2k+2})| \leq \alpha |d(x_{2k}, x_{2k+1})| + \beta |d(x_{2k}, x_{2k+1})| + \gamma |d(x_{2k}, x_{2k+1})| + \gamma |d(x_{2k+1}, x_{2k+2})|.
\]

Therefore we get that
\[
|d(x_{2k+1}, x_{2k+2})| (1 - \gamma) \leq (\alpha + \beta + \gamma) |d(x_{2k}, x_{2k+1})|
\]
\[
|d(x_{2k+1}, x_{2k+2})| \leq \frac{\alpha + \beta + \gamma}{1 - \gamma} |d(x_{2k}, x_{2k+1})|.
\]

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Also we have
\[ d(x_{2k+3}, x_{2k+2}) = d(Sx_{2k+2}, Tx_{2k+1}) \leq \alpha d(x_{2k+2}, x_{2k+1}) + \beta d(x_{2k+2}, Sx_{2k+2}) \]
\[ + \gamma [d(x_{2k+2}, Sx_{2k+2}) + d(x_{2k+1}, Tx_{2k+1})] \]
\[ = \alpha d(x_{2k+2}, x_{2k+1}) + \beta \frac{d(x_{2k+2}, x_{2k+2})}{d(x_{2k+2}, x_{2k+2})} + d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \]
\[ + \gamma [d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})] \]
\[ = \alpha (x_{2k+2}, x_{2k+1}) + \beta \alpha d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \]
\[ + \gamma [d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})] \]
\[ \leq \alpha d(x_{2k+2}, x_{2k+1}) + \beta d(x_{2k+2}, x_{2k+1}) + \gamma [d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})] \].

Since \( d(x_{2k+3}, x_{2k+2}) \leq d(x_{2k+3}, x_{2k+2}) \leq d(x_{2k+3}, x_{2k+2}) + d(x_{2k+1}, x_{2k+2}) \), therefore we get that
\[ |d(x_{2k+3}, x_{2k+2})| \leq \alpha |d(x_{2k+2}, x_{2k+1})| + \beta |d(x_{2k+2}, x_{2k+1})| \]
\[ + \gamma [d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})] \].

Therefore we obtain that \( |d(x_{2k+3}, x_{2k+2})| \leq \frac{\alpha + \beta + \gamma}{\tau} |d(x_{2k+1}, x_{2k+2})| \). If \( \lambda = \frac{\alpha + \beta + \gamma}{\tau} < 1 \), then we have
\[ |d(x_{n+1}, x_{n+2})| \leq \lambda |d(x_{n}, x_{n+1})| \leq \lambda^2 |d(x_{n-1}, x_{n})| \leq \cdots \leq \lambda^n |d(x_0, x_1)| \].

So that for any \( m > n \), we get that
\[ |d(x_n, x_m)| \leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + \cdots + |d(x_{m-1}, x_m)| \]
\[ \leq [\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}] |d(x_0, x_1)| \]
\[ \leq \frac{\lambda^n - \lambda^{m}}{1-\lambda} |d(x_0, x_1)| \].

Hence \( |d(x_n, x_m)| \leq \lambda^n |d(x_0, x_1)| \to 0 \) as \( m, n \to \infty \), which shows that \( \{x_n\} \) is a Cauchy sequence.

Since \( X \) is complete, there exists \( u \in X \) such that \( x_n \to u \). Let on contrary \( u \neq Su \), so that \( d(u, Su) = p > 0 \) and we can have
\[ p = d(u, Su) \leq d(u, x_{2k+2}) + d(x_{2k+2}, Su) \leq d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \]
\[ \leq d(u, x_{2k+2}) + \alpha d(u, x_{2k+1}) + \beta \frac{d(u, Su) + d(x_{2k+1}, Tx_{2k+1})}{d(u, Tx_{2k+1}) + d(x_{2k+1}, Su) + d(u, x_{2k+1})} \]
\[ + \gamma [d(u, Su) + d(x_{2k+1}, Tx_{2k+1})] \],

which on making \( n \to \infty \), give rise \( |d(u, Su)| = 0 \), a contradiction so that \( u = Su \). Similarly, it can be shown that \( u = Tu \).

To prove the uniqueness of common fixed point of \( S \) and \( T \), let \( u^* \) in \( X \) be another common fixed point of \( S \) and \( T \). Then
\[ d(u, u^*) = d(Su, Tu^*) \leq \alpha d(u, u^*) + \beta d(u, Su) + d(u^*, Tu^*) \]
\[ = \alpha d(u, u^*) + \beta d(u, Su) + d(u^*, Su) + d(u, u^*) \]
\[ + \gamma [d(u, Su) + d(u^*, Tu^*)] \],

which again yields that \( |d(u, u^*)| = 0 \) which prove the uniqueness of common fixed point.

This proves the theorem.

The next theorem can be analogously proved in the line of Theorem 2.2 using \( \gamma = 0 \).

**Theorem 2.3.** Let \( (X, d) \) be a complete complex valued metric space and the mappings \( S, T : X \to X \), satisfying
\[ d(Sx; Ty) \leq \alpha d(x, y) + \frac{\beta d(x, Sy) d(y, Ty)}{d(x, Ty) + d(y, Sy) + d(x, y)} \]
for all \( x, y \in X \) such that \( x \neq y \), \( d(x, Ty) + d(y, Sy) + d(x, y) \neq 0 \) where \( \alpha, \beta \) are non-negative real numbers with \( \alpha + \beta < 1 \) or \( d(x, y) = 0 \) if \( d(x, Ty) + d(y, Sy) + d(x, y) = 0 \). Then \( S \) and \( T \) have a unique common fixed point.
The next corollary can be analogously proved in the line of Theorem 2.2 using $S = T$.

**Corollary 2.1.** Let $(X, d)$ be a complete complex valued metric space and let the mapping $T : X \to X$ satisfying

$$d(Tx, Ty) \leq \alpha d(x, y) + \frac{\beta d(x, Tx).d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} + \gamma [d(x, Tx) + d(y, Ty)] ,$$

for all $x, y \in X$ such that $x \neq y$ where $\alpha, \beta$ and $\gamma$ are nonnegative real numbers with $\frac{\alpha + \beta + \gamma}{\alpha} < 1$ or $d(Tx, Ty) = 0$ if $d(x, Ty) + d(y, Tx) + d(x, y) = 0$. Then $T$ has a unique common fixed point.

**Theorem 2.4.** Let $(X, d)$ be a complete complex valued metric space and let the mappings $S, T : X \to X$ satisfying

$$d(Sx, Ty) \leq \alpha d(x, y) + \frac{\beta [d^2(x, Ty) + d^2(y, Sx)]}{d(x, Ty) + d(y, Sx)} + \gamma [d(x, Sx) + d(y, Ty)] ,$$

for all $x, y \in X$ such that $x \neq y$ where $\alpha, \beta$ and $\gamma$ are nonnegative real numbers with $\alpha + 2\beta + 2\gamma < 1$ or $d(Sx, Ty) = 0$ if $d(x, Ty) + d(y, Sx) = 0$. Then the pair $(S, T)$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X$ and define $x_{2k+1} = Sx_{2k}$, $x_{2k+2} = Tx_{2k+1}$, $k = 0, 1, 2, 3, ...$, then

$$d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$$

$$\leq \alpha d(x_{2k}, x_{2k+1}) + \beta \frac{[d^2(x_{2k}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k})]}{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})}$$

$$+ \gamma [d(x_{2k}, Sx_{2k}) + d(x_{2k+1}, Tx_{2k+1})]$$

$$= \alpha d(x_{2k}, x_{2k+1}) + \beta \frac{[d^2(x_{2k}, x_{2k+2}) + d^2(x_{2k+1}, x_{2k+1})]}{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})}$$

$$+ \gamma [d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})].$$

So we have

$$|d(x_{2k+1}, x_{2k+2})| = \alpha |d(x_{2k}, x_{2k+1})| + \beta \frac{|d^2(x_{2k}, x_{2k+2})|}{|d(x_{2k}, x_{2k+2})|}$$

$$+ \gamma [|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|]$$

$$= \alpha |d(x_{2k}, x_{2k+1})| + \beta |d(x_{2k}, x_{2k+2})| + \gamma \left[ \frac{|d(x_{2k}, x_{2k+1})|}{|d(x_{2k}, x_{2k+2})|} \right].$$

As $|d(x_{2k}, x_{2k+2})| \leq |d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|$, therefore we have

$$|d(x_{2k+1}, x_{2k+2})| = \alpha |d(x_{2k}, x_{2k+1})| + \beta |d(x_{2k}, x_{2k+2})| + \gamma [|d(x_{2k}, x_{2k+1})| + |d(x_{2k+1}, x_{2k+2})|].$$

Therefore we obtain that

$$|d(x_{2k+1}, x_{2k+2})| \leq \frac{(\alpha + \beta + \gamma)}{(1 - \beta - \gamma)} |d(x_{2k}, x_{2k+1})| .$$

Also we have

$$d(x_{2k+3}, x_{2k+2}) = d(Sx_{2k+2}, Tx_{2k+1}) \leq \alpha d(x_{2k+2}, x_{2k+1})$$

$$+ \beta \frac{[d^2(x_{2k+2}, Tx_{2k+1}) + d^2(x_{2k+1}, Sx_{2k+2})]}{d(x_{2k+2}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k+2})}$$

$$+ \gamma |d(x_{2k+2}, Sx_{2k+2}) + d(x_{2k+1}, Tx_{2k+1})|$$

$$= \alpha d(x_{2k+2}, x_{2k+1}) + \beta \frac{[d^2(x_{2k+2}, x_{2k+2}) + d^2(x_{2k+1}, x_{2k+3})]}{d(x_{2k+2}, x_{2k+2}) + d(x_{2k+1}, x_{2k+3})}$$

$$+ \gamma |d(x_{2k+2}, x_{2k+3}) + d(x_{2k+1}, x_{2k+2})|.$$
Hence we obtain that
\[ |d(x_{2k+3}, x_{2k+2})| \leq \alpha |d(x_{2k+2}, x_{2k+1})| + \beta |d(x_{2k+1}, x_{2k+3})| + \gamma |d(x_{2k+2}, x_{2k+3})| + |d(x_{2k+1}, x_{2k+2})|. \]
Since \( d(x_{2k+3}, x_{2k+2}) \leq d(x_{2k+3}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2}) \), therefore we obtain that
\[ |d(x_{2k+3}, x_{2k+2})| \leq \alpha |d(x_{2k+2}, x_{2k+1})| + \beta \left[ |d(x_{2k+1}, x_{2k+2})| \right] + \gamma \left[ |d(x_{2k+2}, x_{2k+3})| + |d(x_{2k+1}, x_{2k+2})| \right], \]
i.e., \( |d(x_{2k+3}, x_{2k+2})| \leq \frac{\alpha + \beta + \gamma}{1 - \gamma} |d(x_{2k+1}, x_{2k+2})| \).

If \( \delta = \frac{\alpha + \beta + \gamma}{1 - \gamma} < 1 \), then we have
\[ |d(x_{n+1}, x_{n+2})| \leq \delta |d(x_n, x_{n+1})| \leq \delta^2 |d(x_{n-1}, x_n)| \leq ... \leq \delta^{n+1} |d(x_0, x_1)|, \]
so that for any \( m > n \), we get that
\[ |d(x_n, x_m)| \leq |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})| + ... + |d(x_{m-1}, x_m)| \leq [\delta^n + \delta^{n+1} + ... + \delta^{m-1}] |d(x_0, x_1)| \leq \frac{\delta^n}{1 - \delta} |d(x_0, x_1)|. \]
Hence \( |d(x_n, x_m)| \to 0 \) as \( m, n \to \infty \). Which shows that \( \{x_n\} \) is a Cauchy sequence in \( X \).

Since \( X \) is complete, there exists \( u \in X \) such that \( x_n \to u \). Let on contrary \( u \neq Su \), so that \( d(u, Su) = p > 0 \) and we can have
\[ p = d(u, Su) \leq d(u, x_{2k+2}) + d(x_{2k+2}, Su) = d(u, x_{2k+2}) + d(Tx_{2k+1}, Su) \leq d(u, x_{2k+2}) + \alpha d(u, x_{2k+1}) + \beta \left[ \frac{d^2(u, Tx_{2k+1}) + d^2(x_{2k+1}, Su)}{d(u, Tx_{2k+1}) + d(x_{2k+1}, Su)} \right] + \gamma |d(u, Su) + d(x_{2k+1}, Tx_{2k+1})|, \]
Therefore we get that
\[ |p| \leq |d(u, x_{2k+2})| + \alpha |d(u, x_{2k+1})| + \beta \left[ \frac{d^2(u, x_{2k+2}) + d^2(x_{2k+1}, Su)}{d(u, x_{2k+2}) + d(x_{2k+1}, Su)} \right] + \gamma |d(u, Su) + d(x_{2k+1}, Tx_{2k+1})|, \]
which on making \( n \to \infty \), give rise \( |d(u, Su)| = 0 \), a contradiction so that \( u = Su \). Similarly, it can be shown that \( u = Tu \).

This completes the proof of the theorem.

The next corollary can be analogously proved in the line of Theorem 2.4 using \( S = T \).

**Corollary 2.2.** Let \((X, d)\) be a complete complex valued metric space and let the mapping \( T : X \to X \) be satisfy:
\[ d(Tx, Ty) \leq \alpha d(x, y) + \beta \left[ \frac{d^2(x, Ty) + d^2(y, Tx)}{d(x, Ty) + d(y, Tx)} \right] + \gamma (d(x, Tx) + d(y, Ty)), \]
for all \( x, y \in X \) such that \( x \neq y \), where \( \alpha, \beta \) and \( \gamma \) are non-negative real numbers with \( \alpha + 2\beta + 2\gamma < 1 \) or \( d(Tx, Ty) = 0 \) if \( d(x, Ty) + d(y, x) = 0 \). Then \( T \) has a unique fixed point.

**Example 2.1.** Let \( z = z_1 + iz_2 \in \mathbb{C} \) where \( z_1 = x_1 + iy_1 \), \( z_2 = x_2 + iy_2 \in \mathbb{C} \).
Consider \( X_1 = \{ z \in \mathbb{C} : z_1 \geq 0, z_2 = 0 \} \) and \( X_2 = \{ z \in \mathbb{C} : z_2 \geq 0, z_1 = 0 \} \).
Write \( X = X_1 \cup X_2 \) and define a mapping \( d : X \times X \to \mathbb{C} \) as
\[ d(z_1, z_2) = \left\{ \begin{array}{ll}
(1 + i) \max \{z_1, z_2\}, & z_1, z_2 \in X_1 \\
(1 + i) \{z_1 + z_2\}, & z_1, z_2 \in X_2
\end{array} \right.. \]
Then \((X, d)\) is a complete complex valued metric space.
Taking $T = S$ and $z = z_1 + iz_2$ we define the self mapping $T$ on $X$ as

$$Tz = \begin{cases} \left( \frac{z_1}{2}, 0 \right), & z \in X_1 \\ \left( 0, \frac{z_2}{2} \right), & z \in X_2 \end{cases}$$

Case-I: Let $z_1, z_2 \in X_1$, then we have

$$d(Sz_1, Tz_2) = d\left( \left( \frac{z_1}{2}, 0 \right) , \left( \frac{z_2}{2}, 0 \right) \right) = \frac{1}{2} \left(1 + i\right) \max\{z_1, z_2\} \leq \frac{1}{2} d(z_1, z_2)$$

Case-II: Let $z_1, z_2 \in X_2$, then we have

$$d(Sz_1, Tz_2) = d\left( \left( 0, \frac{z_1}{2} \right) , \left( 0, \frac{z_2}{2} \right) \right) = \frac{1}{2} \left(1 + i\right) \max\{z_1, z_2\} \leq \frac{1}{2} d(z_1, z_2)$$

In each case condition of the above theorem is satisfied with the condition $\alpha = \frac{1}{2}$ and $\beta = 0$ and $z = 0$ is the fixed point of $S$ and $T$.

3 Future Prospect
In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric and other different types of metrics. This may be an active area of research to the future workers in this branch.

4 Conclusion
In this article, we have made more generalizations about the rational contraction mapping and proved some common fixed-point theorems in complex valued metric spaces. We have also proved the existence of common fixed point for a pair of mappings under rational inequality in complex valued metric spaces. Moreover, we have proved an example to justify our results obtained. It is still open to prove some common fixed-point theorems in bicomplex valued $b$-metric space, bicomplex valued fuzzy metric space and in other generalized complex valued metric spaces.

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