ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 54(1) (2024), 113-118**

GRAPHICAL VERSION OF BROUWER'S FIXED POINT THEOREMS OVER METRIC SPACE

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(Received: March 10, 2021*; In format: June* 06, 2021*; Revised: March* 15, 2024*; Accepted: March* 24, 2024*)*

DOI: https://doi.org/10.58250/jnanabha.2024.54114

Abstract

The aim of this paper is to present graphical version of Brouwer's fixed point theorem and to establish some useful results of partial order set and nonexpansive mapping defined on a partial order set. One application of the fixed point theory by combining graph theory is discussed here by using Dijkstra's algorithms which is useful in computer science.

2020 Mathematical Sciences Classification: 47H09, 47H10

Keywords and Phrases: Graphical metric space, Dijkstra's algorithm, fixed point, edge-preserving mapping, nonexpansive mapping

1 Introduction

In 1922, Banach [4] established "Banach contraction principle" that is considered as a source of fixed point in the field of analysis. Banach proved that "Every contraction mapping define on a complete metric space has a unique fixed point". Recall that a point $x \in X$ is called fixed point of a mapping $f : X \to X$ if $f(x) = x$. Later several author's developed fixed point results for contraction mappings over different spaces.

Fixed point theory plays an important role not only in the field of analysis, but also used to find solutions of different mathematical problems like integral equations, differential equations, convex minimization problems, image recovery, signal processing (refer to [7, 9, 28]) etc.

On the other hand, it was crucial to obtain fixed point result for nonexpansive mappings. The first existing result of fixed point for nonexpansive mappings was obtained by Brouwer [6] in 1910. Brouwer proved that A closed unit ball of a finite-dimensional normed linear space has a fixed point. Lots of work has been done to established fixed point results for nonexpansive mappings (refer to [18, 30, 34, 35]).

Fixed point theory has many applications in different fields of mathematics (refer to [10, 17]). One of such field is graph theory. The study of fixed points endowed with graph has increased rapidly in past few years. The fixed point theory by using graph was first studied by Echnique [15] in 2005. After that, in 2006 Espinola and Kirk [16] proved that " Any commuting family of edge-preserving mappings of a connected reflexive graph G that contains no cycles or infinite paths always has at-least one common fixed edge". Also, Espinola and Kirk [16] verified that a commutative family of nonexpansive mappings of a complete R tree X into itself always has a non-empty common fixed points set if X does not contain a geodesic ray. For further results refer to [19, 26, 29].

In 2008, Jachymsky [21] gave a generalization of the Banach contraction principle to a mapping on metric space endowed with graphs. In 2015, Alfuraidan and Khamsi [3] define the concept of multi-valued $G-$ monotone nonexpansive mapping defined on a metric space with a graph. Tiammee [33] proved Browde's convergence theorem for $G-$ nonexpansive mapping in Banach space with a directed graph.

In 2016, Tripak [34] obtained common fixed points for nonexpansive mapping in Banach space endowed with directed graph by using Ishikawa [20] iteration scheme. In 2022, Dewangan et al. [12] established some fixed point theorem in Hilbert spaces by using Mann [24] and normal $S-$iteration scheme.

In the same year, Aggarwal et al. [1] obtained Browder and Gohde fixed point theorem for nonexpansive mapping in $CAT(0)$ space. For more results in this direction, refer to [13, 14, 21, 22, 26, 27, 29] etc.

The aim of this paper is to study some fixed point results for nonexpansive mapping by using graph theory. This paper contains following sections. The first section of the paper contains some results related to partial order set and fixed point results for nonexpansive mappings. The next section contains an application of Dijkstra's algorithm [23]. Recall that Dijkstra's algorithm [23] is one of the known algorithm, which is used in calculation of shortest path in graph theory.

2 Preliminaries

first recall some of the definitions and terminologies, which will be used here onwards and restate a few earlier results exists in the literature.

Definition 2.1 ([23]). *A directed graph or digraph G is defined as an ordered pair $(V(G), E(G))$, where $V(G)$ is a set and $E(G)$ is a binary relation on $V(G)$. The elements of $V(G)$ are called vertices, and the ordered pairs in $E(G)$ are called the edges of the directed graph.*

Definition 2.2 ([25]). *Let (X, d) be a metric space and $G = (V(G), E(G))$ be a directed graph with the set of vertices $V(G) = X$ and a set of edges $E(G)$ that contains all the loops, that is $E(G) \supseteq \{(x, x) : x \in V(G)\}$. Suppose that G has no parallel edges. If $x, y \in V(G)$ then a directed path from x to y is a sequence $\{x_i\}$, $i = 0, 1, ..., M$ of length M of $M + 1$ vertices such that $x_0 = x$, $x_M = y$, where x_0 and x_M are initial and terminal vertices respectively, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, ..., M$. If there is a path between every pair of vertices, then such graph is called connected. In a directed graph if $(u, v) \in E(G)$ and $(v, u) \notin E(G)$, then such a graph is called oriented graph.*

Definition 2.3 ([25]). *Let $G = (V(G), E(G))$ be a directed graph. The inverse of $G = (V(G), E(G))$ is denoted by G^{-1}. It is a graph obtained from G by reversing direction of its edges, i.e., $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$.*

Remark 2.1 ([21]). *Let $G = (V(G), E(G))$ be any graph, then $(V^{'}(G), E^{'}(G))$ is a sub graph of G , if $V^{'}(G) \supseteq V(G)$ and $E^{'}(G) \supseteq E(G)$ and for any edge $(x, y) \in E^{'}(G)$, $x, y \in V^{'}(G)$.*

Remark 2.2. *An undirected graph can be obtained from directed graph G by removing direction of its edges. An undirected graph is called weighted graph if there is a weight associated to each edge.*

Remark 2.3 ([3]). *A relation R on $V(G)$ exists by the rule yRz if there is a (directed) path in G from y to z.*

Definition 2.4 ([16]). *An undirected graph that have a loop at every vertex (i.e., $(a, a) \in E(G)$) for each $a \in V(G)$ and no multiple edges, is called reflexive graph.*

Definition 2.5 ([23]). *The length of a path in weighted graph is defined to be the sum of the lengths of the edges in the path. A path with shortest length between any two vertices is called shortest path.*

Definition 2.6 ([31]). *Let X be a non-empty set endowed with a graph $G = (V(G), E(G))$ and $d_G : X \times X \to R$ be a function satisfying the conditions-*

- $d_G(x, y) \geq 0$ *for all* $x, y \in X$,
- $d_G(x, y) = 0 \Leftrightarrow x = y$,
- $d_G(x, y) = d_G(y, x)$,
- $(xPy)_G$, $z \in (xPy)_G \Rightarrow d_G(x, y) \leq d_G(x, z) + d_G(z, y)$.

Then d_G is called graphical metric on X and (X, d_G) is called graphical metric space.

Example 2.1 ([31]). Every metric space (X, d) with graph G is graphical metric space, where $V(G) = X$ and $E(G) = X \times X$.

Remark 2.4 ([31]). *In a graphical metric space the triangle inequality is satisfied by those points which are situated on some path included in the graphical structure associated with the space.*

Remark 2.5 ([8]). *Recall that the standard metric used on graphs is the distance metric $d(u, v)$ = length of the shortest path joining nodes u and v.*

Definition 2.7 ([2]). *Let $G = (V(G), E(G))$ be a graph. A mapping $T : V(G) \to V(G)$ is called $G-$ edge preserving mapping if $(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)$, $\forall x, y \in V(G)$.*

Example 2.2. Consider a finite connected graph in which each edge has path length 1. Let G_1 be a sub graph of G, then a mapping $f : V_1 \to V$ is an edge preserving mapping, since $d(f(a), f(b)) \leq d(a, b)$. It is clear that f is nonexpansive mapping. Note that a mapping is nonexpansive if $d(f(a), f(b)) \leq d(a, b)$ for all a, b.

Definition 2.8 ([23])**.** *Let P be a nonempty set and $\preceq$ be a binary relation on P. Then, if the binary relation $\preceq$ is reflexive, anti symmetric and transitive on P, then binary relation $\preceq$ is called partial order relation on P and $(P, \preceq)$ is called partial order set or poset.*

Definition 2.9 ([23])**.** *If $\preceq$ is partial ordering on A, then $\preceq$ is called total ordering if for all $x, y \in A$, either $x \preceq y$ or $y \preceq x$.*

Definition 2.10 ([11])**.** *Let $(P, \preceq)$ be a poset and $A \subseteq P$. If $u \in P$ is an element such that $a \preceq u$ for all $a \in A$, then u is called an upper bound of A and it is called least upper bound of A if $u \preceq v$ for all upper bounds v of A. Similarly $u \in P$ is an element such that $u \preceq a$ for all $a \in A$, then u is called a lower bound of A and it is called greatest lower bound of A, if $v \preceq u$ for all lower bounds v of A.*

Definition 2.11 ([32])**.** *Let $(P, \leq)$ be a poset. An element $y \in P$ is called least element of P if $y \leq x$ for all $x \in P$. Similarly, an element $y \in P$ is called greatest element of P if $x \leq y$ for all $x \in P$.*

Lemma 2.1 ([5])**.** *A subset $S \subseteq P$ of a poset P have at-most one greatest and at-most one least element, if they exists.*

Definition 2.12 ([23])**.** *A partially ordered set is called well ordered if every non-empty subset of it has a least element.*

Remark 2.6. *From definition 2.12, it can be concluded that every non-empty well ordered set has a least element.*

Definition 2.13. *If $\preceq$ is partial ordering on A, then $\preceq$ is called total ordering if for all $x, y \in A$, either $x \preceq y$ or $y \preceq x$.*

Definition 2.14 ([2])**.** *Let $(V(G), d)$ be complete metric space, where G is connected graph and $E(G)$ contains all the loops. Let $x_* \in V(G)$, then x_* is called minimal element of $V(G)$ if $x \preceq x_*$ then $x = x_*$ where $\preceq$ is partial order relation on $V(G)$.*

Remark 2.7. *Let $(A, \preceq)$ be a poset. If A has least element, then it is unique and also a minimal element of A.*

3 Main Result

Lemma 3.1. *Let $(V(G), d)$ be a metric space and G is reflexive complete graph and $E(G)$ contains all the loops. Let $f : V(G) \to V(G)$ be a mapping defined by $f(x) = x$. Let $\preceq$ be a binary relation define on $V(G)$ by $x \preceq y \Leftrightarrow$ there is a path between x and y and $f(x) = f(y)$. Then $(V(G), \preceq)$ is poset.*

Proof. (i) Reflexivity is obious.

(ii) Suppose that $x \preceq y$ and $y \preceq x$. Then $(x,y) \in E(G)$ with $f(x) = f(y)$ and $(y,x) \in E(G)$ with $f(y) = f(x)$. Since f is identity mapping, therefore $x = y$.

(iii) Suppose that $x, y, z \in V(G)$ such that $x \preceq y$ and $y \preceq z$. Then $(x, y) \in E(G)$ with $f(x) = f(y)$ and $(y, z) \in E(G)$ with $f(y) = f(z)$. Clearly $f(x) = f(z)$, hence $x \preceq z$ (It is possible because $V(G)$ is complete graph). Hence $(V(G), \preceq)$ is poset.

□

Lemma 3.2. *Consider the poset $(V(G), \preceq)$ as in Lemma 3.1. Suppose that $\preceq$ is total ordering on $V(G)$, then every non-empty subset of $V(G)$ has a least element (equivalently greatest element).*

Proof. To show the existence of least element, induction method is proceed on number of vertices. Suppose that $A \subseteq V(G)$ such that A has only one element. Clearly A has a least element. Now consider result is true for the subsest having n elements. Consider a subset A with $n+1$ elements, then $A = A^{'} \cup \{y\}$ for some $y \in A$ and $A^{'}$ with n elements. By induction, since $A^{'}$ has a least element say x and $\preceq$ is total ordering,. so either $x \preceq y$ or $y \preceq x$. In the first case x is the least element of A. In the second case, we claim that y is the least element of A. If $z \in A$ then either $z \in A^{'}$ or $z = y$ and $y \preceq x \preceq z$. In either case $y \preceq z$ as desired. □

Corollary 3.1. *From Lemma 3.2, $(V(G), \preceq)$ is well ordered set.*

Lemma 3.3. *Let (X, d) be a complete metric space with a graph $G = (V, E)$, where $V(G) = X$ and $E(G) = X \times X$. Consider a poset $(V(G), \preceq)$ as in Lemma 3.1. Then $V(G)$ is complete.*

Proof. To prove that $V(G)$ is complete, it is sufficient to prove that for every $A \subseteq V(G)$, $\sup A$ and $\inf A$ exists. First of all, we will show that if a subset A of poset $(V(G), \preceq)$ has a greatest element, then this element will be supremum of A. Due to Lemma 3.2, A has a greatest element say g, then $x \preceq g$ for all $x \in A$, so g is an upper bound of A. If $b \in V(G)$ is such that it is an upper bound of A, then $g \preceq b$ which implies that g is least upper bound (supemum) of A. This shows that A has a supremum.
Now suppose that $\sup A$ exists for every $A \subseteq V(G)$. If $\inf A$ exists, then result will be proved. For this, we claim that for every $A \subseteq V(G)$,

$$\inf A = \sup lb(A),$$

where $lb(A)$ and $ub(A)$ are lower bound and upper bound of A. Since

$$lb(A) = \{y \in V(G) : \forall x \in A, y \preceq x\}. \tag{3.1}$$

Let $\sup lb(A) = a$. We will show that $\inf A = a$. By Equation 3.1, $x \in A$ and $y \preceq x$, i.e. x is an upper bound of $lb(A)$ and $\sup(lb(A)) \leq ub(lb(A))$, i.e., $\sup(lb(A)) \leq x \Rightarrow a \leq x$. Clearly, a is lower bound of A, i.e., $a \in lb(A)$, so by Equation (3.1), for every $y \in lb(A)$, $y \leq a$. ☐

Lemma 3.4. *Let $G = (V(G), E(G))$ be a connected an undirected graph and $G_1 = (V_1(G), E_1(G))$ be a sub graph of G. Let $f : V_1(G) \to V(G)$ be $G-$edge preserving mapping as well as nonexpansive mapping. Then f is uniformly continuous.*

Proof. Define a metric $d(u, v) = l(u, v)$, ,i.e., d denotes the path length between vertices u and v on V where $u, v \in V$. Then (V, d) is a metric space. Let $a, b \in V$ then $(f(a), f(b)) \in E_1$. Choose $\delta \in R$ with $\delta > 0$ such that $d(a, b) < \delta$. Since f is nonexpansive, $d(f(a), f(b)) \leq d(a, b) < \delta$. Choose $\varepsilon > 0 \in R$ such that $\delta < \varepsilon$, then $d(f(a), f(b)) < \varepsilon$ which implies that f is uniformly continuous. ☐

Lemma 3.5. *Let $(V(G), d)$ be complete metric space where G is a complete reflexive graph and $E(G)$ contains all the loops. Consider the poset $(V(G), \preceq)$ as in Lemma 3.1, then $V(G)$ has a minimal element.*

Proof. Consider a non-empty poset $(V(G), \preceq)$. Then by Lemma 3.2, $(V(G), \preceq)$ is well ordered set and due to Remark 2.6, it has a least element. Due to Remark 2.7, this least element will be minimal element. ☐

Theorem 3.1. *Consider $(V(G), d_*)$ as a complete metric space under metric d_* where d_* denote path length as defined in Remark 2.5 and G is complete reflexive graph and $E(G)$ contains all the loops. Let $\preceq$ be a partial order relation on $V(G)$ and $T : V(G) \to V(G)$ be a nonexpansive mapping. Then T has a fixed point in $V(G)$.*

Proof. Suppose that $\{x_n : x_n \in V(G)\}$ is any Cauchy sequence in $V(G)$, then by completeness of $V(G)$, there exists a point $x \in V(G)$ such that $x_n \to x$ as $n \to \infty$. Now

$$d_*(Tx_n, Tx) \leq d_*(x_n, x).$$

This implies that $Tx_n \to Tx$ as $n \to \infty$. Since $x \in V(G) \Rightarrow Tx \in V(G)$, i.e., $(x, Tx) \in E(G)$. Due to anti symmetry of $\preceq$, $x = Tx$. ☐

The following theorem is graphical version of Brouwer's [6] fixed point theorem.

Theorem 3.2. *Let $G = (V(G), E(G))$ be a finite complete reflexive graph and $E(G)$ contains all the loops. Let us define path length between two vertices $x, y \in V(G)$ by $l(x, y) = |x - y|$ (i.e., the shortest path between x and y) and assign metric d on $V(G)$ by $d(x, y) = l(x, y)$. Let $G^{'}$ be a sub graph which contains those circuits having path length is one. Then $G^{'}$ leaves some vertex fixed.*

Proof. Clearly $(V(G), d)$ is metric space. Let $f : G^{'} \to G$ be $G-$edge preserving mapping. The aim is, if $x_* \in V(G^{'})$ is fixed point of f then $(x_*, f(x_*)) \in E(G)$. Consider a binary relation $\preceq$ on $G^{'}$ as in Lemma 3.1, then $(G^{'}, \preceq)$ is poset due to Lemma 3.1. Now suppose that $x_* \in V(G^{'})$ such that $f(x_*) = x_*$, then $(x_*, x_*) = (x_*, f(x_*)) \in E(G^{'}) \subseteq E(G) \Rightarrow (x_*, f(x_*)) \in E(G)$, i.e., $E(G)$ contains self loop at x_*. This shows that $G^{'}$ has a vertex that is always fixed . ☐

4 Application

A very nice application of fixed point theory in graph theory is discussed here. Dijkstra's algorithm is used to find shortest path in a given graph. First, recall Dijkstra's algorithm.

Suppose that we have to calculate shortest path from vertex a to vertex z in a weighted graph. The algorithm [23] is as follows-

(i) Initially, let $P = \{a\}$ and $T = V - \{a\}$. For every vertex $t \in T$, let $l(t) = w(a, t)$.

(ii) Select the vertex in T that has the smallest index with respect to P. Let x denote such vertex.

(iii) If x is the vertex we wish to reach from a, stop. If not, let $P^{'} = P \cap \{x\}$ and $T^{'} = T - \{x\}$. For every vertex t in $T^{'}$, compute its index with respect to $P^{'}$.

(iv) Repeat step (ii) and (iii) using $P^{'}$ as P and $T^{'}$ as T.

Theorem 4.1. *Let G be a finite weighted undirected graph. Let $d : V(G) \times V(G) \to R$ such that $d(x, y) = l(x, y)$ for $x, y \in V(G)$ and $l(x, y)$ is defined as in Theorem 3.2. Suppose that $f : V(G) \to V(G)$ is $G-$edge preserving mapping. Then f always leave some edge fixed.*

Proof. It is clear that $(V(G), d)$ is a metric space. Using Dijkstra's algorithm, there exists a pair of vertices (an edge) in $E(G)$, that has shortest path length. This edge always remain fixed. □

5 Conclusion

The present paper contains version of Brouwer's fixed point theorem by including graph theory. Some properties of partial ordered set including graph theory is discussed here. One application of fixed point theory is discussed here in finding shortest path in graph, which is most important area of study in computer science.

Acknowledgement. The author is thankful to the Editor and Reviewers for their valuable suggestions.

References

[1] S. Aggarwal, I. Uddin and J. J. Nietro, Existence and convergence of fixed point for a $G-$non-Lipschitzian mapping, *Miskolc Mathematical Notes*, **23**(1) (2022), 2940.

[2] M. R. Alfuraidan, Remarks on Caristi's fixed point theorem in metric spaces with graph, *Fixed Point Theory and Applications*, **240**(1) (2014), 1-8.

[3] M. R. Alfuraidan and M.A. Khamsi, Fixed points of monotone nonexpansive mappings on a hyperbolic metric space with a graph, *Fixed Point Theory and Applications*, **44** (2015), 1-10.

[4] S. Banach, Sur les operations dans ensembles abstraits et leur application aux equations integrales, *Fundamenta Mathematicae*, **3** (1922), 133-181.

[5] J. G. Belinfante, Class notes on poset theory in Carnegie-Mellon University in Pittsburgh, Pensylvania, 1995.

[6] L. E. Brouwer, Uber Abbildungen Von Mannigfaltigken, *Mathematische Annalen*, **71** (1910), 97-115.

[7] F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *Journal of Mathematical Analysis and Applications*, **20**(2) (1967), 197-228.

[8] F. Buckley and F. Harary, Distance in graphs, Addison-Wesley Publishing Company, New York, 1989.

[9] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems*, **20(1)** (2004), 103-120.

[10] A. Bucur, About applications of the fixed point theory, *Scientific Bulletin*, **XXII 1**(43) (2017), 13-17.

[11] J. P. Chauhan and H. K. Pathak, Advanced Discrete Mathematics, Shiksha Sahitya Prakashan, Meerut, 2000.

[12] K. Dewangan and N. Gurudwan, Some convergence results for nonexpansive mappings in Hilbert space endowed with graph, *Bulletin of Calcutta Mathematical Society*, **114**(6) (2022), 853-868.

[13] K. Dewangan and N. Gurudwan, Approximation of fixed points for multi-valued $G-$nonexpansive mappings through Ullah iteration scheme in uniformly convex Banach spaces endowed with graph, *Poincare Journal of Analysis & Applications*, **9**(2) (2022), 337-350.

[14] K. Dewangan, N. Gurudwan and et al., Iterative Approximation of common fixed points for edge-preserving quasi-nonexpansive mappings in Hilbert spaces along with directed graph, *Journal of Mathematics*, **2023** (2023), Article ID 6400676, 9 pages.

[15] F. Echenique, A short and constructive proof of Tarki's fixed point theorem, *International Journal of Game Theory*, **33** (2005), 215-218.

[16] R. Espinola and W. A. Kirk, Fixed point theorems in R trees with applications to graph theory, *Toplology and its Applications*, **153** (2006), 1046-1055.

[17] A. P. Farazjadeh, A. Kaewchareon and S. Plubtieng, An application of fixed point theory to a non linear differential equation, *Abstract and Applied Analysis*, **2014** (2014), 1-7.

[18] C. Garodia, I. Uddin and J. J. Nietro, Mann iteration for monotone nonexpansive mappings in ordered $CAT(0)$ space with an application to integral equations, *Journal of Inequalities and Applications*, **339** (2018), 1-13.

[19] R. J. Hussain and K. Manoj, Some fixed point results for cyclic $(\psi, \phi, Z)-$contraction in partial metric spaces, *Jñānābha*, **53**(1) (2023), 110-117.

[20] S. Ishikawa, Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*, **44**(1) (1974), 147-150.

[21] J. Jachymski, The contraction principle for mappings on a metric space with graph, *Proceedings of American Mathematical Society*, **136**(4) (2008), 1359-1373.

[22] J. Jachymski and I. G. Gwozdz, IFS on a metric space with a graph structure and extension of the Kelisky Rivlin theorem, *Journal of Mathematical Analysis and Applications*, **356** (2009), 453-463.

[23] C. L. Liu and D. P. Mohapatra, Elements of Discrete Mathematics-a computer oriented approach, Tata McGraw Hill Education Pvt. Ltd, New Delhi, 2000.

[24] W. R. Mann, Mean value methods in iteration, *Proceedings of the American Mathematical Society*, **4**(3) (1953), 506-510.

[25] S. K. Mohanta, Common fixed point results in c^*-algebra valued b-metric spaces via diagraphs, *CUBO: A mathematical journal*, **20**(1) (2018), 41-64.

[26] A. Muraliraj and R. Thangathamizh, Some fixed point theorems for iterated contraction maps in intuitionistic fuzzy metric space, *jnanabha*, **52**(1) (2022), 66-68.

[27] R. Nowakowski and I. Rival, Fixed edge theorem for graphs with loops, *Journal of Graph Theory*, **3** (1979), 339-350.

[28] C. I. Podilchuk and R. J. Mammone, Image recovery by convex projections using a least-squares constraint, *Journal of Optical Society of America A*, **7**(3) (1990), 517-521.

[29] G. S. Saluja, Some fixed point theorems for weak contraction mappings in $S-$metric spaces, *jnanabha*, **50**(1) (2020), 20-26.

[30] A. Sharma, Approximating fixed points of nearly asymptotically nonexpansive mappings in $CAT(k)$ spaces, *Arab Journal of Mathematical Sciences*, **24**(2) (2018), 166-181.

[31] S. Shkukla, C. Vetro and S. Radenovic, Graphical metric space: a generalized setting in fixed point theory, *RACSAM*, **111**(3) (2016), 641-655.

[32] J. P. Tremblay and R. Manohar, Discrete mathematical structures with applications to computer science, Tata McGraw-Hill Pub. Comp. Ltd., 2003.

[33] J. Tiammee, A. Kaewkhao, and S. Suantai, On Browders convergence theorem and Halpern iteration process for $G-$nonexpansive mappings in Hilbert spaces endowed with graphs, *Fixed Point Theory and Applications*, **2015** (2015), 1-12.

[34] O. Tripak, Common fixed points of G-nonexpansive mappings on Banach spaces with a graph, *Fixed Point Theory and Applications*, **87** (2016), 1-8

[35] T. J. Xiong and H. Y. Lan, General modified viscosity implicit rules for generalized asymptotically nonexpansive mappings in complete $CAT(0)$ spaces, *Journal of Inequalities and Applications*, **176** (2019), 1-16.