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TWO CLASSES OF TWO DIMENSIONAL MIXED GEGENBAUER-LEGENDRE POLYNOMIALS TO APPLY IN COMPUTATION OF THE REGION OF CONVERGENCE OF ARBITRARY FUNCTION CONTAINING THEM

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Abstract

In this article we introduce interesting two dimensional formulae of Legendre and Gegenbauer polynomials and then study their analytic and algebraic properties to derive some known and unknown results. Again we define two classes of two dimensional Gegenbauer-Legendre polynomials to obtain their series formulae. Finally, we use these results in the computation of the region of convergence of arbitrary function consisting of Gegenbauer-Legendre mixed polynomials.
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1 Introduction

The Legendre polynomials are used in various scientific and potential problems for example in ([4, p. 157, Sec. 86, Eqn. (1)], [7, p. 83, Eqn. (8)]) and presented by following generating function

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)t^n = H_{1/2}(x,t) \text{ (let)}, \ |t|<1, \forall x \in [-1,1]. \tag{1.1}$$

From (1.1), first few Legendre polynomials are found as [4, p. 160]

$$P_0(x)=1, P_1(x)=x, P_2(x)=\frac{1}{2}\left(3x^2-1\right), P_3(x)=\frac{1}{2}\left(5x^3-3x\right). \tag{1.2}$$

A generating function of Gegenbauer polynomials studied in ([4, p. 276, Sec. 143, Eqn. (1)], [7, p.83, Eqn. (7)]), is given by

$$\frac{1}{(1-2xt+t^2)^{\alpha}} = \sum_{n=0}^{\infty} C_n^{\alpha}(x)t^n = H_{\alpha}(x,t) \text{ (let)}, \ |t|<1, \forall x \in [-1,1], \alpha \in \left(-\frac{1}{2},\infty\right)\setminus\{0\}, \tag{1.3}$$

Here in (1.3), we have

$$C_n^{\alpha}(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(\alpha)_{n-k}(2x)^{n-2k}}{k!(n-2k)!}. \tag{1.4}$$

By (1.4) first few Gegenbauer polynomials are found as

$$C_0^{\alpha}(x)=1, C_1^{\alpha}(x)=2\alpha x, C_2^{\alpha}(x)=2\alpha(\alpha+1)x^2-\alpha. \tag{1.5}$$

Now in the field of polynomials to explore new interesting ideas in two and more dimensions, we introduce a two dimensional generating function as provided that $|(xy+1)t| < 1, (xy+1) \neq 0, |x+y| \leq 1, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$.

Clearly in (1.6), on setting for any $\beta, 0 < \beta < 1$,

$$(xy+1) = \beta, x+y = X,$$

we get generalized one dimensional Gegenbauer polynomials defined by (1.3), in the form

$$H_\alpha(x,y,\beta,t) = \frac{1}{(1-2\beta Xt+\beta^2t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(X)(\beta t)^n, \tag{1.7}$$

where, $x = \frac{X \pm \sqrt{X^2+4(1-\beta)}}{2}, y = \frac{X \mp \sqrt{X^2+4(1-\beta)}}{2}$.

Now on making an appeal to the formulae (1.3), (1.6) and (1.7), we find that

$$\lim_{\beta \to 1} H_\alpha\left(\frac{X}{2}, \frac{X}{2}, \beta, t\right) = \frac{1}{(1-2Xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(X)t^n = H_\alpha(X,t). \tag{1.8}$$

Now putting $\alpha = \frac{1}{2}$ in (1.8) and then making an appeal to the formula (1.1), we get

$$\lim_{\beta \to 1} H_{1/2}\left(\frac{X}{2}, \frac{X}{2}, \beta, t\right) = H_{1/2}(X,t) \tag{1.9}$$

In 1978, a different formula for two variables Gegenbauer polynomials has been studied due to Ricci [5] from the generating function

$$\frac{1}{(1-xt+yt^2-t^3)^\alpha} = \sum_{n=0}^{\infty} P_{n+1}^\alpha(x,y)t^n. \tag{1.10}$$

By the Eqn. (1.10) following recursion formulae are computed:

$$P_0^\alpha(x,y) = 0, \; P_1^\alpha(x,y) = 1, \; P_2^\alpha(x,y) = \alpha x; \tag{1.11}$$

$$(n+2)P_{n+3}^\alpha(x,y) = (n+\alpha+1)xP_{n+2}^\alpha(x,y) - (n+2\alpha)yP_{n+1}^\alpha(x,y) + (n+3\alpha-1)P_n^\alpha(x,y). \tag{1.12}$$

But no further results for these polynomials (1.10) were shown in the paper [5].

In this paper employing the formula (1.6), we obtain series of formulae for the double polynomials $G_n^\alpha(x,y)$ and then discuss the analytic and algebraic properties to derive some known and unknown results. Recently, Quintana [3] studied one dimensional two classes of Gegenbauer-Bernoulli's polynomials. Here, we introduce two dimensional two classes of Gegenbauer-Legendre polynomials and use them in computation of the region of convergence of any function containing Gegenbauer-Legendre mixed polynomials.

2 Series formulae due to the double polynomials $G_n^\alpha(x,y)$

In this section, we derive some series formulae for the function $G_n^\alpha(x,y)$ defined in (1.6) given by:

Lemma 2.1. *If $|x+y| \leq 1, |(xy+1)t| < 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$, then the two variables Gegenbauer polynomials are represented by the formula*

$$G_n^\alpha(x,y) = (1+xy)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(\alpha)_{n-k}(2(x+y))^{n-2k}}{k!(n-2k)!} = (1+xy)^n C_n^\alpha(x+y) \tag{2.1}$$

Proof. Consider the formula (1.6) in the form

$$\sum_{n=0}^{\infty} G_n^\alpha(x,y)t^n = [1-(xy+1)t\{2(x+y)-(xy+1)t\}]^{-\alpha}, \tag{2.2}$$

where in the right hand side under the conditions $|x+y| \leq 1, |(xy+1)t| < 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$, using the binomial formula, we get

$$\sum_{n=0}^{\infty} G_n^\alpha(x,y)t^n = \sum_{n=0}^{\infty} (1+xy)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(\alpha)_{n-k}(2(x+y))^{n-2k}}{k!(n-2k)!} t^n \tag{2.3}$$

Now making an appeal to (1.3) in (2.3), we get a relation involving Gegenbauer polynomials (1.6) as

$$\sum_{n=0}^{\infty} G_n^\alpha(x,y)t^n = \sum_{n=0}^{\infty} C_n^\alpha(x+y)(1+xy)^n t^n \tag{2.4}$$

Finally, equating the like powers of t in both sides of (2.4), we obtain (2.1). □

Lemma 2.2. *If* $|x+y| \leq 1, |(xy+1)t| < 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$, *then the two variables Gegenbauer polynomials is represented by the formula*

$$G_n^{\alpha}(x,y) = (2\alpha)_n (1+xy)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left((x+y)^2-1\right)^k ((x+y))^{n-2k}}{2^{2k}\left(\alpha+\frac{1}{2}\right)_k k!(n-2k)!} \tag{2.5}$$

Proof. Considering the formula (1.6) in the form

$$\sum_{n=0}^{\infty} G_n^{\alpha}(x,y)t^n = \{1-(x+y)(xy+1)t\}^{-2\alpha} \left[1 - \frac{(xy+1)^2 t^2 \left\{(x+y)^2-1\right\}}{\{1-(x+y)(xy+1)t\}^2}\right]^{-\alpha} \tag{2.6}$$

and making an appeal to (2.2), we derive

$$\sum_{n=0}^{\infty} G_n^{\alpha}(x,y)t^n = \sum_{n=0}^{\infty} (1+xy)^n t^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(2\alpha)_n (x+y)^{n-2k} \left((x+y)^2-1\right)^k}{2^{2k}\left(\alpha+\frac{1}{2}\right)_k (n-2k)!k!} \tag{2.7}$$

Finally, equating the coefficients of t^n both the sides, we derive the result (2.5). □

Theorem 2.1. *Under the conditions* $|x+y| \leq 1, |(xy+1)t| < 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$, *due to the Lemmas 2.1 and 2.2, there exists a generating function*

$$\sum_{n=0}^{\infty} \frac{G_n^{\alpha}(x,y)}{(2\alpha)_n} \left(\frac{t}{xy+1}\right)^n = e^{(x+y)t} {}_0F_1 \left[\begin{matrix} -; \alpha+\frac{1}{2} \end{matrix} \; \frac{t^2\left((x+y)^2-1\right)}{4}\right] \tag{2.8}$$

Proof. Under the given conditions $|x+y| \leq 1, |(xy+1)t| < 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \setminus \{0,1\}$, making an appeal to the result (2.5) of the Lemma 2.2, we get

$$G_n^{\alpha}(x,y) \frac{t^n}{(2\alpha)_n (1+xy)^n} = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{\left((x+y)^2-1\right)^k ((x+y))^{n-2k}}{2^{2k}\left(\alpha+\frac{1}{2}\right)_k k!(n-2k)!} t^n \tag{2.9}$$

In the formula (2.9) using the series rearrangement techniques [6, pp. 100-102], we obtain

$$\sum_{n=0}^{\infty} G_n^{\alpha}(x,y) \frac{t^n}{(2\alpha)_n (1+xy)^n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\left(\frac{(x+y)^2-1}{4}\right)^k (x+y)^n t^{n+2k}}{\left(\alpha+\frac{1}{2}\right)_k k!n!} \tag{2.10}$$

The formula (2.10) ultimately gives us the result (2.8). □

Thus employing (1.6) and (2.1), first few double Gegenbauer polynomials are as follows

$$G_0^{\alpha}(x,y) = 1, G_1^{\alpha}(x,y) = 2\alpha(1+xy)(x+y), G_2^{\alpha}(x,y) = \left\{2\alpha(\alpha+1)(x+y)^2 - \alpha\right\}(1+xy)^2. \tag{2.11}$$

3 Analytic and algebraic properties of $H_{\alpha}(x,y,t)$ and $G_n^{\alpha}(x,y)$

In this section, we study some analytic and algebraic properties of the two dimensional Gegenbauer polynomials defined by (1.6) and make their applications to obtain matrix equations and interesting functions.

Consider the operators $D_x \equiv \left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2} - (2\alpha+1)x\frac{\partial}{\partial x}\right\}$ and $D_y \equiv \left\{\left(1-y^2\right)\frac{\partial^2}{\partial y^2} - (2\alpha+1)\, y\frac{\partial}{\partial y}\right\}$, and operate them on (1.6) with the use of the differential equation [4, p. 279, Sec. 144, Eqn. (14)], we find

$$\begin{aligned}
D_x H_{\alpha}(x,y,t) &= \sum_{n=0}^{\infty} \left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2} - (2\alpha+1)x\frac{\partial}{\partial x}\right\} G_n^{\alpha}(x,y)t^n \\
&= \left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2} - (2\alpha+1)x\frac{\partial}{\partial x}\right\} \sum_{n=0}^{\infty} C_n^{\alpha}(x+y)((xy+1)t)^n \text{ (By (2.4))} \\
&= \sum_{n=0}^{\infty} C_n^{\alpha}(x+y) \left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2} - (2\alpha+1)x\frac{\partial}{\partial x}\right\} ((xy+1)t)^n \\
&+ \sum_{n=0}^{\infty} ((xy+1)t)^n \left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2} - (2\alpha+1)x\frac{\partial}{\partial x}\right\} C_n^{\alpha}(x+y)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} C_n^{\alpha}(x+y)((xy+1)t)^n((xy+1))^{-n}\left\{\left(1-x^2\right)\frac{\partial^2}{\partial x^2}-(2\alpha+1)x\frac{\partial}{\partial x}\right\}((xy+1))^n \\
&- \sum_{n=0}^{\infty} n(2\alpha+n)C_n^{\alpha}(x+y)((xy+1)t)^n. \qquad (3.1)
\end{aligned}
$$

Similarly as (3.1), we get

$$
\begin{aligned}
D_y H_\alpha(x,y,t) &= \left\{\left(1-y^2\right)\frac{\partial^2}{\partial y^2}-(2\alpha+1)y\frac{\partial}{\partial y}\right\}\sum_{n=0}^{\infty} C_n^{\alpha}(x+y)((xy+1)t)^n \\
&= \sum_{n=0}^{\infty} C_n^{\alpha}(x+y)t^n\left\{\left(1-y^2\right)\frac{\partial^2}{\partial y^2}-(2\alpha+1)y\frac{\partial}{\partial y}\right\}((xy+1))^n \\
&+ \sum_{n=0}^{\infty}((xy+1)t)^n\left\{\left(1-y^2\right)\frac{\partial^2}{\partial y^2}-(2\alpha+1)y\frac{\partial}{\partial y}\right\}C_n^{\alpha}(x+y) \\
&= \sum_{n=0}^{\infty} C_n^{\alpha}(x+y)((xy+1)t)^n((xy+1))^{-n}\left\{\left(1-y^2\right)\frac{\partial^2}{\partial y^2}-(2\alpha+1)y\frac{\partial}{\partial y}\right\}((xy+1))^n \\
&- \sum_{n=0}^{\infty} n(2\alpha+n)C_n^{\alpha}(x+y)((xy+1)t)^n. \qquad (3.2)
\end{aligned}
$$

Making an appeal to (3.1) and (3.2), we further obtain

$$
\begin{aligned}
D_x H_\alpha(x,y,t) - D_y H_\alpha(x,y,t) &= \frac{\left(y^2-x^2\right)}{(xy+1)^2}\sum_{n=0}^{\infty} n(n-1)C_n^{\alpha}(x+y)((xy+1)t)^n \\
&> \frac{\left(y^2-x^2\right)}{(xy+1)^2}\left\{\sum_{n=0}^{\infty} C_n^{\alpha}(x+y)((xy+1)t)^n\right\} - \frac{\left(y^2-x^2\right)}{(xy+1)^2}\{1+2\alpha(x+y)\}. \\
&> \frac{\left(y^2-x^2\right)}{(xy+1)^2}H_\alpha(x,y,t) - \frac{\left(y^2-x^2\right)}{(xy+1)^2}\{1+2\alpha(x+y)\}. \qquad (3.3)
\end{aligned}
$$

Theorem 3.1. *If all conditions of* (1.6) *are followed then the following symmetric condition is followed*

$$
\left|\left\{D_y - D_x + \frac{\left(y^2-x^2\right)}{(xy+1)^2}\right\}H_\alpha(x,y,t)\right| < \left|\frac{\left(y^2-x^2\right)}{(xy+1)^2}\{1+2\alpha(x+y)\}\right|. \qquad (3.4)
$$

Proof. Making an appeal to the formulae (3.1) to (3.3), we establish (3.4). □

Theorem 3.2. *If* $(xy+1)^n \neq 0\ \forall n \in \mathbb{N}\cup\{0\}$ *and* $G_{-n}^{\alpha}(\cdot) = 0 \forall n \in \mathbb{N}$, *then by Lemma 2.1 there exists*

$$
(n+1)G_{n+1}^{\alpha}(x,y) - 2(n+\alpha)(x+y)(xy+1)G_n^{\alpha}(x,y) + (n+2\alpha-1)(xy+1)^2G_{n-1}^{\alpha}(x,y) = 0. \qquad (3.5)
$$

Proof. Considering (1.6), we write

$$
\begin{aligned}
\frac{\partial}{\partial t}H_\alpha(x,y,t) &= \frac{\partial}{\partial t}\frac{1}{(1-2(x+y)(xy+1)t+(xy+1)^2t^2)^{\alpha}} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} G_n^{\alpha}(x,y)t^n \\
&\Rightarrow \frac{(\alpha)\left(2(x+y)(xy+1)-2(xy+1)^2t\right)}{(1-2(x+y)(xy+1)t+(xy+1)^2t^2)^{\alpha+1}} = \sum_{n=0}^{\infty} nG_n^{\alpha}(x,y)t^{n-1} \\
&\Rightarrow \left\{\left(2\alpha(x+y)(xy+1)-2\alpha(xy+1)^2t\right)\right\}\sum_{n=0}^{\infty} G_n^{\alpha}(x,y)t^n \\
&= \left(1-2(x+y)(xy+1)t+(xy+1)^2t^2\right)\sum_{n=0}^{\infty} nG_n^{\alpha}(x,y)t^{n-1} \\
&\Rightarrow \sum_{n=0}^{\infty} nG_n^{\alpha}(x,y)t^{n-1} - \sum_{n=0}^{\infty} 2(n+\alpha)(x+y)(xy+1)G_n^{\alpha}(x,y)t^n
\end{aligned}
$$

$$+\sum_{n=0}^{\infty}(n+2\alpha)(xy+1)^2G_n^{\alpha}(x,y)t^{n+1}=0.$$

$$\Rightarrow\sum_{n=0}^{\infty}(n+1)G_{n+1}^{\alpha}(x,y)t^n-\sum_{n=0}^{\infty}2(n+\alpha)(x+y)(xy+1)G_n^{\alpha}(x,y)t^n$$

$$+\sum_{n=0}^{\infty}(n-1+2\alpha)(xy+1)^2G_{n-1}^{\alpha}(x,y)t^n=0. \tag{3.6}$$

Finally, equating the coefficients of t^n on both the sides of (3.6), we derive the relation (3.5). □

Theorem 3.3. *Under the conditions of the Theorem 3.2, an appeal to (3.5), establishes the matrix equation*

$$\begin{bmatrix} (n+1)C_{n+1}^{\alpha}(x+y) & -2(n+\alpha)(x+y)C_n^{\alpha}(x+y) & (n+2\alpha-1)C_{n-1}^{\alpha}(x+y) \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\times\begin{bmatrix} ((xy+1))^n \\ ((xy+1))^n \\ ((xy+1))^n \end{bmatrix}=\begin{bmatrix} 0 \\ ((xy+1))^n \\ 2((xy+1))^n \end{bmatrix}. \tag{3.7}$$

Proof. Making an appeal to (2.1) in (3.5) we derive

$$\big[(n+1)(1+xy)^nC_{n+1}^{\alpha}(x+y)-2(n+\alpha)(x+y)(1+xy)^nC_n^{\alpha}(x+y)$$

$$+(n+2\alpha-1)(1+xy)^nC_{n-1}^{\alpha}(x+y)\big](1+xy)=0 \tag{3.8}$$

Since by the Theorem 3.2, $(xy+1)^n\neq 0\ \forall n\in\mathbb{N}\cup\{0\}\Rightarrow(xy+1)\neq 0$, the equation (3.8) terminated into the matrix equation (3.7), which proves the required result. □

Theorem 3.4. *Under the conditions of the Theorem* 3.2 *and by the matrix equation* (3.7)*, where* $C_{-j}^{\alpha}(\cdot)=0\ \forall j\in\mathbb{N}$*, a matrix element is defined for all* $j=0,1,2,\ldots,n$ *and* $k=0,1,2,\ldots,n+1$*, as*

$$a_{jk}(x,y)=C_j^{\alpha}(x+y)((xy+1))^k \tag{3.9}$$

Then (3.9) *satisfies an algebraic equation*

$$(j+1)\,a_{j+1k}\,(x,y)-2\,(j+\alpha)\,(x+y)\,a_{jk}\,(x,y)+(j+2\alpha-1)\,a_{j-1k}\,(x,y)=0. \tag{3.10}$$

Proof. Making an appeal to the equations (3.5) and (3.8), in general, we write

$$\big[(j+1)C_{j+1}^{\alpha}(x+y)-2(j+\alpha)(x+y)C_j^{\alpha}(x+y)\ +(j+2\alpha-1)C_{j-1}^{\alpha}(x+y)\big](1+xy)^k=0. \tag{3.11}$$

Employing (3.9), in (3.11) we establish the equation (3.10). □

Verification

We may also verify the Theorem 3.4 as:

Considering the equation (3.10) for $j=0,k=0$, we find

$$a_{10}(x,y)-2\alpha(x+y)a_{00}(x,y)+(2\alpha-1)a_{-10}(x,y)=0. \tag{3.12}$$

Since due to the polynomials given in (1.5) and (3.9), we have

$$a_{10}(x,y)=C_1^{\alpha}(x+y)((xy+1))^0=2\alpha(x+y),a_{00}(x,y)=1,\ \text{and}\ a_{-10}(x,y)=0.$$

Therefore, the equation (3.12) is satisfied.

Again for $j=1,k=1$, we get

$$2a_{21}(x,y)-2(1+\alpha)(x+y)a_{11}(x,y)+2\alpha a_{01}(x,y)=0. \tag{3.13}$$

Now in (3.13), due to (1.5) on setting :

$$a_{21}(x,y)=C_2^{\alpha}(x+y)(xy+1)=2\alpha(\alpha+1)(x+y)^2(xy+1)-\alpha(xy+1)$$
$$a_{11}(x,y)=C_1^{\alpha}(x+y)(xy+1)=2\alpha(x+y)(xy+1),$$
$$a_{01}(x,y)=C_0^{\alpha}(x+y)(xy+1)=(xy+1),$$

we get

$$4\alpha(\alpha+1)(x+y)^2(xy+1)-2\alpha(xy+1)-4\alpha(1+\alpha)(x+y)^2(xy+1)+2\alpha(xy+1)=0.$$

Hence equation (3.13) is satisfied.

Motivated by above theorems 3.1 to 3.4 , further for all $j = 0, 1, 2, \ldots$, we define an interesting symmetric function with respect to x and y such that

$$\Psi_j(x,y) = \sum_{k=0}^{\infty} \frac{a_{jk}(x,y)}{k!} = e^{(xy+1)} C_j^{\alpha}(x+y) \forall j = 0, 1, 2, \ldots. \tag{3.14}$$

Therefore making an appeal to the operators given in (3.1) and the formula (3.14), we obtain

$$\begin{aligned} D_x \Psi_j(x,y) &= \left\{ \left(1-x^2\right) \frac{\partial^2}{\partial x^2} - (2\alpha+1)x \frac{\partial}{\partial x} \right\} \Psi_j(x,y) \\ &= e^{(xy+1)} \left\{ \left(1-x^2\right) \frac{\partial^2}{\partial x^2} - (2\alpha+1)x \frac{\partial}{\partial x} \right\} C_j^{\alpha}(x+y) \\ &+ C_j^{\alpha}(x+y) \left\{ \left(1-x^2\right) \frac{\partial^2}{\partial x^2} - (2\alpha+1)x \frac{\partial}{\partial x} \right\} e^{(xy+1)} \\ &= e^{(xy+1)} \left\{ -j(2\alpha+j) + \left\{ \left(1-x^2\right) y^2 - (2\alpha+1)xy \right\} \right\} C_j^{\alpha}(x+y). \end{aligned} \tag{3.15}$$

Similarly, we get

$$\begin{aligned} D_y \Psi_j(x,y) &= \left\{ \left(1-y^2\right) \frac{\partial^2}{\partial y^2} - (2\alpha+1)y \frac{\partial}{\partial y} \right\} \Psi_j(x,y) \\ &= e^{(xy+1)} \left\{ \left(1-y^2\right) \frac{\partial^2}{\partial y^2} - (2\alpha+1)y \frac{\partial}{\partial y} \right\} C_j^{\alpha}(x+y) \\ &+ C_j^{\alpha}(x+y) \left\{ \left(1-y^2\right) \frac{\partial^2}{\partial y^2} - (2\alpha+1)y \frac{\partial}{\partial y} \right\} e^{(xy+1)} \\ &= e^{(xy+1)} \left\{ -j(2\alpha+j) + \left\{ \left(1-y^2\right) x^2 - (2\alpha+1)xy \right\} \right\} C_j^{\alpha}(x+y). \end{aligned} \tag{3.16}$$

Theorem 3.5. *If by the Theorem 3.4, a matrix element, given in* (3.9), *is extended by* $a_{jk}(x,y) = C_j^{\alpha}(x+y)((xy+1))^k$, *for all* $j = 0, 1, 2, 3, \ldots,$ *and* $k = 0, 1, 2, 3, \ldots,$; *a symmetric function* $\Psi_j(x,y)$ *is defined in* (3.14) *provided that* $C_{-j}^{\alpha}(\cdot) = 0 \ \forall \ j \in \mathbb{N}$, *then there exists an operator* $e^{-xy \frac{(D_x - D_y)}{(y-x)}}$ *such that*

$$e^{-xy \frac{(D_x - D_y)}{(y-x)}} \Psi_j(x,y) = e(y+x) C_j^{\alpha}(x+y). \tag{3.17}$$

Proof. Operating the symmetric function (3.14) due to the operators given in (3.1) and then making an appeal to the formulae (3.15) and (3.16), we obtain (3.17). □

4 Two classes of Two dimensional Gegenbauer-Legendre polynomials and application to determine the region of convergence of any function containing Gegenbauer-Legendre mixed polynomials

In this section to obtain required results, we make an appeal to the formula (1.6) and present a generating function of two classes of two dimensional Gegenbauer-Legendre polynomials in the form

$$\sum_{n=0}^{\infty} \mathfrak{G}_n^{\alpha,1/2}(x,y) t^n = \frac{1}{\left(1 - 2(x+y)(xy+1)t + (xy+1)^2 t^2\right)^{\alpha+\frac{1}{2}}} = H_{\alpha+1/2}(x,y,t) \tag{4.1}$$

provided that $|(xy+1)t| < 1, \forall |x+y| \leq 1, (xy+1) \neq 0, \alpha \in \left(-\frac{1}{2}, \infty\right) \backslash \{0, 1\}$.

Starting with (4.1), we derive various results on using two classes two dimensional polynomials $\mathfrak{G}_n^{\alpha,1/2}(x,y)$ to determine of the region of convergence of any function containing these mixed polynomials.

Theorem 4.1. *By the generating formula* (4.1), *two classes of two dimensional Gegenbauer-Legendre polynomials is expressed as*

$$\mathfrak{G}_n^{\alpha,1/2}(x,y) = \sum_{k=0}^{n} G_{n-k}^{\alpha}(x,y) P_k(x,y) \tag{4.2}$$

Proof. Starting with (4.1) and making an appeal to the formulae (1.6), we get

$$\sum_{n=0}^{\infty} \mathfrak{G}_n^{\alpha,1/2}(x,y) t^n = \frac{1}{\left(1 - 2(x+y)(xy+1)t + (xy+1)^2 t^2\right)^{\alpha+\frac{1}{2}}}$$

$$= \left\{\sum_{n=0}^{\infty} G_n^{\alpha}(x,y)t^n\right\}\left\{\sum_{k=0}^{\infty} P_k(x,y)t^k\right\} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} G_{n-k}^{\alpha}(x,y)P_k(x,y)t^n. \tag{4.3}$$

Here in (4.3), the two variables Legendre polynomials is defined by

$$\sum_{n=0}^{\infty} P_n(x,y)t^n = \frac{1}{(1-2(x+y)(xy+1)t+(xy+1)^2t^2)^{\frac{1}{2}}}$$

$$= \sum_{n=0}^{\infty}(1+xy)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2(x+y))^{n-2k}}{k!(n-2k)!}t^n. \tag{4.4}$$

Therefore equating like powers of the t in (4.3), we derive Gegenbauer-Legendre polynomials as given in (4.2). □

Then by (4.4), we have

$$P_n(x,y) = (1+xy)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2(x+y))^{n-2k}}{k!(n-2k)!} \tag{4.5}$$

Now formula (4.5) gives us first few polynomials as

$$P_0(x,y) = 1, P_1(x,y) = (1+xy)(x+y), P_2(x,y) = (1+xy)^2\left(\frac{3(x+y)^2}{2} - \frac{1}{2}\right), \ldots \tag{4.6}$$

Theorem 4.2. *By the generating function* (4.1)*, another generating formula is obtained as*

$$\sum_{n=0}^{\infty} \mathfrak{G}_n^{\alpha,1/2}(x,y)t^n = \{1-(xy+1)t\}^{-1-2\alpha}\,{}_1F_0\left[\ \alpha+\tfrac{1}{2};\ \tfrac{2(xy+1)t((x+y)-1)}{\{1-(xy+1)t\}^2}\ \right]. \tag{4.7}$$

Proof. Consider the generating formula (4.1), we write

$$\sum_{n=0}^{\infty} \mathfrak{G}_n^{\alpha,1/2}(x,y)t^n = \{1-(xy+1)t\}^{-1-2\alpha}\left(\frac{\{1-(xy+1)t\}^2 - 2(x+y)(xy+1)t + 2(xy+1)t}{\{1-(xy+1)t\}^2}\right)^{-\alpha-\frac{1}{2}}$$

$$= \{1-(xy+1)t\}^{-1-2\alpha}\left(1 - \frac{2(xy+1)t((x+y)-1)}{\{1-(xy+1)t\}^2}\right)^{-\alpha-\frac{1}{2}}$$

$$= \{1-(xy+1)t\}^{-1-2\alpha}\sum_{n=0}^{\infty}\frac{\left(\alpha+\frac{1}{2}\right)_n}{n!}\left(\frac{2(xy+1)t((x+y)-1)}{\{1-(xy+1)t\}^2}\right)^n$$

$$= \{1-(xy+1)t\}^{-1-2\alpha}\,{}_1F_0\left[\ \alpha+\tfrac{1}{2};\ \tfrac{2(xy+1)t((x+y)-1)}{\{1-(xy+1)t\}^2}\ \right]. \tag{4.8}$$

Hence the formula (4.7) is followed. □

Theorem 4.3. *For any β such that $\beta \neq 0$ and if $(xy+1) = \beta, x+y = X \in (-1,1)$ and*

$$F(x,y,t) = \sum_{n=0}^{\infty} \mathfrak{G}_n^{\alpha,1/2}(x,y)t^n, \text{ converges for } |t| < (\beta)^{-1}, \tag{4.9}$$

then the formula of β is given by

$$\beta = \left\{\frac{n!(1+2\alpha+2n)\Gamma(1+2\alpha)}{2^{2\alpha+1}\Gamma(1+\alpha+n)\Gamma(1+\alpha)}\int_{-1}^{1}\left(1-X^2\right)^{\alpha}\left\{\sum_{k=0}^{n} G_{n-k}^{\alpha}(x,y)P_k(x,y)\right\}P_n^{(\alpha,\alpha)}(X)dX\right\}^{1/n}, \tag{4.10}$$

where, $x = \frac{X \pm\sqrt{X^2+4(1-\beta)}}{2}, y = \frac{X \mp\sqrt{X^2+4(1-\beta)}}{2}$ *and* $(x-y)^2 + 4(\beta-1) = X^2$.

Proof. Under the conditions $0 < \beta < 1, (xy+1) = \beta, x+y = X \in (-1,1), \Re(\alpha) > -1$, use the formula (1.6) and the Theorem 4.2, from the statement of the Theorem 4.3 we evaluate some of the results of real x and y given by

$$x = \frac{X \pm\sqrt{X^2+4(1-\beta)}}{2}, y = \frac{X \mp\sqrt{X^2+4(1-\beta)}}{2}, \text{ by which we get } (x-y)^2 + 4(\beta-1) = X^2,$$

and again due to (4.9) we obtain

$$F(x,y,t)=\sum_{n=0}^{\infty}\mathfrak{G}_n^{\alpha,1/2}(x,y)t^n=\{1-\beta t\}^{-1-2\alpha}\sum_{n=0}^{\infty}\frac{\left(\alpha+\frac{1}{2}\right)_n}{n!}\left(\frac{2\beta t(X-1)}{\{1-\beta t\}^2}\right)^n$$

$$=\{1-\beta t\}^{-1-2\alpha}\sum_{n=0}^{\infty}\frac{\left(\alpha+\frac{1}{2}\right)_n{}^{(1+\alpha)_n}}{n!(1+\alpha)_n}\left(\frac{2\beta t(X-1)}{\{1-\beta t\}^2}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{(2\alpha+1)_n}{(1+\alpha)_n}P_n^{(\alpha,\alpha)}(X)\beta^n t^n, \tag{4.11}$$

(see also Rainville [4, p. 276, Sec. 143, Eqn. (2)]).

Now multiply (4.11) by $\left(1-X^2\right)^{\alpha}P_m^{(\alpha,\alpha)}(X)\forall m\in\mathbb{N}\cup\{0\}$ and then integrating both sides with respect to X from -1 to 1 , we get

$$\int_{-1}^{1}\left(1-X^2\right)^{\alpha}F(x,y,t)P_m^{(\alpha,\alpha)}(X)dX$$

$$=\sum_{n=0}^{\infty}t^n\int_{-1}^{1}\left(1-X^2\right)^{\alpha}\mathfrak{G}_n^{\alpha,1/2}(x,y)P_m^{(\alpha,\alpha)}(X)dX$$

$$=\sum_{n=0}^{\infty}\beta^n t^n\frac{(2\alpha+1)_n}{(1+\alpha)_n}\int_{-1}^{1}\left(1-X^2\right)^{\alpha}P_n^{(\alpha,\alpha)}(X)P_m^{(\alpha,\alpha)}(X)dX. \tag{4.12}$$

Then due to last two equalities, we obtain that

$$\sum_{n=0}^{\infty}t^n\left[\int_{-1}^{1}\left(1-X^2\right)^{\alpha}\mathfrak{G}_n^{\alpha,1/2}(x,y)P_m^{(\alpha,\alpha)}(X)dX\right.$$

$$\left.-\beta^n\frac{(2\alpha+1)_n}{(1+\alpha)_n}\int_{-1}^{1}\left(1-X^2\right)^{\alpha}P_n^{(\alpha,\alpha)}(X)P_m^{(\alpha,\alpha)}(X)dX\right]=0. \tag{4.13}$$

Since in the equation (4.13), $t\neq 0$, thus on employing the formula due to Srivastava and Manocha [7, p.71],

$$\int_{-1}^{1}(1-X)^a(1+X)^bP_n^{(a,b)}(X)P_m^{(a,b)}(X)dX=\frac{2^{a+b+1}\Gamma(a+1+n)\Gamma(b+1+n)}{n!(a+b+1+2n)\Gamma(a+b+1+n)}\delta_{mn},$$

where, δ_{mn} is a Kronecker delta and $\min\{\Re(a),\Re(b)\}>-1$, we find that

$$\int_{-1}^{1}\left(1-X^2\right)^{\alpha}\mathfrak{G}_n^{\alpha,1/2}(x,y)P_m^{(\alpha,\alpha)}(X)dX=\beta^n\frac{2^{2\alpha+1}\Gamma(\alpha+1+n)\Gamma(\alpha+1)}{n!(2\alpha+1+2n)\Gamma(2\alpha+1)}\delta_{mn}$$

again here put $m=n$, we obtain

$$\beta^n=\frac{n!(2\alpha+1+2n)\Gamma(2\alpha+1)}{2^{2\alpha+1}\Gamma(\alpha+1+n)\Gamma(\alpha+1)}\int_{-1}^{1}\left(1-X^2\right)^{\alpha}\mathfrak{G}_n^{\alpha,1/2}(x,y)P_n^{(\alpha,\alpha)}(X)dX. \tag{4.14}$$

Finally by the equation (4.14), we obtain the formula of the convergence of the function (4.9). □

5 Conclusions

In the Section 2, we derive various results of two dimensional Lgendre and Gegenbauer polynomials. We use these results in the third Section to determine matrix equation, matrix element and its concerning function to obtain some interesting analytic formulae. Chandel, Agrawal and Kumar [1] used Legendre polynomial in a problem on electrostatic potential in spherical regions. Kumar [2] solved three dimensional Legendre-Sturm-Liouville diffusion and wave problem generated due to fractional time derivative, we may apply our these matrix element and functions in computation of the scientific problems. Srivastava and Karlsson [6] analysed the convergence properties of two and more variable Gaussian hypergeometric series on applying Horn theorem but here we present a different formula of the convergence condition due to orthogonal property of the Jacobi polynomials.

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