STUDY ON MEAN VALUES OF AN ENTIRE FUNCTION AND ITS DERIVATIVES

By

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Abstract

In this paper, we investigate mean values of an entire function and its derivative in the usual notation. It is obvious that generally \(\lambda_1 \leq \lambda\) and \(\rho_1 \leq \rho\). There are entire Dirichlet’s series for which \(\lambda_1 < \lambda\) and \(\rho_1 < \rho\). We have generally to distinguish between the limits as well as types of \(f(s)\) belonging to the same order \(\rho_1\) \((0 < \rho_1 < \infty)\). We obtain some results of mean value \(I_2(\sigma)\) for the mean value of an entire Dirichlet’s series and its derivative.

In this paper we shall establish some interesting results in terms of theorems on mean values of an entire function and its derivatives.


Keywords and Phrases: Generalized order \((\rho)\), Generalized lower order \((\lambda)\), Type of the function \((t)\), Lower type of function \((t)\), Mean value \(I_2(\sigma)\).

1 Introduction


\[
\lim_{\sigma \to \infty} \inf \frac{\sigma \log M_S(\sigma)}{\sigma} = \frac{\rho}{\lambda}.
\]

(1.1)

Hardy [4] and Bergweiler [2] investigated the mean function of the modulus of an analytic function and studied some of its properties. This led to various persons to study the mean functions of the Riemann’s Zeta function [5] and entire function defined by Taylor series. For entire functions \(f \in E\), Gupta [3] investigated the \(r\)th mean function \(I_r, r \in R_+,\) as

\[
I_r(\sigma, f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(s + it)|^r ds, \forall \sigma < \sigma_c
\]

(1.2)

and investigated into a few properties of \(I_r\) for \(r = 2\). The function \(I_2\) is referred to as the quadratic mean function \(f\).

\[
I_2(\sigma, f) = \sum_{n \in N} |a_n|^2 e^{2\sigma \lambda_n}, \forall \sigma < \sigma_c.
\]

(1.3)

Define

\[
I_2(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(s)|^2 ds.
\]

(1.4)

Then due to Taylor [13] for all \(\sigma < \infty\), we have

\[
I_2(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma \lambda_n}.
\]

(1.5)

Also let for any \(k(0 < k < \infty),\)

\[
m_k(\sigma) = \lim_{T \to \infty} \frac{1}{2Te^{k\sigma}} \int_{-\infty}^{\sigma} \int_{-T}^{T} |f(x + it)|^2 e^{kx} dx dt, 0 < k < \infty
\]

(1.6)
Let us also define
\[ m_{2,k}(\sigma, f^{(m)}) = \frac{1}{e^{k\sigma}} \int_{-\infty}^{\sigma} A_{2}(x, f^{(m)}) e^{kx} dx \]
\[ = \lim_{T \to \infty} \frac{1}{2T} e^{k\sigma} \int_{-\infty}^{\sigma} \int_{-T}^{T} |f^{(m)}(x + it)|^{2} e^{kx} dx dt, 0 < k < \infty \]
where \( m = 1, 2, 3, \ldots \).

2 Some Useful Lemmas
In this section, we shall prove some Lemmas which will be useful to establish main interesting theorems.

**Lemma 2.1.** Given \( \varepsilon = \varepsilon(\sigma) > 0 \) which may be a constant, we have
\[ m_{2,k}(\sigma, f^{(1)}) \leq \frac{m_{2,k}(\sigma + \varepsilon)}{\varepsilon^{2}}. \] (2.1)

**Proof.** By the definitions, we have
\[ A_{2}(\sigma) = \left( \sum_{n=1}^{\infty} |a_n|^{2} e^{2\sigma \lambda_n} \right) \text{(Parseval's inequality),} \] (2.2)
\[ m_{2,k}(\sigma) = \sum_{n=1}^{\infty} |a_n|^{2} e^{2\sigma \lambda_n}. \] (2.3)

Similarly, it can be shown that
\[ m_{2,k}(\sigma, f^{(1)}) = \sum_{n=1}^{\infty} \frac{\lambda_n^2 |a_n|^{2} e^{2\sigma \lambda_n}}{2\lambda_n + k}. \] (2.4)

By using the fact that \( \lambda_n^2 \leq \frac{e^{2\lambda_n \sigma}}{\varepsilon^2} \) which is always true, we obtain
\[ m_{2,k}(\sigma, f^{(1)}) \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{|a_n|^{2} e^{2\lambda_n (\sigma + \varepsilon)}}{2\lambda_n + k} \leq \frac{1}{\varepsilon^2} m_{2,k}(\sigma + \varepsilon). \] (2.5)

Hence the result. \( \square \)

**Lemma 2.2.** There exists a monotonic increasing function \( S(\sigma) \) associated with \( m_{2,k}(\sigma) \) such that
\[ \log m_{2,k}(\sigma) = \log m_{2,k}(\sigma) + \int_{\sigma_0}^{\sigma} S(x) dx \] (2.6)
\[ \frac{\rho_{1}}{\lambda_{1}} \leq \lim_{\sigma \to \infty} \sup_{f} \frac{\log(\sigma)}{\sigma} \leq \frac{\rho}{\lambda}. \] (2.7)

**Proof.** First assertion follows from the fact that \( \log m_{2,k}(\sigma) \) is a convex function of \( \sigma \).

To prove (2.7), we get from (2.6) successively
\[ S(\sigma) \leq \int_{\sigma}^{\sigma+1} S(x) dx = \log m_{2,k}(\sigma + 1) - \log m_{2,k}(\sigma) < \log m_{2,k}(\sigma + 1). \] (2.8)

This leads to
\[ \lim_{\sigma \to \infty} \sup_{f} \frac{\log S(\sigma)}{\sigma} \leq \lim_{\sigma \to \infty} \sup_{f} \frac{\log m_{2,k}(\sigma + 1)(\sigma+1)}{\sigma+1} \leq \frac{\rho}{\lambda}. \] (2.9)

From (2.6), we get
\[ \log m_{2,k}(\sigma) = \log m_{2,k}(\sigma_0) + \int_{\sigma_0}^{\sigma} S(x) dx \]
\[ \leq \log m_{2,k}(\sigma_0) + (\sigma - \sigma_0) S(\sigma) \]
\[ \sim \sigma S(\sigma), \quad \sigma \to \infty. \]

Taking the logarithm on both the sides and proceeding to limits, we get
\[ \lim_{\sigma \to \infty} \sup_{f} \frac{\log m_{2,k}(\sigma)}{\sigma} \leq \lim_{\sigma \to \infty} \sup_{f} \frac{\log S(\sigma)}{\sigma}. \]

Combining this
\[ \frac{\rho_{1}}{\lambda_{1}} \leq \lim_{\sigma \to \infty} \sup_{f} \frac{\log m_{2,k}(\sigma)}{\sigma} \leq \frac{\rho}{\lambda}, \] (2.10)

we have
\[ \frac{\rho_{1}}{\lambda_{1}} \leq \lim_{\sigma \to \infty} \sup_{f} \frac{\log S(\sigma)}{\sigma} \] (2.10)

The inequalities (2.9) and (2.10) together give (2.7). \( \square \)
3 Main Theorems

Theorem 3.1. If \( f(s) \) is an entire function, then

\[
\frac{\rho_1}{\lambda_1} \leq \frac{\log \frac{m_{2,k}(\sigma)}{m_{2,k}(\sigma,f)}}{\sigma} \leq \frac{\rho}{\lambda}.
\]

(3.1)

In particular case, when \( \{\lambda_n\} \) satisfies the condition, we have

\[
\lim_{n \to \infty} \sup \frac{\log \frac{m_{2,k}(\sigma,f^{(1)})}{m_{2,k}(\sigma,f)}}{\sigma} \leq \frac{\rho_1}{\lambda_1}.
\]

(3.2)

In fact, Lahiri and Banerjee [8], Ritt order [10] for the truth of "limsup" part of the above conclusion the following appropriate condition is sufficient,

\[
\lim_{n \to \infty} \frac{\log n}{\lambda_n \log \lambda_n} = 0.
\]

(3.3)

Proof. \( m_{2,k}(\sigma,f^{(1)}) = \lim_{T \to \infty} \frac{1}{2T^{2}e^{k\sigma}} \int_{-T}^{T} f(1)(x + it)^{2} e^{kx} dx dt \)

\[
= \lim_{T \to \infty} \frac{1}{2T^{2}e^{k\sigma}} \sigma \int_{-\infty}^{\infty} \int_{-T}^{T} \left| \frac{f(x + it) - f(x - \epsilon x + it)}{\epsilon x} \right|^{2} e^{kx} dx dt
\]

\[
\geq \lim_{T \to \infty} \frac{1}{2T^{2}e^{k\sigma}} \sigma \int_{-\infty}^{\infty} \int_{-T}^{T} \left| \frac{f(x + it) - f(x - \epsilon x + it)}{\epsilon x} \right|^{2} e^{kx} dx dt.
\]

Now, by Minkowski's inequality

\[
\left\{ \int_{-T}^{T} \left( |f(x + it)| - |f(x - \epsilon x + it)| \right)^{2} \right\}^{\frac{1}{2}}
\]

\[
\geq \left\{ \left( \int_{-T}^{T} |f(x + it)|^{2} dt \right)^{\frac{1}{2}} - \left( \int_{-T}^{T} |f(x - \epsilon x + it)|^{2} dt \right)^{\frac{1}{2}} \right\}.
\]

Hence,

\[
m_{2,k}(\sigma,f^{(1)})
\]

\[
\geq \lim_{T \to \infty} \sigma \int_{-\infty}^{\infty} \int_{-T}^{T} \left\{ \left( \int_{-T}^{T} |f(x + it)|^{2} e^{kx} dx dt \right)^{\frac{1}{2}} - \left( \int_{-T}^{T} |f(x - \epsilon x + it)|^{2} e^{kx} dx dt \right)^{\frac{1}{2}} \right\}^{2} e^{k\sigma} dx
\]

Again, by using Minkowski's inequality, we have

\[
m_{2,k}(\sigma,f^{(1)}) \geq \lim_{T \to \infty} \lim_{\epsilon \to 0} \frac{1}{2T^{2}e^{k\sigma}} \sigma \int_{-\infty}^{\infty} \int_{-T}^{T} \left( |f(x + it)|^{2} e^{kx} dx dt \right)^{\frac{1}{2}}
\]

\[
- \left( \int_{-\infty}^{\infty} \int_{-T}^{T} |f(x - \epsilon x + it)|^{2} e^{kx} dx dt \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} \int_{-T}^{T} |f(x - \epsilon x + it)|^{2} e^{kx} dx dt \right)^{\frac{1}{2}} \right\}
\]

\[
\geq \lim_{\epsilon \to 0} \left\{ \frac{m_{2,k}(\sigma)}{\sigma} \right\}^{\frac{1}{2}} - \frac{m_{2,k}(\sigma - \epsilon)}{\epsilon} \right\}^{\frac{1}{2}} \left( \frac{m_{2,k}(\sigma)}{\sigma} \right)^{\frac{1}{2}}.
\]

Now let, \( H(\sigma) = \frac{\log m_{2,k}(\sigma)}{\sigma} \), then since \( \log m_{2,k}(\sigma) \) is steadily increasing convex function of \( \sigma \) for \( \sigma > \sigma_0 \), it follows that \( H(\sigma) \) is a positive increasing function of \( \sigma \) and also defined as,
\[ m_{2,k} \left( \sigma, f^{(1)} \right) \geq \lim_{\varepsilon \to 0} \left\{ \frac{e^{\frac{\sigma H(\sigma)}{2}} - e^{-\frac{\sigma H(\sigma - \varepsilon)}{2}}}{\varepsilon \sigma} \right\}^2 \]
\[ \geq e^{\sigma H(\sigma)} \left( \frac{H(\sigma)}{2} \right)^2 \]
\[ = m_{2,k}(\sigma) \left\{ \log \frac{m_{2,k}(\sigma)}{2\sigma} \right\}^2. \tag{3.4} \]

From this we have, for all large
\[ \log \frac{m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} \geq \log \log \frac{m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} - \log \frac{2\sigma}{\sigma}, \tag{3.5} \]
\[ \lim_{\sigma \to \infty} \sup \frac{\log m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} \geq \frac{\rho_1}{\lambda}. \tag{3.6} \]
in virtue of left half of inequality of (3.1).

Next, we consider the inverse of above inequality. Lemma 2.1 leads to
\[ \log m_{2,k} \left( \sigma, f^{(1)} \right) \leq \log m_{2,k}(\sigma) + \log \frac{1}{\varepsilon^2}, \]
which with (2.6), gives.
\[ \log m_{2,k} \left( \sigma, f^{(1)} \right) \leq \log m_{2,k}(\sigma) + \int_{\sigma}^{\sigma + \varepsilon} S(x) dx + \log \frac{1}{\varepsilon^2} \]
\[ \leq \log m_{2,k}(\sigma) + \varepsilon S(\sigma + \varepsilon) + 2 \log \frac{1}{\varepsilon^2}. \tag{3.7} \]

In (3.7) we choose \( \varepsilon \) in terms of \( \sigma \) as follows. Let the expression in (3.7), be considered as function of \( \varepsilon \) in first instance (with \( \sigma \) fixed for the time being). This expression, for varying \( \varepsilon \), is least when.
\[ S(\sigma + \varepsilon) - \frac{z}{\varepsilon} = 0. \tag{3.8} \]

Suppose \( \varepsilon \) be chosen to satisfy (3.8). Then \( S(\sigma) \) being a monotonic increasing function of \( \sigma, \varepsilon \) satisfied further
\[ \frac{2}{\varepsilon} \geq S(\sigma) \to \infty, (\sigma \to \infty). \]

Also, in view of (3.7), we get
\[ \lim_{\sigma \to \infty} \sup_{\infty} \frac{\log m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} \leq \lim_{\sigma \to \infty} \sup_{\infty} \frac{\log S(\sigma + \varepsilon) (\sigma + \varepsilon)}{\sigma + \varepsilon} \leq \lim_{\sigma \to \infty} \sup_{\infty} \frac{\log S(\sigma)}{\sigma}. \tag{3.9} \]
Combining (2.7) and (3.9), we have
\[ \lim_{\sigma \to \infty} \sup_{\infty} \frac{\log m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} \leq \frac{\rho}{\lambda}. \tag{3.10} \]
Finally, combining (3.6) and (3.10), we obtain
\[ \frac{\rho_1}{\lambda} \leq \lim_{\rho \to \infty} \sup_{\infty} \frac{\log m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma, f)} \leq \frac{\rho}{\lambda}. \]

Thus, we have completed the proof of the theorem.

\[ \textbf{Theorem 3.2.} \text{ If } 0 \leq \rho < \infty, \text{ then } \log m_{2,k}(\sigma) \sim \log m_{2,k} \left( \sigma, f^{(1)} \right). \]
Proof. From (2.6), we have
\[ \frac{\sigma}{2} S\left(\frac{\sigma}{2}\right) \leq \int_{\frac{\sigma}{2}}^{\sigma} S(x) dx = \log m_{2,k}(\sigma) - \log m_{2,k}\left(\frac{\sigma}{2}\right) < \log m_{2,k}(\sigma) \]
or
\[ \frac{\log m_{2,k}(\sigma)}{\sigma} > \frac{1}{2} S\left(\frac{\sigma}{2}\right) \to \infty. \quad (3.11) \]

Since, in case $S\left(\frac{\sigma}{2}\right)$, which is monotonic increasing has finite limit as $\sigma \to \infty$, the series for $m_{2,k}(\sigma)$ must have a finite number of terms, as also the series for $f(s)$.

To prove the theorem, we have only to note that (3.1) gives us, for any small $\varepsilon > 0$ and sufficiently large $\sigma$,
\[ \frac{2(\lambda_s - \varepsilon) \sigma}{\log m_{2,k}(\sigma)} \leq \frac{\log m_{2,k}(\sigma, f(1))}{\log m_{2,k}(\sigma)} - 1 < \frac{2(\sigma - \varepsilon)}{\log m_{2,k}(\sigma)}. \]

Since, by (3.11), the extreme members tend to 0 as $\sigma \to \infty$, the middle member also tends to 0 which proves the theorem.

4 Conclusion
Theorem 3.2 is not only more general form of Juneja and Awasthi [6], but has a proof different from that as well is shorter and more widely applicable.

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References