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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN MULTIPLICATIVE CONE METRIC SPACE

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Abstract

In this article, we first introduce the idea of weakly compatible mappings in multiplicative cone metric space and then prove some common fixed point theorems for two pairs of weakly compatible mappings that satisfy a contractive condition in multiplicative cone metric space with a multiplicative normal cone setting. Examples are also provided to support our findings.
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1 Introduction

Huang and Zhang [5] generalized metric space to cone metric space in 2007, in which they considered convergence and took ordered Banach space at the place of the set of real numbers. Also, with normal cone conditions, they have presented some fixed-point results in this space. Afterward, many researchers studied and proved several fixed and common fixed point (CFP) results in this space; some of them are [3, 6, 13, 15, 17, 19, 20, 21].

Recently, the conception of multiplicative cone metric was presented by Clement Boateng Ampadu [1] in 2017 by replacing the triangular property of a metric with a multiplicative triangular property in cone metric space. Additionally, he established some outcomes [1, 2] in this space.

On the other hand, in 1998, by using the concept of coincidence points of mappings, Jungck and Rohoads [7] defined weakly compatible mappings to generalize the ideas of commutative [8], weakly commutative [18], and compatible mappings [9]. They proved CFP results for these mappings, and they also showed that every pair of compatible mappings is weakly compatible. The perception of weak compatibility is less restrictive than any other commutativity concept. After that, with this concept, many authors have obtained coincidence point and CFP results with different contractive conditions in various spaces; see [4, 10, 11, 12, 14, 16] etc.

In this study, we introduce the idea of weakly compatible (w-compatible) mappings in multiplicative cone metric space, and by employing a new type of contractive condition, we prove a few CFP theorems in multiplicative cone metric space with multiplicative normal cone setting without the use of the concept of mapping continuity. Ours establishes results to generalize and extend several common fixed-point results existing in the literature. To support our findings, examples are also provided.

2 Preliminaries

In 2017, C. Boateng Ampadu [1] gave the perception of multiplicative cone metric space as follows:

Definition 2.1 (1)**.** *A subset $L \subseteq K$ (where K is a real Banach space) is called a multiplicative cone if and only if:*

(i) L is closed, nonempty, and $L \neq \{1\}$,
(ii) $u^m \cdot v^n \in L$, for all $u, v \in L$ and $m, n \geq 0$,
(iii) $u \in L$ and $\frac{1}{u} \in L$ imply $u = 1$ i.e., $L \cap \frac{1}{L} = 1$.

Definition 2.2 (1). *Let a multiplicative cone $L \subseteq K$, then we define partial ordering $\leq$ to L by $u \leq v$ iff $\frac{u}{v} \in L$. Where $u < v$ indicates $u \leq v$ but $u \neq v$, while $u << v$ will stand for $\frac{u}{v} \in int\,(L)$ (interior of L).*

Definition 2.3 (1). *A multiplicative cone $L \subseteq K$ is called multiplicative normal if there exists a constant $\psi > 0$ such that for all $u, v \in K, 1 \leq u \leq v$ implies, $\|u\| \leq \|v\|^{\psi}$. The least positive number satisfying the above is called the multiplicative constant of L. Here $\|\cdot\|$ denotes a multiplicative norm.*

Definition 2.4 (1). *Let $L \subseteq K$ be a multiplicative cone, where K is a real Banach space. Let Φ be any non-empty set, then if the mapping $s : \Phi \times \Phi \to K$ satisfies the following:*

(i) $1 < s(u, v), \forall u, v \in \Phi$ and $s(u, v) = 1$ iff $u \neq v$,

(ii) $s(u, v) = s(v, u) \forall u, v \in \Phi$,

(iii) $s(u, v) \leq s(u, w) \cdot s(w, v) \quad \forall u, v, w \in \Phi$ (multiplicative triangle inequality).

Then pair (Φ, s) represents a multiplicative cone metric space (for short, MCM space) and s is called a multiplicative cone metric on Φ.

Example 2.1. Let $K = R^2, L = \{(u, v) \in K : u, v \geq 1\} \subseteq R^2, \Phi = R$ and mapping $s : \Phi \times \Phi \to K$ be such that, $s(u, v) = \left(\omega^{|u-v|}, \omega^{\lambda|u-v|}\right)$, where $\lambda \geq 0$ is a constant and $\omega > 1$. Then (Φ, s) is multiplicative cone metric space.

Definition 2.5 ((1)). *Let (Φ, s) be MCM space, then the sequence $\{u_n\}$ in Φ, is said to be,*

(i) Multiplicative convergent, if for every $\delta \in K$ with $1 << \delta, \exists N$ s.t., $\forall n > N, s\,(u_n, u) << \delta$, i.e. $\lim_{n\to\infty} u_n = u$, where $u \in \Phi$ is a multiplicative converges point of sequence $\{u_n\}$.

(ii) Cauchy sequence, if for any $\delta \in$ K with $1 << \delta, \exists N$ s.t., $\forall n, m > N, s\,(u_n, u_m) << \delta$.

Definition 2.6 (1). *A multiplicative cone metric space is said to be complete if every multiplicative Cauchy sequence is multiplicative convergent in Φ.*

Definition 2.7 (1). *Let $H_1, H_2 : \Phi \to \Phi$ be two self-mappings on non-empty set Φ, and $u, \tau \in \Phi$ be such that $H_1(u) = H_2(u) = \tau$. Then u is called the coincidence point and τ is called the point of coincidence of the pair (H_1, H_2).*

Definition 2.8. *Let (Φ, s) be a multiplicative cone metric space, and $H_1, H_2 : \Phi \to \Phi$ are two self-mappings of (Φ, s). Then H_1, and H_2 are said to be weakly compatible (w-compatible) if,*

$$H_1(u) = H_2(u), \forall u \in \Phi \text{ implies } H_1H_2(u) = H_2H_1(u)$$

That is, if, $s\,(H_1u, H_2u) = 1 \Rightarrow s\,(H_1u, H_2u) = 1$.

Example 2.2. Let $\Phi = R, K = R$, and $L = \{u \in K : u \geq 1\}$ be a multiplicative cone in K. Let for every u, $v \in \Phi$, and $\omega > 1, s(u, v) = \omega^{|u-v|}$ be a multiplicative metric on Φ, then clearly, we can see that pair (Φ, s) is a MCM space. Suppose $H_1, H_2 : \Phi \to \Phi$ are two self-mappings of (Φ, s) defined by,

$$H_1(u) = u \text{ and } H_2(u) = 2u, \forall u \in \Phi$$

Here, mappings H_1 and H_2 have coincidence point $u = 0$, also $H_1(0) = H_2(0) = 0$ implies $H_1H_2(0) = H_2H_1(0) = 0$. Hence mappings H_1 and H_2 are weakly compatible.

Example 2.3. Let $\Phi = [0, \infty), K = R$, and $L = \{u \in K : u \geq 1\}$ be a multiplicative cone in K. Let for every $u, v \in \Phi$, and $\omega > 1, s(u, v) = \omega^{|u-v|}$ be a multiplicative metric on Φ, then clearly (Φ, s) is a MCM space. Suppose $H_1, H_2 : \Phi \to \Phi$ are two self-mappings of (Φ, s) defined by,

$$H_1(u) = 3u \text{ and } H_2(u) = \frac{u^2}{3}, \forall u \in \Phi$$

Here, mappings H_1 and H_2 have two coincidence points $u = 0, 9$. For $u = 0$, we have $H_1(0) = H_2(0) = 0$ implies $H_1H_2(0) = H_2H_1(0) = 0$, that is mappings H_1 and H_2 commute at coincidence point $u = 0$. But for $u = 9$, we have $H_1(9) = H_2(9) = 9$ implies $H_1H_2(9) = 81 \neq 243 = H_2H_1(0)$, that is mappings H_1 and H_2 does not commute at coincidence point $u = 9$. Hence mappings H_1 and H_2 are not weakly compatible.

Remark 2.1. *Every pair of weakly compatible mappings is commutative, weakly commutative, and compatible, while the opposite isn't always true, as shown by the following illustration:*

Example 2.4. Let $\Phi = [0, \infty), K = R$, and $L = \{u \in K : u \geq 1\}$ be a multiplicative cone in K. Let for every $u, v \in \Phi$, and $\omega > 1, s(u, v) = \omega^{|u-v|}$ be a multiplicative metric on Φ, then clearly (Φ, s) is a multiplicative cone metric space. Suppose $H_1, H_2 : \Phi \to \Phi$ are two self-mappings of (Φ, s) defined by

$$H_1(u) = \begin{cases} u & \text{if } u \in [0, 2) \\ 2 & \text{if } u = 2 \\ 4 & \text{if } u \in (2, \infty) \end{cases} \quad \text{and} \quad H_2(u) = \begin{cases} 4 - u & \text{if } u \in [0, 2) \\ 2 & \text{if } u = 2 \\ 7 & \text{if } u \in (2, \infty) \end{cases}$$

Here, we can see the definition of H_1 and H_2, that we have $H_1(u) = H_2(u) = 2$ only for $u = 2$ and $H_1H_2(u) = H_2H_1(u) = 2$, that is mappings H_1 and H_2 are weakly compatible.

Now, for $u_n = 2 - \frac{1}{n} \in (0, 2)$, we obtain,

$$\lim_{n\to\infty} H_1u_n = \lim_{n\to\infty} H_2u_n = 2$$

but, $\lim_{n\to\infty} s\left(H_1H_2u_n, \quad H_2H_1u_n\right) = \lim_{n\to\infty} \omega^{u_n} = \omega^2 \neq 1$, because $v > 1$. Hence mappings H_1 and H_2 are not compatible. Also, neither commutative nor weakly commutative, since every pair of compatible mappings must be commutative and weakly commutative.

3 3 Main Results

Theorem 3.1. *Let (Φ, s) be a complete MCM space and L be a multiplicative normal cone with multiplicative constant ψ. Let $H_1, H_2, T_1, T_2 : \Phi \to \Phi$ be four self-mappings of (Φ, s), which satisfy the following conditions:*

(1) $H_1(\Phi) \subset T_2(\Phi)$ and $H_2(\Phi) \subset T_1(\Phi)$,

(2) $s\left(H_1u, H_2v\right) \leq \{s\left(T_1u, H_1u\right) \cdot s\left(T_2v, H_2v\right)\}^{\alpha} \cdot \{s\left(T_1u, H_2v\right) \cdot s\left(T_2v, H_1u\right)\}^{\beta} \cdot s^{\gamma}\left(T_1u, T_2v\right)$ for all $u, v \in \Phi$, where $2\alpha + 2\beta + \gamma < 1$, and $\alpha, \beta, \gamma \geq 0$,

(3) (H_1, T_1) and (H_2, T_2) are pairs of weakly compatible mappings,

(4) Subspace $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$ or $T_2(\Phi)$, (one of the subspaces) is complete.

Then mappings H_1, H_2, T_1 and T_2 have a unique CFP b in Φ.

Proof. From (1), we have $H_1(\Phi) \subset T_2(\Phi)$, then for considering a point $u_0 \in \Phi$, we can have a point $u_1 \in \Phi$ such that $H_1u_0 = T_2u_1 = v_0$, since from (1) we also have $H_2(\Phi) \subset T_1(\Phi)$ then clearly for this point u_1 there has to be a point $u_2 \in \Phi$ such that $H_2u_1 = T_1u_2 = v_1$. If we continue to do so, a sequence $\{v_n\}$ in Φ can be defined as,

$$H_1u_{2n} = T_2u_{2n+1} = v_{2n} \quad \text{and } H_2u_{2n+1} = T_1u_{2n+2} = v_{2n+1} \tag{3.1}$$

Now, put $u = u_{2n}$ and $v = u_{2n+1}$ in inequality (2), we obtain

$$\begin{aligned} s\left(v_{2n}, v_{2n+1}\right) &= s\left(H_1u_{2n}, H_2u_{2n+1}\right) \\ &\leq \{s\left(T_1u_{2n}, H_1u_{2n}\right) \cdot s\left(T_2u_{2n+1}, H_2u_{2n+1}\right)\}^{\alpha} \cdot \{s\left(T_1u_{2n}, H_2u_{2n+1}\right) \cdot s\left(T_2u_{2n+1}, H_1u_{2n}\right)\}^{\beta} \\ &\ldots \gamma\left(T_1u_{2n}, T_2u_{2n+1}\right) \\ &= \{s\left(v_{2n-1}, v_{2n}\right) \cdot s\left(v_{2n}, v_{2n+1}\right)\}^{\alpha} \cdot \{s\left(v_{2n-1}, v_{2n+1}\right) \cdot s\left(v_{2n}, v_{2n}\right)\}^{\beta} \cdot s^{\gamma}\left(v_{2n-1}, v_{2n}\right) \end{aligned}$$

This implies that,

$$s\left(v_{2n}, v_{2n+1}\right) \leq \{s\left(v_{2n-1}, v_{2n}\right)\}^{\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)}} = \{s\left(v_{2n-1}, v_{2n}\right)\}^{\rho} \tag{3.2}$$

where, $\rho = \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} < 1$.

Similarly, by using (2), we obtain

$$\begin{aligned} s\left(v_{2n+1}, v_{2n+2}\right) &= s\left(H_1u_{2n+1}, H_2u_{2n+2}\right) \\ &\leq \{s(T_1u_{2n+1}, H_1u_{2n+1}) \cdot s\left(T_2u_{2n+2}\right. \\ &{}_2u_{2n+2}^{\alpha} \cdot \{s\left(T_1u_{2n+1}, H_2u_{2n+2}\right) \cdot s\left(T_2u_{2n+2}, H_1u_{2n+1}\right)\}^{\beta} \cdot s^{\gamma}\left(T_1u_{2n+1}, T_2u_{2n+2}\right) \\ &= \{s\left(v_{2n,}, v_{2n+1}\right) \cdot s\left(v_{2n+1}, v_{2n+2}\right)\}^{\alpha} \cdot \{s\left(v_{2n,}v_{2n+1}\right) \cdot s\left(v_{2n+1}, v_{2n+2}\right)\}^{\beta} \cdot s^{\gamma}\left(v_{2n,}v_{2n+1}\right) \end{aligned}$$

This implies that,

$$s\left(v_{2n+1}, v_{2n+2}\right) \leq \left\{s\left(v_{2n}, \quad v_{2n+1}\right)\right\}^{\frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)}} = \left\{s\left(v_{2n}, \quad v_{2n+1}\right)\right\}^{\rho} \tag{3.3}$$

where, $\rho = \frac{\alpha+\beta+\gamma}{1-(\alpha+\beta)} < 1$.

Hence, for all $n \in N$ from (3.2) and (3.3), we get

$$s\left(v_n, v_{n+1}\right) \leq \left\{s\left(v_{n-1}, v_n\right)\right\}^{\rho} \leq s\left(v_{n-2}, v_{n-1}\right)^{\rho^2} \leq \ldots \leq s\left(v_0, v_1\right)^{\rho^n}$$

Next, by using multiplicative triangle inequality, $\forall n, m \in N$, s.t., $n < m$, we have

$$\begin{aligned} & s\left(v_n, v_m\right) \leq s\left(v_n, v_{n+1}\right) \cdot s\left(v_{n+1}, v_{n+2}\right) \ldots s\left(v_{m-1}, v_m\right) \\ & \leq s\left(v_0, v_1\right)^{\rho^n} \cdot s\left(v_0, v_1\right)^{\rho^{n-1}} \ldots s\left(v_0, v_1\right)^{\rho^{m-1}} \\ & \leq \end{aligned}$$

Now, by the condition of multiplicative normality of cone L, we get

$$\left\|s\left(v_n, v_m\right)\right\| \leq \left\|s\left(v_0 v_1\right)\right\|^{\psi \frac{\rho^n}{1-\rho}}$$

Since $\rho < 1$ it follows that, $\lim_{n,m\to\infty} s\left(v_n, v_m\right) = 1$. Hence $\{v_n\}$ is a multiplicative Cauchy sequence in Φ.

Now, let subspace $T_1(\Phi)$ is complete. Then there has to be a point $b \in T_1(\Phi)$ s.t.,

$$\lim_{n\to\infty} v_{2n+1} = \lim_{n\to\infty} H_2 u_{2n+1} = \lim_{n\to\infty} T_1 u_{2n+2} = b \tag{3.4}$$

Consequently, we can find a point $\lambda \in \Phi$, such that $T_1\lambda = b$. Since $\{v_{2n+1}\}$ is a multiplicative convergent subsequence of a multiplicative Cauchy sequence $\{v_n\}$. Therefore, sequence $\{v_n\}$ and the subsequence $\{v_{2n}\}$ also converges, so we get,

$$\lim_{n\to\infty} v_{2n} = \lim_{n\to\infty} H_1 u_{2n} = \lim_{n\to\infty} T_2 u_{2n+1} = b \tag{3.5}$$

Now, we claim that $H_1\lambda = b$.

On putting $u = \lambda$ and $v = u_{2n+1}$ in inequality (2), we obtain

$$\begin{aligned} s\left(H_1\lambda, H_2 u_{2n+1}\right) \leq & \left\{s\left(T_1\lambda, H_1\lambda\right) \cdot s\left(T_2 u_{2n+1}, H_2 u_{2n+1}\right)\right\}^{\alpha} \cdot \left\{s\left(T_1\lambda, H_2 u_{2n+1}\right) \cdot s\left(T_2 u_{2n+1}, H_1\lambda\right)\right\}^{\beta} \\ & s^{\gamma}\left(T_1\lambda, T_2 u_{2n+1}\right) \end{aligned}$$

Letting $n \to \infty$ and using (3.4) and (3.5), we obtain

$$\begin{aligned} & s\left(H_1\lambda, b\right) \leq \left\{s\left(b, H_1\lambda\right)\right\}^{\alpha} \cdot \left\{s\left(b, H_1\lambda\right)\right\}^{\beta} \cdot s^{\gamma}(b, b) \\ & s\left(H_1\lambda, b\right) \leq \left\{s\left(H_1\lambda, \quad b\right)\right\}^{\alpha+\beta} \end{aligned}$$

This implies that, $s^{1-(\alpha+\beta)}\left(H_1\lambda, \quad b\right) = 1$ as $\alpha + \beta \neq 1$. Therefore, $H_1\lambda = b$. Hence, we have $b = H_1\lambda = T_1\lambda$ that is λ is a coincidence point of H_1 and T_1. Since $b = H_1\lambda \in H_1(\Phi) \subset T_2(\Phi)$, So clearly, there has to be a point $\mu \in \Phi$ s.t., $b = T_2\mu$.

Now, we shall show that $H_2\mu = b$.

On putting $u = u_{2n}$ and $v = \mu$ in inequality (2), we obtain

$$\begin{aligned} & s\left(H_1 u_{2n}, H_2\mu\right) \leq \left\{s\left(T_1 u_{2n}, H_1 u_{2n}\right) \cdot s\left(T_2\mu, H_2\mu\right)\right\}^{\alpha} \cdot \left\{s\left(T_1 u_{2n}, H_2\mu\right) \cdot s\left(T_2\mu, H_1 u_{2n}\right)\right\}^{\beta} \\ & \cdot s^{\gamma}\left(T_1 u_{2n}, T_2\mu\right) \end{aligned}$$

Letting $n \to \infty$ and using (3.4) and (3.5), we obtain

$$\begin{aligned} & s\left(b, H_2\mu\right) \leq \left\{s\left(b, H_2\mu\right)\right\}^{\alpha} \cdot \left\{s\left(b, H_2\mu\right)\right\}^{\beta} \cdot s^{\gamma}(b, b) \\ & s\left(b, H_2\mu\right) \leq \left\{s\left(b, H_2\mu\right)\right\}^{\alpha+\beta} \end{aligned}$$

This implies that,

$$s^{1-(\alpha+\beta)}(b, H_2\mu) = 1 \text{ as } \alpha+\beta \neq 1$$

Therefore, $H_2\mu = b$. Hence, $b = H_2\mu = T_2\mu$ that is μ is a coincidence point of mappings H_2 and T_2. Also, (H_1, T_1) and (H_2, T_2) are pairs of weakly compatible mappings, therefore we have,

$$H_1 b = H_1(T_1\lambda) = T_1(H_1\lambda) = T_1 b, \text{ and } H_2 b = H_2(T_2\mu) = T_2(H_2\mu) = T_2 b \tag{3.6}$$

Now, we prove that $H_1 b = b$.
On putting $u = b$ and $v = u_{2n+1}$ in inequality (2), we obtain

$$\begin{aligned} s(H_1 b, H_2 u_{2n+1}) \leq & \{s(T_1 b, H_1 b) \cdot s(T_2 u_{2n+1}, H_2 u_{2n+1})\}^{\alpha} \cdot \{s(T_1 b, H_2 u_{2n+1}) \cdot s(T_2 u_{2n+1}, H_1 b)\}^{\beta} \\ & s^{\gamma}(T_1 b, T_2 u_{2n+1}) \end{aligned}$$

Taking $n \to \infty$ and by using (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} & s(H_1 b, b) \leq \{s(H_1 b, H_1 b) \cdot s(b, b)\}^{\alpha} \cdot \{s(H_1 b, b) \cdot \mathrm{s}(b, H_1 b)\}^{\beta} \cdot s^{\gamma}(H_1 b, b) \\ & \mathrm{s}(H_1 b, b) \leq \{s(H_1 b, b)\}^{2\beta+\gamma} \end{aligned}$$

This implies that, $s^{1-(2\beta+\gamma)}(H_1 b, b) = 1$ as $2\beta+\gamma \neq 1$. Therefore, $H_1 b = b$. Hence, we have,

$$H_1 b = T_1 b = b \tag{3.7}$$

Next, we prove that, $H_2 b = b$.
On putting $u = u_{2n}$ and $v = b$ in inequality (2), we obtain

$$\begin{aligned} s(H_1 u_{2n}, H_2 b) \leq & \{s(T_1 u_{2n}, H_1 u_{2n}) \cdot s(T_2 b, H_2 b)\}^{\alpha} \cdot \{s(T_1 u_{2n}, H_2 b) \cdot s(T_2 b, H_1 u_{2n})\}^{\beta} \\ & s^{\gamma}(T_1 u_{2n}, T_2 b) \end{aligned}$$

Taking $n \to \infty$ and by using (3.4) , (3.5) and (3.6), we obtain

$$\begin{aligned} & s(b, H_2 b) \leq \{s(b, b) \cdot s(H_2 b, H_2 b)\}^{\alpha} \cdot \{s(b, H_2 b) \cdot s(b, b)\}^{\beta} \cdot s^{\gamma}(b, H_2 b) \\ & s(b, H_2 b) \leq \{s(b, H_2 b)\}^{2\beta+\gamma} \end{aligned}$$

This implies that, $s^{1-(2\beta+\gamma)}(b, H_2 b) = 1$ as $2\beta+\gamma \neq 1$. Therefore, $H_2 b = b$. Hence, we have,

$$H_2 b = T_2 b = b \tag{3.8}$$

Thus, from (3.7) and (3.8), we have $H_1 b = H_2 b = T_1 b = T_2 b = b$.
i.e., b is a CFP of mappings H_1, H_2, T_1, and T_2.
The proof is similar for cases in which subspace $H_1(\Phi)$ or $H_2(\Phi)$ or $T_2(\Phi)$ is complete, therefore omitted. Finally, to prove uniqueness, assume h be another CFP of mappings H_1, H_2, T_1 and T_2, then from inequality (2) we have,

$$\begin{aligned} s(b, h) &= s(H_1 b, H_2 h) \\ &\leq \{s(T_1 b, H_1 b) \cdot \mathrm{s}(T_2 h, H_2 h)\}^{\alpha} \cdot \{s(T_1 b, H_2 h) \cdot s(T_2 h, H_1 b)\}^{\beta} \cdot s^{\gamma}(T_1 b, T_2 h) \\ &= \{s(b, b) \cdot s(h, h)\}^{\alpha} \cdot \{s(b, h) \cdot s(h, b)\}^{\beta} \cdot s^{\gamma}(b, h) \\ s(b, h) &\leq \{s(b, h)\}^{2\beta+\gamma} \end{aligned}$$

i.e., we obtain $b = h$, which shows the uniqueness of CFP for all four mappings H_1, H_2, T_1, and T_2.

Example 3.1. Let $\Phi = [0, 1], K = R$, and $L = \{u \in K : u \geq 1\}$ be a multiplicative cone in K. Let for every $u, v \in \Phi$, and $\omega > 1, s(u, v) = \omega^{|u-v|}$ be a multiplicative metric on Φ, then clearly (Φ, s) is complete multiplicative cone metric space. Consider four self-mappings $H_1, H_2, T_1, T_2 : \Phi \to \Phi$ defined by,

$$H_1 u = H_2 u = \begin{cases} \frac{1}{4} & \text{if } u \in \left[0, \frac{1}{4}\right], \\ \frac{1}{6} & \text{if } u \in \left(\frac{1}{4}, 1\right] \end{cases} \quad \text{and} \quad T_1 u = T_2 u = u, \forall u \in \Phi \tag{3.9}$$

Then,
(i) $H_1(\Phi) \subset T_2(\Phi)$ and $H_2(\Phi) \subset T_1(\Phi)$, that is condition (1) holds.
(ii) Now, we shall show that the inequality (2) is satisfied.
Case (a): If $u, v \in \left[0, \frac{1}{4}\right]$. Then we have

$$s\left(H_1u, H_2v\right) = s\left(\frac{1}{4}, \frac{1}{4}\right) = \omega^{\left|\frac{1}{4}-\frac{1}{4}\right|} = 1$$

Hence, inequality (2) holds in this case.
Case (b): If $u, v \in \left(\frac{1}{4}, 1\right)$. Then we have

$$s\left(H_1u, H_2v\right) = s\left(\frac{1}{6}, \frac{1}{6}\right) = \omega^{\left|\frac{1}{6}-\frac{1}{6}\right|} = 1$$

Hence, inequality (2) holds in this case.
Case (c): If $u \in \left[0, \frac{1}{4}\right]$ and $v \in \left(\frac{1}{4}, 1\right]$. Then we derive

$$s\left(H_1u, H_2v\right) = s\left(\frac{1}{4}, \frac{1}{6}\right) = \omega^{\left|\frac{1}{4}-\frac{1}{6}\right|} = \omega^{\frac{1}{12}}, s\left(T_1u, H_1u\right) = s\left(u, \frac{1}{4}\right) = \omega^{\left|u-\frac{1}{4}\right|}$$
$$s\left(T_2v, H_2v\right) = s\left(v, \frac{1}{6}\right) = \omega^{\left|v-\frac{1}{6}\right|}, s\left(T_1u, H_2v\right) = s\left(u, \frac{1}{6}\right) = \omega^{\left|u-\frac{1}{6}\right|}$$
$$s\left(T_2v, H_1u\right) = s\left(v, \frac{1}{4}\right) = \omega^{\left|v-\frac{1}{4}\right|}, s\left(T_1u, T_2v\right) = s(u, v) = \omega^{|u-v|}$$

Hence, for $\alpha = \frac{1}{5}, \beta = \frac{1}{5}$, and $\gamma = \frac{1}{10}$, we get

$$\begin{aligned}
&\left\{s\left(T_1u, H_1u\right) \cdot s\left(T_2v, H_2v\right)\right\}^{\alpha} \cdot \left\{s\left(T_1u, H_2v\right) \cdot s\left(T_2v, H_1u\right)\right\}^{\beta} \cdot s^{\gamma}\left(T_1u, T_2v\right) \\
&= \omega^{\alpha\left|u-\frac{1}{4}\right|+\alpha\left|v-\frac{1}{6}\right|+\beta\left|u-\frac{1}{6}\right|+\beta\left|v-\frac{1}{4}\right|+\gamma|u-v|} \\
&\geq \omega^{\alpha\left|u-\frac{1}{4}\right|+\alpha\left|v-\frac{1}{6}\right|+\beta\left|u-\frac{1}{4}\right|+\beta\left|\frac{1}{4}-\frac{1}{6}\right|+\beta\left|v-\frac{1}{6}\right|+\beta\left|\frac{1}{6}-\frac{1}{4}\right|+\gamma\left|u-\frac{1}{4}\right|+\gamma\left|\frac{1}{4}-v\right|} \\
&\geq \omega^{\alpha\left|u-\frac{1}{4}\right|+\alpha\left|v-\frac{1}{6}\right|+\beta\left|u-\frac{1}{4}\right|+\beta\left|\frac{1}{12}\right|+\beta\left|v-\frac{1}{6}\right|+\beta\left|\frac{1}{12}\right|+\gamma\left|u-\frac{1}{4}\right|+\gamma\left|v-\frac{1}{6}\right|+\gamma\left|\frac{1}{6}-\frac{1}{4}\right|} \\
&= \omega^{(\alpha+\beta+\gamma)\left|u-\frac{1}{4}\right|+(\alpha+\beta+\gamma)\left|v-\frac{1}{6}\right|+2\beta\left|\frac{1}{12}\right|+\gamma\left|\frac{1}{12}\right|} \\
&\geq \omega^{\frac{1}{12}} = s\left(H_1u, H_2v\right)
\end{aligned}$$

Hence, inequality (2) holds in this case.
Case (d): If $v \in \left[0, \frac{1}{4}\right]$ and $u \in \left(\frac{1}{4}, 1\right]$. Then we derive

$$s\left(H_1u, H_2v\right) = s\left(\frac{1}{6}, \frac{1}{4}\right) = \omega^{\left|\frac{1}{6}-\frac{1}{4}\right|} = \omega^{\frac{1}{12}}, s\left(T_1u, H_1u\right) = s\left(u, \frac{1}{6}\right) = \omega^{\left|u-\frac{1}{6}\right|}$$
$$s\left(T_2v, H_2v\right) = s\left(v, \frac{1}{4}\right) = \omega^{\left|v-\frac{1}{4}\right|}, s\left(T_1u, H_2v\right) = s\left(u, \frac{1}{4}\right) = \omega^{\left|u-\frac{1}{4}\right|}$$
$$s\left(T_2v, H_1u\right) = s\left(v, \frac{1}{6}\right) = \omega^{\left|v-\frac{1}{6}\right|}, s\left(T_1u, T_2v\right) = s(u, v) = \omega^{|u-v|}$$

Hence, for $\alpha = \frac{1}{5}, \beta = \frac{1}{5}$, and $\gamma = \frac{1}{10}$, we get

$$\begin{aligned}
&\left\{s\left(T_1u, H_1u\right) \cdot s\left(T_2v, H_2v\right)\right\}^{\alpha} \cdot \left\{s\left(T_1u, H_2v\right) \cdot s\left(T_2v, H_1u\right)\right\}^{\beta} \cdot s^{\gamma}\left(T_1u, T_2v\right) \\
&= \omega^{\alpha\left|u-\frac{1}{6}\right|+\alpha\left|v-\frac{1}{4}\right|+\beta\left|u-\frac{1}{4}\right|+\beta\left|v-\frac{1}{6}\right|+\gamma|u-v|} \\
&\geq \omega^{\alpha\left|u-\frac{1}{6}\right|+\alpha\left|v-\frac{1}{4}\right|+\beta\left|u-\frac{1}{6}\right|+\beta\left|\frac{1}{6}-\frac{1}{4}\right|+\beta\left|v-\frac{1}{4}\right|+\beta\left|\frac{1}{4}-\frac{1}{6}\right|+\gamma\left|u-\frac{1}{6}\right|+\gamma\left|v-\frac{1}{6}\right|} \\
&\geq \omega^{\alpha}\left|u-\frac{1}{6}\right|+\alpha|v-4|+\beta\left|u-\frac{1}{6}\right|+\beta\left|\frac{1}{12}\right|+\beta\left|v-\frac{1}{4}\right|+\beta\left|\frac{1}{12}\right|+\gamma\left|u-\frac{1}{6}\right|+\gamma\left|v-\frac{1}{4}\right|+\gamma\left|\frac{1}{4}-\frac{1}{6}\right| \\
&= \omega^{(\alpha+\beta+\gamma)\left|u-\frac{1}{6}\right|+(\alpha+\beta+\gamma)\left|v-\frac{1}{4}\right|+2\beta\left|\frac{1}{12}\right|+\gamma\left|\frac{1}{12}\right|} \\
&\geq \omega^{\frac{1}{12}} = s\left(H_1u, H_2v\right)
\end{aligned}$$

Hence, inequality (2) holds in this case also.

(iii) We can see clearly from (3.9) that (H_1, T_1) and (H_2, T_2) are pairs of w-compatible mappings as both the pairs commute at their coincidence point $\frac{1}{4}$ i.e.,

$$H_1\left(\frac{1}{4}\right) = T_1\left(\frac{1}{4}\right) = \frac{1}{4} \text{ implies that } H_1T_1\left(\frac{1}{4}\right) = T_1H_1\left(\frac{1}{4}\right) = \frac{1}{4}$$
$$H_2\left(\frac{1}{4}\right) = T_2\left(\frac{1}{4}\right) = \frac{1}{4} \text{ implies that } H_2T_2\left(\frac{1}{4}\right) = T_2H_2\left(\frac{1}{4}\right) = \frac{1}{4}$$

(iv) It is clear from (3.9), that one of the subspaces $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$ or $T_2(\Phi)$ is complete.

Hence, all four conditions of the Theorem 3.1, are fulfilled. Also, we have $H_1\left(\frac{1}{4}\right) = H_2\left(\frac{1}{4}\right) = T_1\left(\frac{1}{4}\right) = T_2\left(\frac{1}{4}\right) = \frac{1}{4}$ i.e., mappings H_1, H_2, T_1, and T_2 have the unique $CFP \quad b = \frac{1}{4}$.

Corollary 3.1. *Let $H_1, H_2, T_1 : \Phi \to \Phi$ are three self-mappings of complete MCM space (Φ, s) with multiplicative normal cone L, satisfying the following conditions:*

(i) $H_1(\Phi) \subset T_1(\Phi)$ and $H_2(\Phi) \subset T_1(\Phi)$,

(ii) $s(H_1u, H_2v) \leq \{s(T_1u, H_1u) \cdot s(T_1v, H_2v)\}^{\alpha} \cdot \{s(T_1u, H_2v) \cdot s(T_1v, H_1u)\}^{\beta} \cdot s^{\gamma}(T_1u, T_1v)$ for all $u, v \in \Phi$, where $2\alpha + 2\beta + \gamma < 1$, and $\alpha, \beta, \gamma \geq 0$,

(iii) (H_1, T_1) and (H_2, T_1) are pairs of weakly compatible mappings,

(iv) Subspace $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$, (One of the subspaces) is complete.

Then mappings H_1, H_2, and T_1 have a unique CFP b in Φ.

Proof. On putting $T_1 = T_2$ in Theorem 3.1, as a result, we get this corollary. □

Corollary 3.2. *Let $H_1, H_2 : \Phi \to \Phi$ are two self-mappings of complete MCM space (Φ, s) with multiplicative normal cone L, satisfying the following conditions:*

(i) $H_1(\Phi) \subset H_2(\Phi)$,

(ii) $s(H_1u, H_2v) \leq \{s(H_1u, H_2u) \cdot s(H_1v, H_2v)\}^{\alpha} \cdot \{s(H_1u, H_2v) \cdot s(H_1v, H_2u)\}^{\beta} \cdot s^{\gamma}(H_2u, H_2v)$ for all $u, v \in \Phi$, where $2\alpha + 2\beta + \gamma < 1$, and $\alpha, \beta, \gamma \geq 0$,

(iii) (H_1, H_2) is a pair of weakly compatible mappings,

(iv) Subspace $H_1(\Phi)$ or $H_2(\Phi)$, (one of the subspaces) is complete.

Then mappings H_1 and H_2 have a unique CFP b in Φ.

Proof. In the Theorem 3.1, if we put $T_1 = T_2 = I_{\Phi}$, (where I_{Φ} is the identity mapping on Φ), we get this result.

Next, we prove the following theorem to drop the condition of completeness of subspaces as follows. □

Theorem 3.2. *Let (Φ, s) be a complete MCM space and L be a multiplicative normal cone with multiplicative constant ψ. Let $H_1, H_2, T_1, T_2 : \Phi \to \Phi$ be four self-mappings of (Φ, s), which satisfy conditions (1) - (3) of the Theorem 3.1 and one of the subspaces $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$ or $T_2(\Phi)$ is a closed subset of Φ. Then mappings H_1, H_2, T_1, and T_2 have a unique CFP b in Φ.*

Proof. From (1), we have $H_1(\Phi) \subset T_2(\Phi)$, then for considering a point $u_0 \in \Phi$, we can have a point $u_1 \in \Phi$ such that $H_1u_0 = T_2u_1 = v_0$, since from (1) we also have $H_2(\Phi) \subset T_1(\Phi)$ then clearly for this point u_1 there has to be a point $u_2 \in \Phi$ such that $H_2u_1 = T_1u_2 = v_1$. If we continue to do so, a sequence $\{v_n\}$ in Φ can be defined as,

$$H_1u_{2n} = T_2u_{2n+1} = v_{2n} \quad \text{and } H_2u_{2n+1} = T_1u_{2n+2} = v_{2n+1} \tag{3.10}$$

Then, from the Theorem 3.1, sequence $\{v_n\}$ is a multiplicative Cauchy sequence in Φ. Also, by multiplicative completeness of Φ, we have a point $b \in \Phi$ s.t., $\lim_{n\to\infty} v_n = b$.

Therefore, all the sub-sequences of $\{H_1u_{2n}\}, \{T_2u_{2n+1}\}, \{H_2u_{2n+1}\}$, and $\{T_1u_{2n+2}\}$ of sequence $\{v_n\}$ also multiplicative converges to $b \in \Phi$.

Now, let $T_2(\Phi)$ is a closed subset of Φ, therefore, we have $T_2\eta = b$ for $\eta \in \Phi$.

Subsequently, we have,

$$\lim_{n\to\infty} H_1u_{2n} = \lim_{n\to\infty} H_2u_{2n+1} = \lim_{n\to\infty} T_1u_{2n+2} = \lim_{n\to\infty} T_2u_{2n+1} = b = T_2\eta \tag{3.11}$$

Now, we claim that $H_2\eta = b$.
On putting $u = u_{2n}$ and $v = \eta$ in inequality (2), we obtain

$$s\left(H_1u_{2n}, H_2\eta\right) \leq \left\{s\left(T_1u_{2n}, H_1u_{2n}\right) \cdot s\left(T_2\eta, H_2\eta\right)\right\}^{\alpha} \cdot \left\{s\left(T_1u_{2n}, H_2\eta\right) \cdot s\left(T_2\eta, H_1u_{2n}\right)\right\}^{\beta}$$
$$s^{\gamma}\left(T_1u_{2n}, T_2\eta\right)$$

Letting $n \to \infty$ and using (3.11), we get

$$s\left(b, H_2\eta\right) \leq \left\{s\left(b, H_2\eta\right)\right\}^{\alpha} \cdot \left\{s\left(b, H_2\eta\right)\right\}^{\beta} \cdot s^{\gamma}(b, b)$$
$$s\left(b, H_2\eta\right) \leq s^{\alpha+\beta}\left(b, H_2\eta\right)$$

This implies that, $s^{1-(\alpha+\beta)}\left(b, H_2\eta\right) = 1$ as $\alpha + \beta \neq 1$. Therefore, $H_2\eta = b$.
Hence, we have,

$$b = T_2\eta = H_2\eta \tag{3.12}$$

Since $b = H_2\eta \in H_2(\Phi) \subset T_1(\Phi)$, so there has to be a point $\kappa \in \Phi$ s.t.,

$$b = H_2\eta = T_1\kappa \tag{3.13}$$

Now, we claim that $H_1\kappa = b$.
On putting $u = \kappa$ and $v = \eta$ in inequality (2) and using (3.12) and (3.13), we obtain

$$s\left(H_1\kappa, H_2\eta\right) \leq \left\{s\left(T_1\kappa, H_1\kappa\right) \cdot s\left(T_2\eta, H_2\eta\right)\right\}^{\alpha} \cdot \left\{s\left(T_1\kappa, \quad H_2\eta\right) \cdot s\left(T_2\eta, \quad H_1\kappa\right)\right\}^{\beta} \cdot s^{\gamma}\left(T_1\kappa, T_2\eta\right)$$

$$s\left(H_1\kappa, b\right) \leq \left\{s\left(b, H_1\kappa\right)\right\}^{\alpha} \cdot \left\{s\left(b, H_1\kappa\right)\right\}^{\beta} \cdot s^{\gamma}(b, b)$$
$$s\left(H_1\kappa, b\right) \leq s^{\alpha+\beta}\left(H_1\kappa, b\right)$$

This implies that, $s^{1-(\alpha+\beta)}\left(H_1\kappa, \quad b\right) = 1$ as $\alpha + \beta \neq 1$. Therefore, $H_1\kappa = b$.
Hence, we have,

$$b = H_1\kappa = T_1\kappa \tag{3.14}$$

Since (H_1, T_1) and (H_2, T_2) are pairs of weakly compatible mappings and clearly from (3.12) and (3.14), κ and η are their coincidence point respectively, therefore we have,

$$H_1b = H_1\left(T_1\kappa\right) = T_1\left(H_1\kappa\right) = T_1b, \text{ and } H_2b = H_2\left(T_2\eta\right) = T_2\left(H_2\eta\right) = T_2b \tag{3.15}$$

Now, on putting $u = \kappa$ and $v = b$ in inequality (2) and using (3.14) and (3.15), we derive

$$s\left(H_1\kappa, H_2b\right) \leq \left\{s\left(T_1\kappa, H_1\kappa\right) \cdot s\left(T_2b, H_2b\right)\right\}^{\alpha} \cdot \left\{s\left(T_1\kappa, H_2b\right) \cdot s\left(T_2b, H_1\kappa\right)\right\}^{\beta} \cdot s^{\gamma}\left(T_1\kappa, T_2b\right)$$
$$s\left(H_1\kappa, H_2b\right) \leq \left\{s\left(H_1\kappa, H_1\kappa\right) \cdot s\left(H_2b, H_2b\right)\right\}^{\alpha} \cdot \left\{s\left(H_1\kappa, H_2b\right) \cdot s\left(H_2b, H_1\kappa\right)\right\}^{\beta} \cdot s^{\gamma}\left(H_1\kappa, H_2b\right)$$
$$s\left(H_1\kappa, H_2b\right) \leq s^{2\beta+\gamma}\left(H_1\kappa, H_2b\right)$$

This implies that, $s^{1-(2\beta+\gamma)}\left(H_1\kappa, \quad H_2b\right) = 1$ as $2\beta + \gamma \neq 1$. Therefore, $H_1\kappa = H_2b$.
Hence, we have, $b = H_1\kappa = H_2b$. Then from (3.15), we get

$$b = H_2b = T_2b \tag{3.16}$$

Thus b is a CFP of H_2 and T_2. Next, we show that b is also a CFP of H_1 and T_1.
On putting $u = b$ and $v = \eta$ in inequality (2) and using (3.12) and (3.15), we get

$$s\left(H_1b, H_2\eta\right) \leq \left\{s\left(T_1b, H_1b\right) \cdot s\left(T_2\eta, H_2\eta\right)\right\}^{\alpha} \cdot \left\{s\left(T_1b, H_2\eta\right) \cdot s\left(T_2\eta, H_1b\right)\right\}^{\beta} \cdot s^{\gamma}\left(T_1b, T_2\eta\right)$$
$$s\left(H_1b, b\right) \leq \left\{s\left(H_1b, H_1b\right) \cdot s(b, b)\right\}^{\alpha} \cdot \left\{s\left(H_1b, b\right) \cdot s\left(b, H_1b\right)\right\}^{\beta} \cdot s^{\gamma}\left(H_1b, b\right)$$
$$s\left(H_1b, b\right) \leq s^{2\beta+\gamma}\left(H_1b, b\right)$$

This implies that, $s^{1-(2\beta+\gamma)}(H_1b, \quad b) = 1$ as $2\beta + \gamma \neq 1$. Therefore, $H_1b = b$, so from (3.15), we get

$$H_1b = T_1b = b \tag{3.17}$$

Thus b is also a CFP of H_1 and T_1.
Thus, from (3.16) and (3.17), we have

$$H_1b = H_2b = T_1b = T_2b = b$$

i.e., b is a CFP of all four mappings H_1, H_2, T_1, and T_2.

The proof is similar for cases in which subspace $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$ is closed and therefore omitted. Uniqueness: From inequality (2) we can see easily. □

Example 3.2. In Example 3.1, we have one of the subspaces $H_1(\Phi)$ or $H_2(\Phi)$ or $T_1(\Phi)$ or $T_2(\Phi)$ is complete, therefore it has to be closed. Since "the subspace of a complete multiplicative cone metric space is complete if and only if it is closed." Hence, all the conditions of the Theorem 3.2 are fulfilled, and $b = \frac{1}{4}$ is a unique CFP for all four mappings H_1, H_2, T_1, and T_2 i.e., $H_1\left(\frac{1}{4}\right) = H_2\left(\frac{1}{4}\right) = T_1\left(\frac{1}{4}\right) = T_2\left(\frac{1}{4}\right) = \frac{1}{4}$.

4 Conclusion

In this paper, we introduce the weak compatibility conception to multiplicative cone metric space, which is the wider class of mappings in all other existing commutativity concepts, and by using properties of these mappings with a new type of contractive condition and without using the notion of continuity of mappings, we have proved some CFP theorems for two pairs of w-compatible maps. Our presented results generalize numerous prevailing fixed-point results in the literature and also extend the scope of study of CFP theorems in multiplicative cone metric space.

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